1. Compute the sum \( S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2} \) for \( x_i = \frac{i}{101} \).

2. Given the following triangular arrangement of circles:

Each of the numbers 1, 2, …, 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

3. Let \( ABC \) be a triangle. Let \( M \) and \( N \) be the points in which the median and the angle bisector, respectively, at \( A \) meet the side \( BC \). Let \( Q \) and \( P \) be the points in which the perpendicular at \( N \) to \( NA \) meets \( MA \) and \( BA \), respectively, and \( O \) the point in which the perpendicular at \( P \) to \( BA \) meets \( AN \) produced. Prove that \( QO \perp BC \).

4. Let \( n, k \) be given positive integers with \( n > k \). Prove that

\[
\frac{1}{n+1} \cdot \frac{n^n}{k^n(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.
\]

**Solution.** We note that

\[
\frac{n!}{k!(n-k)!} = \binom{n}{k}.
\]

Now

\[
n^n = (k + (n-k))^n = \sum_{r=0}^{n} \binom{n}{r} k^r(n-k)^{n-r}
\] (1)
Since \( n, k \in \mathbb{N} \) with \( n > k \), each term of (1) is positive. In particular, \( n^n \) is greater than the \( k \)th term of (1), i.e.

\[
n^n > \binom{n}{k} k^k (n-k)^{n-k} \\
\frac{n^n}{k^k (n-k)^{n-k}} > \binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

5. Given a permutation \( (a_0, a_1, \ldots, a_n) \) of the sequence 0, 1, \ldots, \( n \), a transposition of \( a_i \) with \( a_j \) is called legal if \( a_i = 0 \) for \( i > 0 \), and \( a_{i-1} + 1 = a_j \). The permutation \( (a_0, a_1, \ldots, a_n) \) is called regular if after a number of legal transpositions it becomes \( (1, 2, \ldots, n, 0) \). For which numbers \( n \) is the permutation \( (1, n, n - 1, \ldots, 3, 2, 0) \) regular?