Euclidean Algorithm Problems

1. For each of the following pairs of integers \(a, b\) use the Euclidean Algorithm to find \(d = (a, b)\) and find a pair of integers \(x, y\) such that \(ax + by = d\).
   
   (i) \(a = 85, b = 41\);  
   
   (ii) \(a = 2613, b = 637\).

2. Show that if there exist integers \(x, y\) such that \(ax + by = 1\) then \((a, b) = 1\).

3. Show that \((3k + 2, 5k + 3) = 1\) for any integer \(k\).

4. Show that \((a, a + 2) = 2\) if \(a\) is even and \((a, a + 2) = 1\) otherwise.

5. Show that if \((a, b) = 1\) then \((a + b, a - b) = 1\) or 2.

6. Find all solutions to the following Diophantine Equations.
   
   (i) \(2x + 5y = 11\).  
   
   (ii) \(12x + 18y = 50\).  
   
   (iii) \(202x + 74y = 7638\).

   Does equation (iii) have a solution in positive integers \(x, y\)?

7. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost 25c each and oranges cost 18c each, how many of each type of fruit did the grocer order?

8. An apartment block has units at two rates: most rent at \$87/week, but a few rent at \$123/week. When all are rented the gross income is \$8733/week. How many units of each type are there?

*9. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.

One is in her teens and ages are in completed years. How old are they?