1. If \( S_n = 2^1 + 2^2 + 2^3 + \cdots + 2^n \),

   (i) what is the 10\textsuperscript{th} term of the series?

   \textbf{Solution.} \( 2^{10} \)

   (ii) what is the value of \( S_5 \)?

   \textbf{Solution.} \( S_5 = 2^1 + 2^2 + 2^3 + \cdots + 2^5 = 2^1 \cdot \frac{2^5 - 1}{2 - 1} = 62. \)

   (iii) express the series in sigma notation.

   \textbf{Solution.} \( S_n = 2^1 + 2^2 + 2^3 + \cdots + 2^n = \sum_{k=1}^{n} 2^k. \)

2. Write down in expanded form:

   (i) \( \sum_{i=1}^{4} x_i^2 \)

   \textbf{Solution.} \( \sum_{i=1}^{4} x_i^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \)

   (ii) \( \sum_{i=1}^{5} (-1)^i i \)

   \textbf{Solution.} The important thing to observe here is the sign alternation, which is the effect of \((-1)^i:\)

   \( \sum_{i=1}^{5} (-1)^i i = -1 + 2 - 3 + 4 - 5. \)

   (iii) \( \sum_{i=1}^{7} i^2 \)

   \textbf{Solution.} \( \sum_{i=1}^{7} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 7^2 \)

   (iv) \( \sum_{i=1}^{4} (i + 2)(i + 3) \)

   \textbf{Solution.} \( \sum_{i=1}^{4} (i + 2)(i + 3) = 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 + 6 \cdot 7 \)
3. Let \( S_n = 1 - 4 + 16 - 64 + \cdots + (-1)^{n-1}4^{n-1} \).

(i) Find a formula for \( S_n \).

**Solution.** \( S_n \) is a *geometric series* with *first term* 1 and *common ratio* -4. So

\[
S_n = 1 \cdot \frac{1 - (-4)^n}{1 - (-4)} = \frac{1 - (-4)^n}{5}.
\]

(ii) Express \( S_n \) in sigma notation.

**Solution.** \( S_n = \sum_{k=1}^{n} (-4)^{k-1} \).

4. Find the \( n^{th} \) term of the series 17 + 13 + 9 + 5 + 1 - 3 - \cdots.

**Solution.** The series is *arithmetic* with *first term* 17 and *common difference* -4. So the \( n^{th} \) term is

\[
a_n = 17 + (n - 1) \cdot -4 = 21 - 4n.
\]

5. Find the sum of the following series.

(i) 7 + 13 + 19 + 25 + \cdots + 115

**Solution.** The series is *arithmetic* with *first term* \( a_1 = 7 \) and *common difference* \( d = 6 \). We need to determine the number of terms \( n \); the difference of the last and first terms,

\[
115 - 7 = (n - 1)d
\]

\[
n - 1 = \frac{115 - 7}{6} = 18
\]

\[
n = 19
\]

\[
\therefore \text{the sum } S = \frac{n}{2}(a_1 + a_n)
\]

\[
= \frac{19}{2}(7 + 115)
\]

\[
= 1159
\]

(ii) 17 + 13 + 9 + 5 + 1 - 3 - \cdots - 107

**Solution.** The series is *arithmetic* with *first term* \( a_1 = 17 \) and *common difference* \( d = -4 \). We need to determine the number of terms \( n \),

\[
-107 - 17 = (n - 1)d
\]

\[
n - 1 = \frac{-124}{-4} = 31
\]

\[
n = 32
\]

\[
\therefore \text{the sum } S = \frac{n}{2}(a_1 + a_n)
\]

\[
= \frac{32}{2}(17 + -107)
\]

\[
= -1440
\]
(iii) 1 + 2 + 3 + \cdots + 99

**Solution.** 1 + 2 + 3 + \cdots + 99 = \frac{99}{2}(1 + 99) = 4950.

(iv) 1 + 2 - 3 + 4 - 5 + \cdots + 1990 - 1991

**Solution.**

\[1 + 2 - 3 + 4 - 5 + \cdots + 1990 - 1991 = 1 - 1 + \cdots - 1,\]

there are \(\frac{1990}{2} = 995\) \((-1)\)s

\[= 1 - 995 = -994\]

6. Find the value of the series \(a_1 + a_2 + a_3 + \cdots + a_{100}\) where \(a_1 = 1, a_2 = 2 + 3, a_3 = 4 + 5 + 6, \) etc.

