Theorem 1. If a triangle is isosceles (i.e. it has two equal sides) then the angles opposite the equal sides are equal. Also, if two angles of a triangle are equal then the two sides opposite the equal angles are equal, so that the triangle is isosceles.

Lines and angles

In the following diagram the two horizontal lines are parallel. The line cutting the two parallel lines is called a transversal. Angles A and B are called alternate angles, A and C are corresponding angles, and angles A and D are supplementary angles. Alternate angles and corresponding angles are equal, and pairs of supplementary angles sum to 180°.

Theorem 2. The sum of the interior angles of a triangle is 180°.

Theorem 3. An exterior angle of a triangle equals the sum of the two non-adjacent interior angles.

Theorem 4. The sum of the interior angles of an n-sided polygon is 180(n − 2)°.

Quadrilaterals

Theorem 5. The opposite sides and opposite interior angles of a parallelogram are equal.

Theorem 5a. If a quadrilateral has opposite sides equal then it is a parallelogram.

Theorem 5b. If a quadrilateral has opposite interior angles equal then it is a parallelogram.

Theorem 6. The diagonals of a parallelogram bisect each other.

Theorem 7. The diagonals of a rhombus are perpendicular.
Special Triangle Theorems

Theorem 8. (Pythagoras’ Theorem) In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Theorem 9. (Sine Rule) In a triangle $ABC$ where $a, b, c$ are the lengths of the sides opposite the vertices $A, B, C$, respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where $R$ is the circumradius of $\triangle ABC$.

Theorem 10. (Cosine Rule) In a triangle $ABC$ where $a, b, c$ are the lengths of the sides opposite the vertices $A, B, C$, respectively,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Congruence of triangles

Triangles may be determined to be congruent by any of the following rules.

- **SSS Rule** If three sides of one triangle are equal to the three sides of another, then the triangles are congruent.

- **SAS Rule** If two sides and the included angle of one triangle are equal to the two sides and the included angle of another, then the triangles are congruent.

- **ASA Rule** If two angles and the included side of one triangle are equal to the two angles and the included side of another, then the triangles are congruent.

- **RHS Rule** If the hypotenuse and one other side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle, then the triangles are congruent.

Note that when we say two triangles $ABC$ and $XYZ$ are congruent we mean that the correspondence of vertex $A$ to $X$, $B$ to $Y$ and $C$ to $Z$ determines the congruence. We denote that two triangles $ABC$ and $XYZ$ are congruent by writing $\triangle ABC \cong \triangle XYZ$.

Similarity of triangles

Each of the congruence rules has a corresponding similarity rule, by replacing side-length equality by proportionality. Thus, triangles may be determined to be similar by any of the following rules.

- **SSS Rule** If three sides of one triangle are in the same proportion as the three sides of another, then the triangles are similar.

- **SAS Rule** If two sides of one triangle are in the same proportion as the two sides of another, and the included angles of the sides that correspond are equal then the triangles are similar.

- **AA Rule** (or AAA Rule) If two angles of one triangle are equal to two angles of another, then the triangles are similar. (The equality of the two remaining corresponding angles are then necessarily equal.)
• **RHS Rule** If the hypotenuse and one other side of a right-angled triangle are in the same proportion as the hypotenuse and one side of another right-angled triangle, then the triangles are similar.

As with congruence, when we say two triangles $ABC$ and $XYZ$ are similar we mean that the correspondence of vertex $A$ to $X$, $B$ to $Y$ and $C$ to $Z$ determines the similarity. We denote that two triangles $ABC$ and $XYZ$ are similar by writing $\triangle ABC \sim \triangle XYZ$.

**Theorem 11.** If a line joins the midpoints of two sides of a triangle then that line is parallel to the third side and its length is equal to one half of the length of the third side.

**Theorem 12.** A line parallel to one side of a triangle divides the other two sides in the same proportion.

**Theorem 13.** The bisector of one side of a triangle divides the opposite side in the same ratio as the other two sides.

**Areas and perimeters**

**Theorem 14.** The area of a parallelogram is equal to $bh$ where $b$ is the length of its base and $h$ is its height (the perpendicular distance from the base to the parallel side opposite).

**Theorem 15.** The area of a triangle is equal to $\frac{1}{2}bh$ where $b$ is the length of its base and $h$ is its height (the perpendicular distance from the base to the vertex opposite).

