Algebra: Inequalities

No doubt, you are very familiar with the symbols

\[ > \quad \geq \quad < \quad \leq \]

but you probably have not thought much about the rules they obey. Let us start with some properties of real numbers.

- A real number can only be one of positive, negative or 0. Put another way, for a real number \( r \), one of \( r \) or \( -r \) is positive or else \( r = 0 \).

- The sum or product of two positive numbers is positive.

- Of course, for any real number \( r \), \( r + 0 = r \) and \( r.0 = 0 \).

Now, recognise that \( a > b \) means that \( a - b \) is positive. Also \( a \geq b \) means that either \( a > b \) or \( a = b \). (Sometimes, it is useful to interpret \( a = b \) as: \( a - b \) is 0.) Of course, \( a < b \) means \( b > a \); and \( a \leq b \) means \( b \geq a \).

So now let’s look at some rules that involve \( > \) and \( \geq \) (and \( < \) and \( \leq \)). In each rule \( a, b, c, d \) are real numbers. The proofs will seem obvious – notice in each case we have used just real number properties (the main ones we use are mentioned above.)

- If \( a > b \) then \( a + c > b + c \). \((\text{Note that } c \text{ is allowed to be negative.})\)

  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Now \( a - b = (a + c) - (b + c) \). So \((a + c) - (b + c)\) is positive, i.e. \( a + c > b + c \).

- If \( a > b \) and \( c \) is positive then \( ac > bc \).

  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Also, let \( c \) be positive. Thus, \((a - b)c = ac - bc \) is positive, i.e. \( ac > bc \).

- If \( a > b \) and \( c \) is negative then \( ac < bc \).

  **Proof.** Let \( a > b \) and \( c \) be negative, i.e. \( a - b \) and \( -c \) are positive. Thus, \((a - b)(-c) = bc - ac \) is positive, i.e. \( bc > ac \) (or equivalently \( ac < bc \)).

- Always \( a^2 \geq 0 \). (The *minimum value* property of a square.)

  **Proof.** If \( a \) is positive then \( a.a = a^2 \) is positive. If \( -a \) is positive then \((-a)(-a) = a^2 \) is positive. If \( a \) is 0 then \( a.a = a^2 \) is 0. Hence \( a^2 \) is positive or 0, i.e. \( a^2 \geq 0 \).

- If \( a > b \) and \( b > c \) then \( a > c \). \(\text{(Transitivity property)}\)

  **Proof.** Let \( a > b \) and \( b > c \), i.e. \( a - b \) and \( b - c \) are positive. Hence \((a - b) + (b - c) \) is positive. But \((a - b) + (b - c) = a - c \). Hence \( a - c \) is positive, i.e. \( a > c \).

- If \( a > b \) and \( c > d \) then \( a + c > b + d \).

  **Proof.** Let \( a > b \) and \( c > d \), i.e. \( a - b \) and \( c - d \) are positive. Hence \((a - b) + (c - d) \) is positive. But \((a - b) + (c - d) = (a + c) - (b + d) \). Hence \((a + c) - (b + d) \) is positive, i.e. \( a + c > b + d \).
• If $0 < a < b$ then $\frac{1}{a} > \frac{1}{b} > 0$.

**Proof.** Exercise.

• If $0 < a < 1$ and $n$ is a natural number then $0 < a^n < 1$.

**Proof.** Exercise. (Hint: use Mathematical Induction.)

Observe that if we let $a = \frac{x}{y}$, $b = 1$ and $c = y$ then the second rule becomes:

If $\frac{x}{y} > 1$ and $y$ is positive then $x > y$.

Thus, we may prove that $x > y$ by showing either

• $x - y$ is positive; or

• $\frac{x}{y} > 1$ provided that $y$ is positive.