**Solution.** Observe that \(a_1\) has one term, \(a_2\) has two terms, \ldots, and in general \(a_k\) has \(k\) terms, so that \(a_1 + a_2 + \cdots + a_k\) has \(1 + 2 + \cdots + k = \frac{k(k+1)}{2}\) terms, and hence

\[a_1 + a_2 + \cdots + a_k = 1 + 2 + \cdots + \frac{k(k+1)}{2} = \frac{k(k+1)/2}{2}
\]

Thus

\[a_1 + a_2 + a_3 + \cdots + a_{100} = \frac{100 \cdot 101/2}{2} \left(1 + \frac{100 \cdot 101}{2}\right) = \frac{5050}{2} \cdot 5051 = 12753775.\]

7. If \(a, b, c, d\) is an arithmetic sequence such that

\[a + b + c + d = 8\]  \hspace{1cm} (1)

\[ad + bc = -2,\]  \hspace{1cm} (2)

find the values of \(a, b, c\) and \(d\).

**Solution.** Since \(a, b, c, d\) is an arithmetic sequence, let the common difference be \(\delta\), so that the sequence may be written as \(a, a + \delta, a + 2\delta, a + 3\delta, \) and hence

\[8 = a + b + c + d = 4a + (1 + 2 + 3)\delta,\]

by (1)

\[4 = 2a + 3\delta\]

\[2a = 4 - 3\delta\]  \hspace{1cm} (3)

\[-2 = ad + bc = a(a + 3\delta) + (a + \delta)(a + 2\delta),\]

by (2)

\[-2 = 2a^2 + 6a\delta + 2\delta^2\]

\[-4 = 4a^2 + 12a\delta + 4\delta^2\]

\[= (4 - 3\delta)^2 + 6(4 - 3\delta)\delta + 4\delta^2,\]

substituting (3)

\[= 16 + (-24 + 24)\delta + (9 - 18 + 4)\delta^2\]

\[-20 = -5\delta^2\]

\[\delta^2 = 4\]

\[\delta = \pm 2\]

If \(\delta = 2\) then, by (3), \(a = (4 - 6)/2 = -1,\) and we get the sequence \(-1, 1, 3, 5,\)

If \(\delta = -2\) then, by (3), \(a = (4 + 6)/2 = 5,\) and we get the sequence \(5, 3, 1, -1\) (the other solution in reverse).
8. The houses of a street are numbered consecutively from 1 to 49. Show that there is a value of \( x \) such that the sum of the numbers of the houses before the house numbered \( x \) is equal to the sum of the numbers of the houses after it, and find this value of \( x \).

9. Show that if \( k, n \in \mathbb{N} \), then the sum of all positive integers less than \( kn \) which are not divisible by \( k \) is

\[
\frac{1}{2}k(k-1)n^2.
\]

Solution. In sigma notation, we are required to find

\[
\sum_{\substack{1 \leq j < kn \\backslash \ \mid k}} j = \sum_{j=1}^{kn} j - \sum_{\substack{1 \leq j < kn \\backslash \ \mid k}} j
\]

\[
= \sum_{j=1}^{kn} j - n \sum_{m=1}^{n} km, \quad \text{since } k \mid j \Rightarrow j = km \text{ for some } m,
\]

\[
= \sum_{j=1}^{kn} j - k \sum_{m=1}^{n} m
\]

\[
= \frac{kn}{2} (1 + kn) - k \cdot \frac{n}{2} (1 + n)
\]

\[
= \frac{kn}{2} (kn - n)
\]

\[
= \frac{kn}{2} (k-1)n = \frac{1}{2}k(k-1)n^2
\]

10. If \( a, b, c \) is an arithmetic sequence, prove that

\[
\frac{1}{\sqrt{a} + \sqrt{b}}, \frac{1}{\sqrt{c} + \sqrt{a}}, \frac{1}{\sqrt{b} + \sqrt{c}}
\]

is also an arithmetic sequence.

