Pólya Problems 18 to 21 with Solutions


Solution.

\[
\begin{align*}
AB &= AE, \\
\therefore \triangle BAE &\text{ is isosceles} \\
\therefore \angle AEB &= \angle ABE \\
BC &= ED, \quad \text{(given)} \\
EC &= BD, \quad \text{(given)} \\
BE &= EB \\
\therefore \triangle BCE &\cong \triangle EDB, \quad \text{(by the SSS Rule)} \\
\therefore \angle EBC &= \angle EBD \\
AE &= AB, \quad \text{(given)} \\
\angle AED &= \angle AEB + \angle EBD \\
&= \angle ABE + \angle EBC \\
&= \angle ABC \\
ED &= BC, \quad \text{(given)} \\
\therefore \triangle AED &\cong \triangle ABC, \quad \text{(by the SAS Rule)} \\
\therefore AD &= AC
\end{align*}
\]

19. (2006) $ABCD$ is a quadrilateral with $AB = BC$. The line joining the midpoints $E$ and $F$ of $AD$ and $CD$, respectively, meets $BA$ produced and $BC$ produced in $X$ and $Y$, respectively. Prove that $AX = CY$.

Solution.

\[
\begin{align*}
1 &= \frac{DE}{EA} = \frac{DF}{FC}, \quad \text{(E, F are midpts of AD, CD, resp.)} \\
\therefore EF &\parallel AC \\
\therefore \angle XYB &= \angle ACB \text{ and} \\
\angle YXB &= \angle CAB \\
\therefore \triangle XYB &\sim \triangle ACB, \quad \text{(by the AA Rule)} \\
\therefore 1 &= \frac{AB}{BC} = \frac{XB}{BY}, \quad \text{(since AB = BC given)} \\
\therefore XB &= BY \\
\therefore XA &= XB - AB \\
&= BY - BC \\
&= CY
\end{align*}
\]
20. (2007) In a quadrilateral $ABCD$, the diagonal $AC$ bisects $\angle BAD$ and $AB = BC = CD$. Prove that the other diagonal $BD$ bisects $\angle ADC$.

Solution.

$$AB = BC,$$
$$\therefore \triangle ABC \text{ is isosceles}$$
$$\therefore \angle CAB = \angle ACB$$

Draw $DB$ and let the point of intersection with $AC$ be $O$.

$$\angle DAO = \angle DAC = \angle CAB,$$
$$\angle DAB = \angle ACB = \angle OCB;$$
$$\therefore \angle DAO = \angle BCO$$
$$\angle DOA = \angle BOC;$$
$$\therefore \triangle DAO \sim \triangle BCO$$
$$\therefore \angle ADO = \angle CBO$$
$$\therefore \triangle DCB \text{ is isosceles}$$
$$\therefore \angle CDO = \angle CDB = \angle CBD = \angle CBO$$
$$\therefore \triangle ADO = \triangle CDO$$
i.e. $OD = DB$ bisects $ADC$.

21. (2006) $P$ and $Q$ are midpoints of the sides $XY$ and $XZ$, respectively, of $\triangle XYZ$, and $PZ$ and $QY$ intersect in $R$. Prove that the area of $\triangle YRZ$ equals the area of the quadrilateral $XPRQ$.

Solution. Write $(AB \ldots K)$ for the area of polygon $AB \ldots K$, and let

$$a = (XPRQ), \quad b = (YRP), \quad c = (ZRQ), \quad d = (YRZ).$$

Then $\triangle ZPX$ and $\triangle ZPY$ have equal bases $PX$ and $PY$, respectively, and a common altitude from $Z$, and consequently, equal areas, i.e.

$$a + c = (ZPX) = (ZPY) = b + d \quad (1)$$

Similarly,

$$a + b = (YQX) = (YZ) = c + d \quad (2)$$

$$\therefore 2a + b + c = b + c + 2d, \quad (1) + (2)$$
$$\therefore 2a = 2d$$
$$\therefore a = d$$
i.e. the area of $\triangle YRZ$ equals the area of the quadrilateral $XPRQ$. 

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