1. Let $D$ be a point on side $BC$ of triangle $ABC$. Let $K, L$ be the circumcentres of triangles $ABD$ and $ADC$, respectively. Prove that triangles $ABC$ and $AKL$ are similar.

Solution. Firstly,

\[
\begin{align*}
KA &= KD, & \text{radii of circumcircle of } \triangle ABD \\
LA &= LD, & \text{radii of circumcircle of } \triangle ADC \\
KL &= KL, & \text{common side} \\
\therefore \triangle AKL & \cong \triangle DKL, & \text{by the SSS Rule.}
\end{align*}
\]

Hence $\angle AKL = \angle DKL = \frac{1}{2} \angle AKD$ and $\angle ALC = \angle DLK = \frac{1}{2} \angle ALD$.

We recall the result:

The angle subtended at a circle’s centre is twice the angle subtended at the the circumference on the same arc.

\[
\begin{align*}
\angle ABC &= \angle ABD = \frac{1}{2} \angle AKD = \angle AKL, & \text{angles on arc } AD \text{ for circumcircle of } \triangle ABD \\
\angle ACB &= \angle ACD = \frac{1}{2} \angle ALD = \angle ALC, & \text{angles on arc } AD \text{ for circumcircle of } \triangle ADC \\
\triangle ABC & \sim \triangle AKL, & \text{by the AA Rule.}
\end{align*}
\]

2. Prove that, among any fifteen composite numbers selected from the first 2006 positive integers, there will be two that are not relatively prime.

Solution. Since $2006 < 47^2$, every composite number $\leq 2006$ has a prime divisor $< 47$. There are precisely 14 primes $< 47$, namely

\[2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.\]

Here are some intriguing statistics to remember. There are . . .

10 primes less than 30,
15 primes less than 50, and
25 primes less than 100.

Check this for yourself!

Hence, by the pigeonhole principle, at least two of a set of $15 = 14 + 1$ composites in \{1, 2, \ldots, 2006\} are divisible by the same prime.
3. For each integer $n$, let $a_n$ be the integer nearest to $\sqrt{n}$.
Prove that, for each positive integer $n$, the equation

$$a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2)$$

holds.

**Solution.** We prove the result by induction. Define

$$P(n) : a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2).$$

- For $n = 1$, $a_1 = a_2 = 1$. Note that $1^2 + 1 = 2$. So

  $$\text{LHS of } P(1) = a_1 + a_2 = 1 + 1 = 2(1^2) = \text{RHS of } P(1)$$

  So $P(1)$ holds.

- Now we show $P(k) \implies P(k + 1)$. The extra terms on the LHS of $P(k + 1)$ are: $a_{k^2+k+1}, \ldots, a_{(k+1)^2+(k+1)}$. Observe that:

  $$(k + \frac{1}{2})^2 = k^2 + \frac{1}{4} < k^2 + k + 1$$

  $\therefore k + \frac{1}{2} < a_{k^2+k+1}$

  $\therefore k + 1 \leq a_{k^2+k+1}$

  $$(k + 1 + \frac{1}{2})^2 = (k + 1)^2 + (k + 1) + \frac{1}{4} > (k + 1)^2 + (k + 1)$$

  $\therefore a_{(k+1)^2+(k+1)} < (k + 1) + \frac{1}{2}$

  $\therefore a_{(k+1)^2+(k+1)} \leq k + 1$

  Also, $a_{k^2+k+1} \leq a_{k^2+k+2} \leq \cdots \leq a_{(k+1)^2+(k+1)}$. So, in fact all these terms are equal to $k + 1$, i.e.

  $$a_{k^2+k+1} = a_{k^2+k+2} = \cdots = a_{(k+1)^2+(k+1)} = k + 1.$$ 

  So, assume $P(k)$. Then, we have

  $$\text{LHS of } P(k + 1) = \sum_{i=1}^{(k+1)^2+(k+1)} a_i$$

  $$= \sum_{i=1}^{k^2+k} a_i + \sum_{i=k^2+k+1}^{(k+1)^2+(k+1)} a_i$$

  $$= 2(1^2 + 2^2 + \cdots + k^2) + \sum_{i=k^2+k+1}^{(k+1)^2+(k+1)} (k + 1),$$

  since $\sum_{i=1}^{k^2+k} a_i = \text{LHS of } P(k) = \text{RHS of } P(k)$ (inductive hypothesis)

  $$= 2(1^2 + 2^2 + \cdots + k^2) + ((k + 1)^2 + (k + 1) - (k^2 + k + 1) + 1)(k + 1)$$

  $$= 2(1^2 + 2^2 + \cdots + k^2) + (2k + 2)(k + 1)$$

  $$= 2(1^2 + 2^2 + \cdots + k^2) + 2(k + 1)^2$$

  $$= 2(1^2 + 2^2 + \cdots + k^2 + (k + 1)^2)$$

  $$= \text{RHS of } P(k + 1)$$

  So we have shown that $P(k) \implies P(k + 1)$ for $k \in \mathbb{N}$.

  So, by induction, we have $P(n)$ holds for all $n \in \mathbb{N}$, i.e.

  $$a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2), \text{ for all } n \in \mathbb{N}.$$