The Pigeon-Hole Principle: Problems with some Solutions

Greg Gamble

One day I might get around to completing the solutions, and perhaps add the solutions to the Questions in the notes.

1. In a group of 8 people show that at least two have their birthday on the same day of the week.

   Solution. The people (8 of them) are the pigeons and the weekdays (7 of them) are the pigeon-holes. By the PHP, there is a weekday such that at least 2 people have that day as their birthday.

2. Three numbers are chosen at random. Their sum is 19. Show at least one number is 7 or more.

   Solution. Equivalently, put 19 pigeons into 3 pigeon-holes. By the PHP, one pigeon-hole contains at least $6 + 1 = 7$ pigeons.

3. A box contains 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books. How many must we choose to ensure that we have 12 books in the same language?

   Solution. The languages (5 of them) are the pigeon-holes and the books are the pigeons. By the PHP, we must put 11 books in each pigeon-hole and then add 1 to be sure we have at least 12 books in one language. But two pigeon-holes will take less than 11, so we need $10 + 11 + 8 + 11 + 11 + 1 = 52$ books.

4. There are 30 students in a class. While doing a keyboarding test one student made 12 mistakes, while the rest made fewer mistakes. Show that at least 3 students made the same number of mistakes.

   Solution. There are 29 students (the pigeons) who made fewer than 12 mistakes, i.e. each of these students made one of 0 mistakes, 1 mistake, 2 mistakes, ..., 11 mistakes. Thus we make a pigeon-hole for 0 mistakes, a pigeon-hole for 1 mistake, etc. in all, 12 pigeon-holes. Now $29 > 2 \times 12$. So, by the PHP, at least one pigeon-hole has at least $2 + 1 = 3$ students (pigeons).

5. A teacher starts each year with 3 jokes. Over 12 years the teacher never repeated the same triple of jokes. What is the smallest number of jokes the teacher must have in her repertoire for this to be possible?

   Solution. The PHP is used fairly trivially here ... By the PHP the teacher must have at least 12 different triples of jokes. The rest is a combinatorics problem. We need to find the smallest number of jokes $n$ such that the number of different triples we can form from $n$, is at least 12.
For $n = 3$, only $\binom{3}{3} = 1$ triples can be formed.
For $n = 4$, $\binom{4}{3} = 4$ triples can be formed.
For $n = 5$, $\binom{5}{3} = 10$ triples can be formed.
For $n = 6$, $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$ triples can be formed.
Thus, the teacher must know at least 6 jokes, since $10 < 12 < 20$.

6. A canteen has 95 tables with a total of 465 seats. Can we be sure that there is a table with at least 6 seats?

**Solution.** The seats (465 of them) are the pigeons and the tables (95 of them) are the pigeon-holes. Now, $5 \times 95 > 465$. So, it is possible to have only tables with 5 or fewer seats. Thus we cannot be sure of having a table with at least 6 seats.

7. Prove that of any 5 points chosen in an equilateral triangle of side-length 1, there are two points whose distance apart is at most $\frac{1}{2}$.

**Solution.** This becomes a PHP problem where the points (5 of them) are the pigeons if the triangle can be split up into $4 = 5 - 1$ equal areas (the pigeon-holes). Can we do this? . . . Of course, we can:

Thus, by the PHP, at least one of the smaller triangles contains at least two points. The furthest apart two points can be in one of these triangles, is at two vertices . . . and each pair of vertices is $\frac{1}{2}$ a unit apart. Thus, there are two points whose distance apart is at most $\frac{1}{2}$.

In a similar fashion, we can split the triangle into $1 + 3 + 5 = 9$ equal equilateral triangles. Then, by the PHP, we can show that with 10 points, there are two points whose distance apart is at most $\frac{1}{3}$.

. . . In general, we can split the triangle into $1 + 3 + 5 + \cdots + 2n - 1 = n^2$ equal equilateral triangles. Then, by the PHP, we can show that with $n^2 + 1$ points, there are two points whose distance apart is at most $\frac{1}{n}$.

8. Suppose we have 27 distinct positive odd numbers . . . all less than 100. Show there is a pair of numbers whose sum is 102.

**Solution.** There are 50 positive odd numbers less than 100; 48 of them form pairs that add up to 102:

$$\{3,99\}, \{5,97\}, \ldots, \{49,53\}.$$

Form pigeon-holes for each of these 24 pairs and a pigeon-hole each for the remaining unpaired numbers: 1 and 51. That’s 26 pigeon-holes in all. The 27 given odd numbers are our pigeons. By the PHP, we must put at least 2 numbers (pigeons) in at least one pigeon-hole. (In fact, by the way we defined our pigeon-holes the words at least 2 can be replaced by exactly 2.)

