Pólya Problems 1 to 6 with Solutions

1. (2006) Factor:
   
   (i) \( x^4 - 7x^2y^2 + y^4 \)

   **Solution.** First we try to bind up the \( x^4 \) and \( y^4 \) terms using an \((A + B)^2\) or \((A - B)^2\) factorisation and then finish off with *difference of squares*:
   
   \[
   x^4 - 7x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 9x^2y^2 \\
   = (x^4 + y^4)^2 - (3xy)^2 \\
   = (x^4 + y^4 + 3xy)(x^4 + y^4 - 3xy)
   \]

   (ii) \( (a + b)^2 + (b + c)^2 + (c + a)^2 - a^2 - b^2 - c^2 \)

   **Solution.** There’s a number of ways to do this one. Let us simply expand it and observe that
   
   \[
   (a_1 + a_2 + \cdots + a_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2 + 2a_1a_2 + 2a_1a_3 + 2a_1a_n + 2a_2a_3 + \cdots + 2a_{n-1}a_n
   \]
   
   where each possible product \( a_ia_j \) occurs exactly once (with coefficient 2). Sometimes this is written
   
   \[
   \left( \sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{1 \leq i < j \leq n} 2a_ia_j.
   \]
   
   Here’s an approach to the problem just using \((A + B)^2\) expansion/factorisation:
   
   \[
   (a + b)^2 + (b + c)^2 + (c + a)^2 - a^2 - b^2 - c^2 \\
   = (a + b)^2 + b^2 + c^2 + 2bc + c^2 + a^2 + 2ac - a^2 - b^2 - c^2 \\
   = (a + b)^2 + c^2 + 2bc + 2ac \\
   = (a + b)^2 + 2(a + b)c + c^2 \\
   = ((a + b) + c)^2 \\
   = (a + b + c)^2
   \]

2. (2007) Factor:
   
   (i) \( x^8 + 2x^4y^4 + 9y^8 \)

   **Solution.** First we try to bind up the \( x^4 \) and \( y^4 \) terms using an \((A + B)^2\) or \((A - B)^2\) factorisation and then finish off with *difference of squares*:
   
   \[
   x^8 + 2x^4y^4 + 9y^8 = x^8 + 6x^4y^4 + 9y^4 - 4x^4y^4 \\
   = (x^4 + 3y^4)^2 - (2x^2y^2)^2 \\
   = (x^4 + 3y^4 + 2x^2y^2)(x^4 + 3y^4 - 2x^2y^2)
   \]
   
   This is as far as we can go over \( \mathbb{Q} \). Further factorisation is possible over \( \mathbb{C} \).
(ii) \( a^4 + b^4 + c^2 - 2(a^2b^2 + a^2c + b^2c) \)

**Solution.** As usual there are many ways to do this one, but let’s stick to using the order two identities in \( A \) and \( B \). It’s helpful to make the observation that:

\[
(A + B)^2 - (A - B)^2 = 4AB.
\]

\[
a^4 + b^4 + c^2 - 2(a^2b^2 + a^2c + b^2c) = a^4 + b^4 - 2a^2b^2 + c^2 - 2c(a^2 + b^2)
= (a^2 - b^2)^2 - (a^2 + b^2)^2 + c^2 - 2c(a^2 + b^2) + (a^2 + b^2)^2
= -4a^2b^2 + (c - (a^2 + b^2))^2
= (c - (a^2 + b^2))^2 - (2a^2b^2)^2
= (c - (a^2 + b^2))^2 + 2a^2b^2(c - (a^2 + b^2) - 2a^2b^2)
= (c - (a - b)^2)(c - (a + b)^2)
\]

It was unnecessary to go past the second last step, but the form of the last expression is quite attractive ;-).

3. (2006) Prove that

\[
1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}
\]

is divisible by 15.

**Solution.** The idea is to show the expression is divisible by each of the prime divisors of 15, namely 3 and 5.

The Pólya approach is to use the identities:

\[
A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + A^{n-k}B^k + \cdots + B^{n-1})
\]

\[
A^n + B^n = (A + B)(A^{n-1} - A^{n-2}B + \cdots + (-1)^kA^{n-k}B^k + \cdots + B^{n-1}), \quad n \text{ odd.}
\]

Note that the first identity hold for all \( n \in \mathbb{N} \), but the second only for odd natural numbers \( n \) (since \((-1)^{n-1}\) needs to be +1).

\[
1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99} = (1 + 2)(1^{98} + \cdots + 2^{98}) + 3^{99} + (4 + 5)(4^{98} + \cdots + 5^{98})
= 3 \cdot (1^{98} + \cdots + 2^{98}) + 3^{98} \cdot (4^{98} + \cdots + 5^{98})
\]

which demonstrates that 3 divides the given expression.

\[
= (1 + 4)(1^{98} + \cdots + 4^{98}) + (2 + 3)(2^{98} + \cdots + 3^{98}) + 5^{99}
= 5 \cdot (1^{98} + \cdots + 4^{98}) + (2^{98} + \cdots + 3^{98}) + 5^{98}
\]

which demonstrates that 5 divides the given expression.

Thus \( \text{lcm}(3, 5) = 15 \) divides the given expression.

Alternatively, one can use modulo arithmetic, which primarily relies on the fact that:

If \( a \equiv b \pmod{m} \) then \( a^n \equiv b^n \pmod{m} \) for all \( n \in \mathbb{N} \).