**Theorem 16.** (Heron’s Theorem) If the lengths of the sides of a triangle are $a$, $b$ and $c$, so that the semiperimeter $s = (a + b + c)/2$ then the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

**Theorem 17.** The area of a circle of radius $r$ is $\pi r^2$ and its circumference is $2\pi r$.

**Circles**

**Theorem 18.** If $AB$ is an arc of a circle then angles subtended at the circumference opposite $AB$ are equal and are equal to half the angle subtended at the centre, i.e. in the diagram $\angle ACB = \angle ADB = \frac{1}{2}\angle AOB$.

**Theorem 19.** If $AB$ is an arc of a circle and $C$ is any point on the circumference of the circle then $\angle ACB$ is a right angle.
Theorem 20. If $A$ and $B$ are points on the circumference of a circle with centre $O$ and $C$ is an exterior point of the circle such that $BC$ is a tangent to the circle then $\angle ABC = \frac{1}{2} \angle AOB$.

Theorem 21. A line from the centre of a circle perpendicular to a chord bisects the chord and its arc.

Theorem 22. If $A$ is a point on the circumference of a circle then a tangent to the circle at $A$ is perpendicular to a radius of the circle to $A$.

Theorem 23. The two tangents drawn to a circle drawn from an exterior point of the circle have the same length.

Theorem 24. If two circles touch at a single point then this point and the centres of the circles are collinear.

Theorem 25. If two circles intersect at two points then the line through their centres is the perpendicular bisector of their common chord.

Theorem 26. Opposite angles of a cyclic quadrilateral sum to 180° and if a pair of opposite angles of a quadrilateral sum to 180° then it is cyclic.

Theorem 27. The centre of the circumcircle of a triangle is the intersection of the perpendicular bisectors of the sides of the triangle.

Glossary

Altitude The line through a vertex of a triangle that is perpendicular to the opposite side. A triangle has three altitudes; they are concurrent, meeting at the triangle’s orthocentre.

Arc Any portion of the circumference of a circle.

Centroid The point at which the three medians of a triangle concur. The centroid trisects each of the medians, i.e. splits each median in the ratio 2 : 1. More generally, the centroid of a figure is its centre of mass.

Cevian A line segment in a triangle joining a vertex and a point on the side opposite the vertex.

Chord A line segment whose endpoints lie on the circumference of a circle.

Circumcentre, circumcircle The three perpendicular bisectors of the sides of a triangle concur at the circumcentre of the triangle, which is the centre of the circumcircle, the circle that passes through the three vertices of the triangle.

Collinear This means lying on the same straight line. Several points are collinear if you can draw a single straight line through all of them.

Concurrent This means going through the same point. Several lines are concurrent if they all intersect in the same point.

Congruent Two polygons are congruent if they have the same size and shape (i.e. if one were to shift and/or reflect one polygon the vertices of the two polygons could be made to line up exactly); in particular corresponding sides are of the same length.
Convex  A set $S$ of points on a line, plane or in space is convex if for any points $A, B$ in $S$, all points on the line segment $AB$ are in $S$. We say a polygon is convex if any line segment between points on the boundary of the polygon only intersects the interior of the polygon, i.e. all its interior angles are less than $180^\circ$, e.g. any regular polygon is convex.

Cyclic  A quadrilateral is cyclic if a circle may be drawn that passes through each of its four vertices.

Diameter  A chord of a circle that passes through the circle’s centre.

Edge  A side of a geometrical figure, or more generally, a line segment that joins two vertices.

Equilateral  A triangle is equilateral if all its sides are of equal length. An equilateral triangle necessarily has all its angle equal to $60^\circ$.

Euler line  The line in a triangle on which the orthocentre, centroid and circumcentre lie.

Incentre, incircle, inradius  The three internal bisectors of the angles of a triangle concur at the incentre of the triangle, which is the centre of the incircle, the circle that touches each side of the triangle, i.e. each side of the triangle is a tangent to the incircle. The radius of the incircle is the triangle’s inradius.

Isosceles  A triangle is isosceles if two of its sides are of equal length, in which case, the two angles not included by the sides of equal length are equal.

Line  In plane geometry, a line always means a straight line that is infinite in both directions.