Solution. We are given that \( a, b, c \) is an arithmetic sequence, i.e. we have \( c - b = b - a \).

Now we need to consider the corresponding differences in terms of the second sequence.

\[
\frac{1}{\sqrt{b} + \sqrt{c}} - \frac{1}{\sqrt{c} + \sqrt{a}} = \frac{(\sqrt{c} + \sqrt{a}) - (\sqrt{b} + \sqrt{c})}{(\sqrt{b} + \sqrt{c})(\sqrt{c} + \sqrt{a})}
\]

\[
= \frac{\sqrt{a} - \sqrt{b}}{(\sqrt{b} + \sqrt{c})(\sqrt{c} + \sqrt{a})}
\]

\[
= \frac{(\sqrt{a} - \sqrt{b})(\sqrt{b} - \sqrt{c})(\sqrt{c} - \sqrt{a})}{(b - c)(c - a)}
\]

\[
= \frac{1}{\sqrt{c} + \sqrt{a}} - \frac{1}{\sqrt{a} + \sqrt{b}} = \frac{(\sqrt{a} + \sqrt{b}) - (\sqrt{c} + \sqrt{a})}{(\sqrt{c} + \sqrt{a})(\sqrt{a} + \sqrt{b})}
\]

\[
= \frac{\sqrt{b} - \sqrt{c}}{(\sqrt{c} + \sqrt{a})(\sqrt{a} + \sqrt{b})}
\]

\[
= \frac{(\sqrt{b} - \sqrt{c})(\sqrt{c} - \sqrt{a})(\sqrt{a} - \sqrt{b})}{(c - a)(a - b)}
\]

Now observe that the expressions for (4) and (5) are the same, since \( c - b = b - a \) implies \( b - c = a - b \). Hence the second sequence is also arithmetic.
11. If \( a \) is the first term of an arithmetic sequence and \( b \) is the \( n \)th term of the sequence, what is the \( r \)th term of the sequence?

**Solution.** Let \( d \) be the common difference of the arithmetic sequence. Then

\[
b = a + (n - 1)d, \quad \text{i.e.} \quad d = \frac{b - a}{n - 1}.
\]

So, the \( r \)th term \( a_r \) of the sequence is given by:

\[
a_r = a + (r - 1)d = a + \frac{r - 1}{n - 1}(b - a) = \frac{(n - 1)a + (r - 1)(b - a)}{n - 1} = \frac{(n - r)a + (r - 1)b}{n - 1}.
\]

12. If \( S_n = \sum_{k=1}^{n} k(k + 1) \), find a formula for \( S_n \) in terms of \( n \).

**Solution.**

\[
\sum_{k=1}^{n} k(k + 1) = \sum_{k=1}^{n}(k^2 + k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{1}{6}n(n + 1)(2n + 1) + \frac{1}{2}n(n + 1) = \frac{1}{6}n(n + 1)((2n + 1) + 3) = \frac{1}{3}n(n + 1)(n + 2).
\]

13. If \( S_n = \sum_{k=1}^{n} k(k + 1)(k + 2) \) show that \( S_n = \frac{1}{4}n(n + 1)(n + 2)(n + 3) \).

**Solution.** Let us instead prove the more general result

\[
\sum_{k=1}^{n} k(k + 1) \cdots (k + m - 1) = \frac{1}{m + 1} n(n + 1)(n + 2) \cdots (n + m), \quad (6)
\]

for \( m \in \mathbb{N} \).

