1. The quadratic equation $x^2 - 3x - 5 = 0$ has roots $\alpha, \beta$. Determine $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.

**Solution.** Since

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha \beta$$

the zeros of $x^2 - (\alpha + \beta)x + \alpha \beta$ are $\alpha, \beta$. Comparing coefficients with the quadratic we find

$$\alpha + \beta = 3$$
$$\alpha \beta = -5.$$

(Remember,

"$x^2 - 3x - 5 = 0$ has roots $\alpha, \beta$"

means the same thing as

"$x^2 - 3x - 5$ has zeros $\alpha, \beta$".)

Now

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \beta$$
$$= 3^2 - 2(-5) = 9 + 10 = 19.$$

$$\alpha^{-2} + \beta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2 \beta^2}$$
$$= \frac{\alpha^2 + \beta^2}{(\alpha \beta)^2}$$
$$= \frac{19}{(-5)^2} = \frac{19}{25}.$$

2. Solve $2\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) = 10$.

**Solution.** Firstly, let

$$y = x + \frac{1}{x}$$

Then

$$2\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) = 10$$
becomes

\[ 2y^2 - y = 10. \]

So

\[
0 = 2y^2 - y - 10 \\
= (2y - 5)(y + 2) \\
= 2(y - \frac{5}{2})(y + 2).
\]

Hence the solutions for \( y \) are \( y_1 = \frac{5}{2} \) or \( y_2 = -2 \).

Rearranging (1) we get

\[ x^2 - yx + 1 = 0 \]

So, for \( y = y_1 \) we get

\[ x^2 - \frac{5}{2}x + 1 = 0. \]

Hence

\[
0 = 2x^2 - 5x + 2 \\
= (2x - 1)(x - 2) \\
= 2(x - \frac{1}{2})(x - 2).
\]

So, two solutions of (2) are \( \frac{1}{2} \) or 2.

For \( y = y_2 \) in (3) we get

\[ x^2 + 2x + 1 = 0 \]

i.e. \( (x + 1)^2 = 0 \)

So, \(-1\) is a double root of (2).

Thus, the solutions of (2) are \(-1, \frac{1}{2}\) or 2.

3. The \textit{quadratic polynomial} \( ax^2 + bx - 4 \) leaves remainder 12 on division by \( x - 1 \) and has \( x + 2 \) as a factor. Find \( a, b \) and the \textit{zeros} of the polynomial.

**Solution.** Let \( p(x) = ax^2 + bx - 4 \). Then by the Remainder Theorem, since \( p(x) \) leaves remainder 12 on division by \( (x - 1) \),

\[
p(1) = 12 \\
i.e. \ a + b - 4 = 12 \\
a + b = 16 \quad (4)
\]

Similarly, by the Factor Theorem, since \( (x + 2) \) is a factor of \( p(x) \),

\[
p(-2) = 0 \\
i.e. \ 4a - 2b - 4 = 0 \\
2a - b = 2 \quad (5)
\]
Adding (4) and (5) gives

\[ 3a = 18 \]
\[ a = 6, \]

and substitution back into (4) gives

\[ 6 + b = 16 \]
\[ b = 10. \]

We have already observed that one zero of \( p(x) \) is \(-2\). The product of the zeros of \( p(x) \) is the constant coefficient divided by the leading coefficient, \(-4/a\). Hence the other zero is \( 2/a = \frac{1}{3} \). Thus, \( a = 6, b = 10 \) and the zeros are: \(-2, \frac{1}{3}\). One should now check that:

\[ 6x^2 + 10x - 4 = 6(x + 2)(x - \frac{1}{3}). \]

4. Find a quadratic equation with roots \( 2 + \sqrt{3} \) and \( 2 - \sqrt{3} \).

**Solution.** Let \( \alpha = 2 + \sqrt{3} \) and \( \beta = 2 - \sqrt{3} \). Then

\[ \alpha + \beta = 4 \]
\[ \alpha \beta = (2 + \sqrt{3})(2 - \sqrt{3}) \]
\[ = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1. \]

So, by Viète’s Theorem,

\[ p(x) := x^2 - (\alpha + \beta)x + \alpha \beta \]
\[ = x^2 - 4x + 1 \]

is a quadratic polynomial with zeros \( 2 + \sqrt{3} \) and \( 2 - \sqrt{3} \), i.e.

\[ x^2 - 4x + 1 = 0 \]

is a quadratic equation with roots \( 2 + \sqrt{3} \) and \( 2 - \sqrt{3} \). (Any nonzero multiple of this equation is also a solution.)

