1. Let $a, b, c \in \mathbb{N}$. Prove that
\[ \frac{bc}{a^2b + c} \leq \frac{b + c}{(1 + a)^2}. \]
When does equality hold?

2. Let $ABC$ be an acute-angled triangle, and let $D$ be the point on $AB$ (extended if necessary) such that $AB \perp CD$. Let $t_A$ and $t_B$ be the tangents to the circumcircle of $\triangle ABC$, through $A$ and $B$, respectively. Let $E$ and $F$ be the points on $t_A$ and $t_B$, respectively, such that $CE \perp t_A$ and $CF \perp t_B$.
Prove that
\[ \frac{CD}{CE} = \frac{CF}{CD}. \]

3. Determine all odd $x \in \mathbb{N}$ for which there are $y, z \in \mathbb{N}$ satisfying both
   (i) $8x + (2y - 1)^2 = z^2$ and
   (ii) $9 \leq 3(y + 1) \leq x$.

4. Kate and Len play the following game using a heap of 2008 cards numbered
   \[ 1, 2, 3, \ldots, 2008. \]
Len draws $\ell$ cards from the heap, records all the numbers he has drawn and then returns the cards to the heap. Then Kate draws $k$ cards from the heap and records all the numbers she has drawn. Finally, they calculate all the non-zero differences between pairs of numbers they have recorded. Kate makes a list her differences $\Delta K$, while Len’s list of differences is $\Delta L$.
   (a) Prove that $\Delta K$ and $\Delta L$ have at least one natural number in common if $k\ell \geq 4015$.
   (b) If $k\ell = 4014$, must $\Delta K$ and $\Delta L$ have a natural number in common? Give reasons for your answer.

5. Let $c$ be a circle with centre $O$, $T$ a point on $c$, $t$ the tangent to $c$ at $T$ and $P$ a point on $t$ other than $T$. Let $\ell$ be the line through $P$ and $O$, and let $Q$ be the point, other than $P$, on $\ell$ such that $QO = OP$. A line through $Q$ intersects the circle in the two different points $U$ and $V$, and $TU$ and $TV$ intersect $\ell$ in $R$ and $S$, respectively.
Prove that $RO = SO$. 

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