Then the result we were required to prove is the case \( m = 3 \). The general result (6) follows from looking at the expression

\[
\sum_{k=1}^{n} k(k + 1) \cdots (k + m) - \sum_{k=0}^{n-1} k(k + 1) \cdots (k + m)
\]

in two ways:
\[ n(n + 1) \cdots (n + m) = \sum_{k=1}^{n} k(k + 1) \cdots (k + m) - \sum_{k=0}^{n-1} k(k + 1) \cdots (k + m) \]
\[ = \sum_{k=0}^{n-1} (k + 1) \cdots (k + m)(k + m + 1) - \sum_{k=0}^{n-1} k(k + 1) \cdots (k + m) \]
\[ = \sum_{k=0}^{n-1} (k + 1) \cdots (k + m)((k + m + 1) - k) \]
\[ = \sum_{k=0}^{n-1} (k + 1) \cdots (k + m)(m + 1) \]
\[ = (m + 1) \sum_{k=0}^{n-1} (k + 1) \cdots (k + m) \]
\[ = (m + 1) \sum_{k=1}^{n} k(k + 1) \cdots (k + m - 1) \]

Finally, dividing through by \( m + 1 \) gives us the result.

14. Find the number of tennis balls that can be arranged in a pyramidal pile on a square base, each side of the base containing 10 balls.

15. Find the number of tennis balls that can be arranged in a pyramidal pile on a triangular base, each side of the base containing 12 balls.

16. (i) Show that \( \frac{1}{k} - \frac{1}{k + 1} = \frac{1}{k(k + 1)} \).

Solution. \( \frac{1}{k} - \frac{1}{k + 1} = \frac{k + 1 - k}{k(k + 1)} = \frac{1}{k(k + 1)} \).

(ii) Write down the first 5 terms of the series \( \sum_{k=1}^{n} \frac{1}{k(k + 1)} \).

Solution. The 1st 5 terms are \( \frac{1}{1 \cdot 2} = \frac{1}{2}, \frac{1}{2 \cdot 3} = \frac{1}{6}, \frac{1}{3 \cdot 4} = \frac{1}{12}, \frac{1}{4 \cdot 5} = \frac{1}{20}, \frac{1}{5 \cdot 6} = \frac{1}{30} \).

(iii) Use the result in (i) to show that \( \sum_{k=1}^{n} \frac{1}{k(k + 1)} = 1 - \frac{1}{n + 1} \).

Solution. Doing it long-hand,
\[ \sum_{k=1}^{n} \frac{1}{k(k + 1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} \]
\[ = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n + 1} \]
\[ = 1 - \frac{1}{n + 1} \]
Doing it again, using sigma notation,

\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) \\
= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} \\
= 1 - \frac{1}{n+1}
\]

(iv) Evaluate \( \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \frac{1}{7 \cdot 8} + \cdots + \frac{1}{20 \cdot 21} \).

**Solution.** We could see this directly, by

\[
\frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \frac{1}{7 \cdot 8} + \cdots + \frac{1}{20 \cdot 21} = \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{8} + \cdots + \frac{1}{20} - \frac{1}{21} \\
= \frac{1}{5} - \frac{1}{21} = \frac{21 - 5}{105} = \frac{16}{105}
\]

Alternatively, we could use the previous result:

\[
\frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \frac{1}{7 \cdot 8} + \cdots + \frac{1}{20 \cdot 21} = \sum_{k=5}^{20} \frac{1}{k(k+1)} = \sum_{k=1}^{20} \frac{1}{k(k+1)} - \sum_{k=1}^{4} \frac{1}{k(k+1)} \\
= \left( 1 - \frac{1}{21} \right) - \left( 1 - \frac{1}{5} \right) \\
= \frac{1}{5} - \frac{1}{21} = \frac{21 - 5}{105} = \frac{16}{105}
\]

17. Show that, for all positive integers \( n \),

\[
1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^n(n - 1)^2 + (-1)^{n+1}n^2 \\
= (-1)^{n+1}(1 + 2 + 3 + \cdots + n).
\]

18. Simplify

\[
\left( \frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \cdots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n} \right)^{\frac{1}{2}}.
\]