Since at least one pigeon-hole has two numbers, we must have at least one pair of numbers whose sum is 102.
9. The numbers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is at least 17.

**Solution.** Label the numbers in the order they appear $a, b, c, \ldots, j$. Then the triples of consecutive numbers are:

$$
\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \ldots, \{h, i, j\}, \{i, j, a\}, \{j, a, b\}.
$$

There are 10 of them ... and their sums are respectively:

$$a + b + c, b + c + d, c + d + e, \ldots, h + i + j, i + j + a, j + a + b.$$

Each of the numbers from 1 to 10 is in exactly 3 of the triples. So the sum of the triple sums is:

$$
(a + b + c) + (b + c + d) + (c + d + e) + \cdots + (h + i + j) + (i + j + a) + (j + a + b) = 3(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 3 \cdot 10 \cdot 11 / 2 = 165.
$$

We must show one of the triple sums is 17 or more. This problem is now similar to Problem 2: we have 10 triples and the sum of those triples is 165. So, we equivalently place 165 pigeons in 10 pigeon-holes. By the PHP since $165 > 10 \times 16$, at least one pigeon-hole has at least $16 + 1$ pigeons. Translating: there is at least one triple of consecutive numbers whose sum is at least 17.

10. Six swimmers training together either swam in a race or watched the others swim. At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others?

11. There are 15 people at a party. Some of them exchange handshakes with some of the others. Prove that at least two people have shaken the same number of hands.

**Solution.** In mathematics, “some” means “at least one”. Thus we should interpret “some” to possibly mean as many as 15.

The maximum number of hands that an individual can shake is $14 = 15 - 1$. Thus create a pigeon-hole for each of 0, 1, \ldots, 14 corresponding to “numbers of hands shaken by a party-goer”, where the party-goers are the pigeons.

We have two cases.

Case 1: pigeon-hole 14 has a “pigeon” (party-goer). Then everyone’s hand has been shaken, and so pigeon-hole 0 is empty.

Case 2: pigeon-hole 14 is empty.

In either case, there are only 14 pigeon-holes that can contain pigeons (15 of them), and so by the PHP one of those pigeon-holes must contain two pigeons. Hence at least two party-goers have shaken the same number of hands.
12. A computer is used for 99 hours over a period of 12 days. Prove that on some pair of consecutive days the computer was used at least 17 hours.

**Solution.** Form a pigeon-hole for each bracket of two consecutive days, i.e. a pigeon-hole for each of the 1st–2nd days, the 2nd–3rd days, . . . , the 11th–12th days, a total of 11 pigeon-holes.

Now, every “fragment of time” of the 99 hours appears in two pigeon-holes. So, if we total the times in all the pigeon-holes we will have $2 \times 99 = 198$ hours which is greater than $17 \times 11 = 187$ hours. Thus at least one pigeon hole has at least 17 hours. Hence the computer was used at least 17 hours on some pair of consecutive days.

13. Show that given any 17 natural numbers it is possible to choose 5 whose sum is divisible by 5.

14. A circle is divided into 8 equal sectors. Half are coloured red and half are coloured blue. A smaller circle is also divided into equal sectors, half coloured red and half coloured blue. The smaller circle is placed concentrically on the larger. Prove that no matter how the red and blue sectors are chosen it is always possible to rotate the smaller circle so that at least 4 colour matches are obtained. (The diagram below shows an example.)

![Diagram of circles with red (r) and blue (b) sectors]

15. Five microcomputers are to be connected to three printers. How many connections are necessary between computers and printers in order to ensure that whenever any three computers require a printer the printers are available?

16. Prove that, of any 5 points chosen within a square of side-length 2, there are two whose distance apart is at most $\sqrt{2}$.

17. A disk of radius 1 is completely covered by 7 identical smaller disks. (They may overlap.) Show that the radius of each of the smaller disks must not be less than $\frac{1}{2}$.

18. A **graph** consists of vertices (singular: vertex) and edges. Vertices are usually represented by filled-in dots and each edge starts and finishes at a vertex. The degree of a vertex is the number of edges that start (or finish) at that vertex.

Suppose a graph has 9 vertices such that each vertex has degree 5 or 6. Prove that at least 5 vertices have degree 6 or at least 6 vertices have degree 5.

19. How many trees can farmer Fred plant on his 100 m square field if they are to be no closer than 10 m apart? (Neglect the thickness of the trees.)