10. Given that
\[
\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{r(r-1)(r-2) \cdots 1},
\]
(i) find \( \binom{21}{3} \) and \( \binom{12}{5} \).
(ii) Show that \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

11. (i) Show that the sum of the coefficients of the \( r \)th and \( (r+1) \)st term in the expansion of \((1 + x)^n\) is equal to the \( (r+1) \)st term in the expansion of \((1 + x)^{n+1}\). (To avoid awkward numbering of the terms, let us call the \( x^0 \) term, the zeroth term, so that the \( r \)th term is the term involving \( x^r \).)
(ii) Prove the same result using the factorial expression of \( \binom{n}{r} \).
(iii) How is the result connected with Pascal’s Triangle?

12. Given the Binomial Theorem result:
\[
(1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n,
\]
prove each of the following.
(i) \( \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \)
(ii) \( \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0 \)
(iii) \( \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + 8\binom{n}{3} + \cdots + 2^n \binom{n}{n} = 3^n \)
(iv) \( \binom{n}{0} - 2\binom{n}{1} + 4\binom{n}{2} - 8\binom{n}{3} + \cdots + (-1)^n 2^n \binom{n}{n} = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases} \)

13. Prove that
\[
\binom{n}{r} = \frac{n-r+1}{r} \binom{n}{r-1},
\]
and hence find the value of \( r \) that maximises \( \binom{n}{r} \).

14. Observe that in any row of Pascal’s Triangle that the sum of the odd-indexed elements is equal to the sum of the even-indexed elements, i.e.
\[
\binom{n}{1} + \binom{n}{3} + \cdots = \binom{n}{0} + \binom{n}{2} + \cdots.
\]
Prove this result.
\textit{Hint.} You’ve already done it!
15. Show

(i) $3 | 5^{39} - 2^{39}$.
(ii) $5 | 2^{99} + 3^{99}$.
(iii) $5 | 2^{98} + 3^{98}$.

(iv) $7 | 2^{99} + 3^{99} + 4^{99} + 5^{99}$.
(v) $10 | 2^{99} - 4^{99} - 7^{99} + 9^{99}$.

16. Find a short expression for the following.

(i) $1 + x + x^2 + \cdots + x^n$ for all positive integers $n$.
(ii) $1 - x + x^2 - \cdots + x^n$ for all even positive integers $n$.

17. Factor $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

18. If $x$ and $y$ are positive integers find all solutions of

$$x^2 - 871 = y^6.$$ 

19. If $\left( a - \frac{1}{a} \right)^2 = 3$ and $a - \frac{1}{a} > 0$, evaluate

(i) $a^3 - \frac{1}{a^3}$
(ii) $a^4 + \frac{1}{a^4}$

20. If $a$ is the difference between any quantity and its reciprocal, and $b$ is the difference between the square of the same quantity and the square of its reciprocal, show that

$$a^2(a^2 + 4) = b^2.$$ 

21. Prove that $1991 | 3500^n - 728^n - 785^n + 4^n$ for all $n \in \mathbb{N}$. 

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