5. June solved a quadratic equation of the form:

\[ ax^2 + bx + c = 0 \]

and got 2 as a root. Kay switched the \( b \) and the \( c \) and got 3 as a root. What was June’s equation?

**Solution.** Observe that the problem is unchanged if the equation is multiplied by a nonzero constant. So the problem does not have a unique solution (if it has at least one), and we may as well assume \( a = 1 \). Let \( p(x) := x^2 + bx + c \). Then June’s equation is \( p(x) = 0 \). It has 2 as a root. So, by the Factor Theorem,

\[ p(2) = 0 \]
\[ i.e. \ 2^2 + 2b + c = 0 \]
\[ 4 + 2b + c = 0 \]

(6)
Now let \( q(x) := x^2 + cx + b \). Then Kay’s equation is \( q(x) = 0 \). It has 3 as a root. So again, by the Factor Theorem,

\[
q(3) = 0
\]

i.e. \( 3^2 + 3c + b = 0 \)

\[
9 + 3c + b = 0 \tag{7}
\]

Multiplying (7) by 2 and subtracting (6) eliminates \( b \) giving

\[
14 + 5c = 0
\]

\[
c = -\frac{14}{5}
\]

Substitution back in (7) gives

\[
9 - \frac{3}{5} \cdot \frac{14}{5} + b = 0
\]

\[
b = -\frac{3}{5}.
\]

Thus June’s equation could be \( x^2 - \frac{3}{5}x - \frac{14}{5} = 0 \). (\( 5x^2 - 3x - 14 = 0 \) is just as acceptable as a solution.)

6. The equation \( x^2 + ax + (b + 2) = 0 \) has real roots. What is the least value that \( a^2 + b^2 \) could be?

**Solution.** Since \( x^2 + ax + (b + 2) = 0 \) has real roots, its *discriminant* is non-negative, i.e.

\[
a^2 - 4(b + 2) \geq 0.
\]

Hence

\[
a^2 + b^2 \geq b^2 + 4(b + 2)
\]

\[
\geq (b + 2)^2 + 4
\]

\[
\geq 4,
\]

(since the square \( (b + 2)^2 \) is non-negative). Thus the least value that \( a^2 + b^2 \) can be is 4. (It is achieved for \( a = 0 \) and \( b = -2 \), in which case the equation has real roots that are both 0.)

*7. If \( a, b \) are odd integers, prove that the equation

\[
x^2 + 2ax + 2b = 0
\]

has no rational roots.

**Solution.** Suppose the equation has rational roots then its *discriminant* \( \Delta := (2a)^2 - 4.2b \) must be a square integer. So

\[
\Delta = 2^2(a^2 - 2b).
\]

and hence \( a^2 - 2b \) must be a square integer \( N^2 \), say. So

\[
a^2 - 2b = N^2
\]

\[
a^2 - N^2 = 2b
\]

\[
(a - N)(a + N) = 2b \tag{8}
\]
Now the RHS of (8) is divisible by 2, but not 4. On the other hand
\[ a - N \equiv a + N \pmod{2} \]
So either \(a - N\) and \(a + N\) are both odd or they are both even; i.e. the LHS of (8) is either odd or divisible by 4. Thus we have a contradiction. Hence the assumption that the equation had rational roots is false. So the equation has no rational roots. (Note, this argument did not use the fact that \(a\) was odd ... so \(a\) need only have been an integer. Also note that if one root is rational then both roots are rational.)
An alternative strategy for proving the result would be to show that \(a^2 - 2b \equiv 3 \pmod{4}\) and that no square integer can be congruent to 3 modulo 4.