Line segment  A piece of a line of a definite length with two ends.

Locus  The line, curve or region traced out by a point satisfying certain conditions, e.g. if a point moves with fixed distance from a fixed point then its locus is a circle.

Median  A line joining the vertex of a triangle to the midpoint of the opposite side. A triangle has three medians; they concur at the centroid of the triangle.

Medial triangle of a triangle $ABC$. Triangle formed by joining the midpoints of the sides of $\triangle ABC$.

Nine-point circle  The feet of the three altitudes of a triangle $ABC$ (i.e. the vertices of its orthic triangle), the midpoints of the sides of $\triangle ABC$ (i.e. the vertices of its medial triangle), and the midpoints of the line segments from the vertices of $\triangle ABC$ to the orthocentre of $\triangle ABC$, lie on the same circle; this circle is known as the nine-point circle of $\triangle ABC$. Its radius is $\frac{1}{2}R$, where $R$ is the radius of the circumcircle of $\triangle ABC$. Its centre is the midpoint of the Euler line of $\triangle ABC$.

Orthogonal  Same as perpendicular.

Orthocentre  The common intersection point of the three altitudes of a triangle.

Orthic triangle of a triangle $ABC$. Triangle formed by joining the feet of the altitudes of $\triangle ABC$.

Parallelogram  A quadrilateral that has two pairs of parallel sides.
Pedal point, pedal triangle A pedal point is a point $P$ inside a triangle $ABC$ from which perpendiculars are dropped to the three sides of $\triangle ABC$. A triangle formed by joining the feet of the three perpendiculars dropped from a pedal point is called a pedal triangle. The orthic triangle is the pedal triangle formed when the pedal point $P$ is the orthocentre of $\triangle ABC$. The medial triangle is the pedal triangle formed when the pedal point $P$ is the circumcentre of $\triangle ABC$. In the case where $P$ lies on the circumcircle of $\triangle ABC$, the feet $Q, R, S$ of the perpendiculars to the sides of $\triangle ABC$ are collinear, so that the ‘pedal triangle’ formed is degenerate; the line through $Q, R, S$, in this case is known as a simson.

Perpendicular At right angles.

Plane figure A geometrical figure consisting of vertices and edges that can be drawn in the plane; a 2-dimensional object.

Polygon A plane figure whose edges are connected end to end in a loop. A polygon with $n$ sides is sometimes called an $n$-gon. (Technically, a gon is an angle, but an $n$-gon has just as many sides as it has angles, so could just as easily have been called an $n$-lateral.) Trigon and trilateral are uncommon synonyms for triangle. 4-gons are generally referred to as quadrilaterals and sometimes as quadrangles. And we have pentagon (5-gon), hexagon (6-gon), heptagon (7-gon), octagon (8-gon), nonagon (9-gon), decagon (10-gon), dodecagon (12-gon), etc.

Quadrangle, quadrilateral A polygon with 4 sides (and therefore 4 angles).

Radius (plural: radii) A line segment from the centre to the circumference of a circle.

Ray The part of a line that lies on one side of a point.

Regular A polygon is regular if all its sides are equal and all its angles are equal.

Rhombus A parallelogram whose sides are all of equal length.

Secant A line that intersects a circle in two distinct points. A chord is just the segment of a secant that joins the two points of intersection with the circle.

Sector The area bounded by an arc of a circle and the two radii joining the arc.

Similar Two polygons are similar if angles at corresponding vertices are equal (if the two polygons are $ABC \ldots$ and $XYZ \ldots$ then $A$ corresponds to $X$, $B$ corresponds to $Y$, etc.), in which case corresponding sides are in the same proportion.

Simple A simple plane figure is one that does not cross itself.

Simson line, simson If $P$ lies on the circumcircle of a triangle $ABC$ then the feet $Q, R, S$ of the perpendiculars drawn to the (extensions of the) sides of $\triangle ABC$ are collinear. The line through $Q, R$ and $S$ is the Simson line or simson of the point $P$ with respect to triangle $\triangle ABC$. Also see pedal point.

Tangent A line in the same plane as a circle that intersects (i.e. touches) the circle at exactly one point.

Trapezium, trapezoid A quadrilateral that has one pair of opposite sides parallel.

Vertex (plural: vertices) A “corner” of a geometrical figure, i.e. a point at which edges meet.