This follows from a binomial expansion of \((a + km)^n\), since \( a \equiv b \pmod{m} \) implies that \( b = a + km \) for some integer \( k \).
Recall that: \( N \equiv 0 \pmod{m} \) is equivalent to \( m \mid N \).
\[
\begin{align*}
1^99 + 2^99 + 3^99 + 4^99 + 5^99 & \equiv 1^99 + (-1)^99 + 0^99 + 1^99 + (-1)^99 \pmod{3} \\
& \equiv 0 \pmod{3}
\end{align*}
\]
So the given expression is congruent to 0 modulo 3, and hence is divisible by 3.
\[
\begin{align*}
1^99 + 2^99 + 3^99 + 4^99 + 5^99 & \equiv 1^99 + 2^99 + (-2)^99 + (-1)^99 + 0^99 \pmod{5} \\
& \equiv 0 \pmod{5}
\end{align*}
\]
So the given expression is congruent to 0 modulo 5, and hence is divisible by 5.
As before we finish by deducing that \( \text{lcm}(3, 5) = 15 \) divides the given expression.

4. (2007) Prove that
\[
20^{22} - 17^{22} + 4^{33} - 1
\]
is divisible by 174.

Solution. Let \( N = 20^{22} - 17^{22} + 4^{33} - 1 \).
Observe that 174 = 2 \cdot 3 \cdot 29. So we show that each of 2, 3 and 29 divides \( N \), by showing that \( N \) is congruent to 0 modulo each of these primes.
\[
\begin{align*}
N &= 20^{22} - 17^{22} + 4^{33} - 1 \equiv 0^{22} - 1^{22} + 0^{33} - 1 \pmod{2} \\
& \equiv 0 \pmod{2}
\end{align*}
\]
So \( 2 \mid N \).
\[
\begin{align*}
N &= 20^{22} - 17^{22} + 4^{33} - 1 \equiv (-1)^{22} - (-1)^{22} + 1^{33} - 1 \pmod{3} \\
& \equiv 0 \pmod{3}
\end{align*}
\]
So \( 3 \mid N \).
\[
\begin{align*}
N &= 20^{22} - 17^{22} + 4^{33} - 1 \equiv (-9)^{22} - (-12)^{22} + (4^3)^{11} - 1 \pmod{29} \\
& \equiv ((-9)^2)^{11} - ((-12)^2)^{11} + (4^3)^{11} - 1 \pmod{29} \\
& \equiv 81^{11} - 144^{11} + 64^{11} - 1 \pmod{29} \\
& \equiv (-6)^{11} - (-1)^{11} + 6^{11} - 1 \pmod{29} \\
& \equiv -6^{11} + 1 + 6^{11} - 1 \pmod{29} \\
& \equiv 0 \pmod{29}
\end{align*}
\]
So \( 29 \mid N \).
So \( \text{lcm}(2, 3, 29) = 174 \mid N = 20^{22} - 17^{22} + 4^{33} - 1 \).

5. (2006) Find all possible integers \( n \) such that the fraction
\[
\frac{n^3 - 1}{n^2 + 11n - 12}
\]
reduces to an integer.
Solution. Firstly, let the given expression be $N$. Then

$$N = \frac{n^3 - 1}{n^2 + 11n - 12} = \frac{(n - 1)(n^2 + n + 1)}{(n - 1)(n + 12)} = \frac{n^2 + n + 1}{n + 12}.$$ 

Now we perform the Division Algorithm on $n^2 + n + 1$ in order to find its remainder on division by $n + 12$:

\[
\begin{array}{rcl}
\frac{n^2 + n + 1}{n + 12} & & \\
\text{(dividend)} & & \\
\text{rem. 133} & & \\
\end{array}
\]

\[
\begin{array}{rcl}
(n + 12) & & \\
\text{(divisor)} & & \\
\frac{n^2 + 12n}{n^2 + n + 1} & & \\
\frac{-11n + 1}{-11n - 132} & & \\
\frac{133}{133} & & \\
\end{array}
\]

Thus,

$$N = \frac{n^2 + n + 1}{n + 12} = \frac{(n + 12)(n - 11) + 133}{n + 12} = n - 11 + \frac{133}{n + 12}.$$ 

Hence $N$ is an integer only if $n + 12$ is a divisor of 133. Factorising 133 (the prime divisor candidates to check are 2, 3, 5, 7, 11, since $13^2 > 133$), we find $133 = 7 \times 19$. Thus all the integer divisors of 133 are of form $\pm 7^e 19^f$, for $e, f \in \{0, 1\}$, i.e. the integer divisors of 133 (and hence the possible values of $n + 12$) are:

$$1, -1, 7, -7, 19, -19, 133, -133.$$ 

Thus

$$n + 12 = 1 \implies n = -11$$
$$n + 12 = -1 \implies n = -13$$
$$n + 12 = 7 \implies n = -5$$
$$n + 12 = -7 \implies n = -19$$
$$n + 12 = 19 \implies n = 7$$
$$n + 12 = -19 \implies n = -31$$
$$n + 12 = 133 \implies n = -121$$
$$n + 12 = -133 \implies n = -145.$$ 

Hence $n \in \{-11, -13, -5, -19, 7, -31, -121, -145\}.$