8. Prove that for any non-negative \(a, b,\)
\[ \frac{a + b}{2} \geq \sqrt{ab}. \]
**Solution.** Since \(a, b\) are non-negative, \(\sqrt{a}, \sqrt{b}\) both exist, and since a square is necessarily non-negative
\[ (\sqrt{a} - \sqrt{b})^2 \geq 0. \]
Expanding and rearranging we get:
\[ a + b - 2\sqrt{a} \cdot \sqrt{b} \geq 0 \]
So ... \(a + b \geq 2\sqrt{ab}\)
\[ \frac{a + b}{2} \geq \sqrt{ab}. \]

9. Prove that for any non-negative \(x, y, z,\)
\[ x^2 + y^2 + z^2 \geq xy + yz + xz. \]
**Solution.** By the result of the previous question,
\[ \frac{x^2 + y^2}{2} \geq xy \]
\[ \frac{y^2 + z^2}{2} \geq yz \]
\[ \frac{x^2 + z^2}{2} \geq xz. \]
The sum of the left hand expressions is greater than or equal to the sum of the right hand expressions (by the transitivity property of \(\geq\)). So
\[ x^2 + y^2 + z^2 \geq xy + yz + xz. \]
(Incidentally, the non-negative restriction on \(x, y, z\) is unnecessary.)

10. Prove that for any natural number \(n \geq 2,\)
\[ \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1. \]
**Hint:** First prove
\[ \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}. \]
Solution. Observe that for $k > 0$
\[
\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}.
\]
Hence
\[
\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} = 1 - \frac{1}{n} = \frac{n-1}{n}.
\]

Now, for all $k \geq 2$
\[
\frac{1}{k^2} < \frac{1}{(k-1)k}.
\]
So
\[
\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n} < 1.
\]

11. Prove that for any positive $a$ and $b$
\[
\frac{2}{a + \frac{1}{b}} \leq \sqrt{ab} \leq \sqrt{\frac{a^2 + b^2}{2}}.
\]

Solution. The statement actually comprises three inequalities:
\[
\frac{2}{a + \frac{1}{b}} \leq \sqrt{ab} \tag{9}
\]
\[
\sqrt{ab} \leq \frac{a + b}{2} \tag{10}
\]
\[
\frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}} \tag{11}
\]
for positive $a, b$. We proved (10) in Problem 8. (under slightly weaker restrictions). Thus we need only prove (9) and (11). To prove (9) we start with (10) with positive $a, b$ and then invert:
\[
\sqrt{ab} \leq \frac{a + b}{2} \implies \frac{2}{a + \frac{1}{b}} \leq \frac{2ab}{2} = ab \cdot \frac{1}{\sqrt{ab}} = \sqrt{ab}.
\]
Hence
\[
\frac{2}{a + \frac{1}{b}} = \frac{2ab}{b + a} \leq ab \cdot \frac{1}{\sqrt{ab}} = \sqrt{ab}.
\]
Now we prove (11). Start with an *oldy* but a *goody*, and rearrange:

\[(a - b)^2 \geq 0\]
\[a^2 + b^2 \geq 2ab\]

Now add \(a^2 + b^2\) to both sides, and follow by halving both sides and taking the square root of both sides:

\[2a^2 + 2b^2 \geq a^2 + b^2 + 2ab\]
\[= (a + b)^2\]

\[\frac{a^2 + b^2}{2} \geq \frac{(a + b)^2}{4}\]
\[\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a + b}{2}\]

Suppose \(f\) is a function. Then we are allowed to take \(f\) of both sides of an inequality if \(f\) is an *increasing* function and \(f\) is defined for the possible values of the LHS and the RHS. Here we took the *positive square-root* of both sides. Now, for non-negative \(x, y\)

\[\text{if } x \geq y \text{ then } \sqrt{x} \geq \sqrt{y}.\]

So the *positive square-root* function is an *increasing* function. Also both sides of the inequality were positive here.

Thus we have proved

\[\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab} \leq \frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}.\]