Algebra: Inequalities

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No doubt, you are very familiar with the symbols

\[ > \; \geq \; < \; \leq \]

but you probably have not thought much about the rules they obey. Let us start with some properties of real numbers.

- A real number can only be one of positive, negative or 0. Put another way, for a real number \( r \), one of \( r \) or \( -r \) is positive or else \( r = 0 \).
- The sum or product of two positive numbers is positive.
- Of course, for any real number \( r \), \( r + 0 = r \) and \( r \cdot 0 = 0 \).

Now, recognise that \( a > b \) means that \( a - b \) is positive. Also \( a \geq b \) means that either \( a > b \) or \( a = b \). (Sometimes, it is useful to interpret \( a = b \) as: \( a - b \) is 0.) Of course, \( a < b \) means \( b > a \); and \( a \leq b \) means \( b \geq a \).

So now let’s look at some rules that involve \( > \) and \( \geq \) (and \( < \) and \( \leq \)). In each rule \( a, b, c, d \) are real numbers. The proofs will seem obvious – notice in each case we have used just real number properties (the main ones we use are mentioned above.)

- If \( a > b \) then \( a + c > b + c \). (Note that \( c \) is allowed to be negative.)

  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Now \( a - b = (a + c) - (b + c) \). So \( (a + c) - (b + c) \) is positive, i.e. \( a + c > b + c \).

- If \( a > b \) and \( c \) is positive then \( ac > bc \).

  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Also, let \( c \) be positive. Thus, \( (a - b)c = ac - bc \) is positive, i.e. \( ac > bc \).

- If \( a > b \) and \( c \) is negative then \( ac < bc \).

  **Proof.** Let \( a > b \) and \( c \) be negative, i.e. \( a - b \) and \( -c \) are positive. Thus, \( (a - b)(-c) = bc - ac \) is positive, i.e. \( bc > ac \) (or equivalently \( ac < bc \)).
• Always \(a^2 \geq 0\). (The minimum value property of a square.)

**Proof.** If \(a\) is positive then \(a \cdot a = a^2\) is positive. If \(-a\) is positive then \((-a) \cdot (-a) = a^2\) is positive. If \(a\) is 0 then \(a \cdot a = a^2\) is 0. Hence \(a^2\) is positive or 0, i.e. \(a^2 \geq 0\).

• If \(a > b\) and \(b > c\) then \(a > c\). (Transitivity property)

**Proof.** Let \(a > b\) and \(b > c\), i.e. \((a - b)\) and \((b - c)\) are positive. Hence \((a - b) + (b - c) = a - c\) is positive, i.e. \(a > c\).

• If \(a > b\) and \(c > d\) then \(a + c > b + d\).

**Proof.** Let \(a > b\) and \(c > d\), i.e. \(a - b\) and \(c - d\) are positive. Hence \((a - b) + (c - d) = (a + c) - (b + d)\) is positive, i.e. \(a + c > b + d\).

• If \(0 < a < b\) then \(\frac{1}{a} > \frac{1}{b} > 0\).

**Proof.** Exercise.

• If \(0 < a < 1\) and \(n\) is a natural number then \(0 < a^n < 1\).

**Proof.** Exercise. (Hint: use Mathematical Induction.)

Observe that if we let \(a = x/y\), \(b = 1\) and \(c = y\) then the second rule becomes:

If \(\frac{x}{y} > 1\) and \(y\) is positive then \(x > y\).

Thus, we may prove that \(x > y\) by showing either

• \(x - y\) is positive; or

• \(\frac{x}{y} > 1\) provided that \(y\) is positive.

**Example 1.** (i) If \(x, y\) are distinct positive numbers then

\[x^3 + y^3 > x^2y + xy^2.\]

**Proof.** We will show that \((x^3 + y^3) - (x^2y + xy^2)\) is positive. Now

\[
(x^3 + y^3) - (x^2y + xy^2) = x^3y - x^2y + y^3 - xy^2 = x^2(x - y) + y^2(y - x) = (x^2 - y^2)(x - y) = (x + y)(x - y)^2.
\]

Now, by our properties of real numbers and our rules, both \(x + y\) and \((x - y)^2\) are positive, and hence their product is positive, i.e. \(x^3 + y^3 > x^2y + xy^2\).

(ii) If \(x > y > 0\) then

\[4x^3(x - y) > x^4 - y^4.\]
Proof. Since $x > y > 0$ we have $x > 0$ (using the transitivity property). Now $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$ and each of $x - y$, $x + y$ and $x^2 + y^2$ is positive. (Check the details!) Hence $x^4 - y^4$ is positive. We are now in a position to prove the result by showing that

$$\frac{4x^3(x - y)}{x^4 - y^4} > 1.$$ 

But,

$$\frac{4x^3(x - y)}{x^4 - y^4} = \frac{4x^3(x - y)}{(x - y)(x^3 + x^2y + xy^2 + y^3)} = \frac{4x^3}{x^3 + x^2y + xy^2 + y^3} \quad \text{since } x - y \neq 0$$

$$= \frac{4}{1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}} \quad \text{since } x \neq 0$$

$$> 1$$

The last step is valid since $0 < \frac{y}{x} < 1$. (Check all the skipped details!)

Thus, we may deduce that $4x^3(x - y) > x^4 - y^4$. \qed