**Plane Geometry: Ceva’s Theorem with Problems**

**Definition.** A *cevian* of a triangle is a line segment joining a vertex of the triangle to a point on the opposite side. Thus, if $X$, $Y$, $Z$ are points on sides $BC$, $CA$, $AB$, respectively, of $\triangle ABC$, then the line segments $AX$, $BY$, $CZ$ are *cevians*.

**Theorem (Ceva).** Cevians $AX$, $BY$, $CZ$ of $\triangle ABC$ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$  

**Convention.** We denote the area of a figure by the vertex labelling of the figure in parentheses, e.g. area of $\triangle ABC$ is denoted by $(ABC)$, the area of quadrilateral $PQRS$ is denoted by $(PQRS)$, etc.

**Problems.**

1. Prove for any $\triangle ABC$, even if $B$ or $C$ is obtuse, that

$$a = b \cos C + c \cos B.$$

Thus use the Sine Rule to deduce the *addition formula*:

$$\sin(B + C) = \sin B \cos C + \sin C \cos B.$$

2. Prove for $\triangle ABC$,

$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$  

3. Prove for $\triangle ABC$, $(ABC) = \frac{abc}{4R}$.

4. For $\triangle ABC$, let $p$ and $q$ be the radii of two circles through $A$, touching $BC$ at $B$ and $C$, respectively. Prove $pq = R^2$.

5. If $X$, $Y$, $Z$ are the midpoints of sides $BC$, $CA$, $AB$, respectively, of $\triangle ABC$, prove the cevians $AX$, $BY$, $CZ$ are concurrent.

The cevians $AX$, $BY$, $CZ$ here, are the *medians* of $\triangle ABC$ and the point at which they concur is the *centroid* or *centre of gravity* of $\triangle ABC$.

6. Prove cevians perpendicular to the opposite sides are concurrent.

Such cevians of a triangle are its *altitudes* and the point at which they concur is the *orthocentre*.

7. Let $\triangle ABC$ and $\triangle A'B'C'$ be non-congruent triangles whose corresponding sides are parallel. Prove the three lines $AA'$, $BB'$ and $CC'$ (extended) are concurrent. Such triangles are said to be *homothetic*. 
8. Let $AX$ be a cevian of $\triangle ABC$ of length $p$ dividing $BC$ into segments $BX = m$ and $XC = n$. Prove

$$a(p^2 + mn) = b^2 m + c^2 n.$$ 

This result is known as Stewart’s Theorem.

*Hint.* Use the Cosine Rule on each of $\triangle ABX$ and $\triangle AXC$, in each case taking the cosine of the angle at $X$. What relationship do the cosines of supplementary angles have?

9. Prove that the medians of a triangle dissect the triangle into six smaller triangles of equal area.

*Hint.* Suppose in the diagram that $x, y, z$ represent the areas of the smaller triangles they lie in. Show the two triangles marked $x$ have the same area, by showing they have a common altitude, and similarly for $y$ and $z$. Then in a similar way show $2x + z = 2y + z$. Continue from there.

10. Prove the medians of a triangle divide each other in the ratio 2 : 1, i.e. the medians of a triangle *trisect* one another.

11. Prove that each (internal) angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides, e.g. if $AX$ is the cevian of $\triangle ABC$ that bisects the angle at $A$ internally, then $BX : XC = c : b$.

*Hint.* Use the Sine Rule on each of $\triangle ABX$ and $\triangle AXC$, in each case taking the sine of the angle at $X$. What relationship do the sines of supplementary angles have?

12. The angle bisector of the angle between two sides is the locus of points that are equidistant from the sides making the angle. One consequence of this is that any pair of internal angle bisectors of a triangle meet at a point that is equidistant from all three sides of the triangle, and hence in fact the three internal angle bisectors are concurrent.

The point at which the angle bisectors of a triangle concur is the *incentre* $I$, the common (perpendicular) distance from $I$ to the three sides is the *inradius* $r$, and the circle with centre $I$ and radius $r$ thus touches each side tangentially and is called the *incircle* of the triangle.

Find an alternative proof that the (internal) angle bisectors of a triangle concur, using Ceva’s Theorem and the result of the previous problem.