**Example 1.** (i) If $x, y$ are distinct positive numbers then

$$x^3 + y^3 > x^2y + xy^2.$$  

**Proof.** We will show that $(x^3 + y^3) - (x^2y + xy^2)$ is positive. Now

$$(x^3 + y^3) - (x^2y + xy^2) = x^3y - x^2y + y^3 - xy^2 = x^2(x - y) + y^2(y - x)$$

$$= (x^2 - y^2)(x - y)$$

$$= (x + y)(x - y)^2.$$ 

Now, by our properties of real numbers and our rules, both $x + y$ and $(x - y)^2$ are positive, and hence their product is positive, i.e. $x^3 + y^3 > x^2y + xy^2$.

(ii) If $x > y > 0$ then

$$4x^3(x - y) > x^4 - y^4.$$  

**Proof.** Since $x > y > 0$ we have $x > 0$ (using the transitivity property). Now $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$ and each of $x - y, x + y$ and $x^2 + y^2$ is positive. (Check the details!) Hence $x^4 - y^4$ is positive. We are now in a position to prove the result by showing that

$$\frac{4x^3(x - y)}{x^4 - y^4} > 1.$$ 

But,

$$\frac{4x^3(x - y)}{x^4 - y^4} = \frac{4x^3(x - y)}{(x - y)(x^3 + x^2y + xy^2 + y^3)}$$

$$= \frac{4x^3}{x^3 + x^2y + xy^2 + y^3}$$ since $x - y \neq 0$

$$= \frac{4}{1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}}$$ since $x \neq 0$

$$> 1$$

The last step is valid since $0 < \frac{y}{x} < 1$. (Check all the skipped details!)

Thus, we may deduce that $4x^3(x - y) > x^4 - y^4$. 

2
1 Absolute values

Absolute values are often most easily treated from a geometric point of view. In particular, the absolute value of a number measures its distance from 0. We can extend this idea to interpret

$$|x - a|$$

as the distance of $x$ from $a$. Thus to solve

$$|x + 1| < 3$$

we may first rewrite it as

$$|x - (-1)| < 3$$

and interpret it as: the distance of $x$ from $-1$ is less than 3 giving us $-4 < x < 2$. (To see this, draw a number line.)

Algebraically, we have the following definition for $|x|$,

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

and note that for a positive real number $a$ we have that

$$|x| < a \text{ if and only if } -a < x < a.$$

Exercises.

1. (i) $|x + 7| > 3$
   (ii) $|2x - 7| < 2$
   (iii) $|x - 2| \geq |2x + 3|
   (iv) $1 - x \geq |x - 1|

2 Triangle Inequality

The name of this inequality comes from the geometric observation that the length of a side of triangle must lie between the difference and sum of the other two sides:

$$||x| - |y|| < |x + y| < |x| + |y|$$

for any real numbers $x, y$.

3 Squares are never negative

We identified this property earlier, but it’s so important it bears repeating and putting it in its own section.

The square of a real number is never negative, i.e.

$$x^2 \geq 0, \text{ with equality } \iff x = 0,$$

or more generally

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0, \text{ with equality } \iff x_1 = x_2 = \cdots = x_n = 0.$$
Exercises – squares are non-negative.

2. Prove that for any non-negative \( a, b \),
\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

This result is AM-GM (see the next section) for the case \( n = 2 \).

3. For arbitrary \( a, b, c \in \mathbb{R} \), prove \( a^2 + b^2 + c^2 \geq ab + bc + ca \).

4. Let \( a, b, c, d \in \mathbb{R} \). Prove that the numbers \( a - b^2, b - c^2, c - d^2, d - a^2 \) cannot all be larger than \( \frac{1}{4} \).

4 Arithmetic, Geometric and Harmonic Means

For a positive real number sequence \( x_1, x_2, \ldots, x_n \), these means are defined by

**The Arithmetic Mean (AM)**

\[
\text{AM}(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

**The Geometric Mean (GM)**

\[
\text{GM}(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}
\]

**The Harmonic Mean (HM)**

\[
\text{HM}(x_1, x_2, \ldots, x_n) = \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)^{-1} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}
\]

**Theorem 2 (AM-GM-HM).** Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. Then

\[
\text{AM}(x_1, x_2, \ldots, x_n) \geq \text{GM}(x_1, x_2, \ldots, x_n) \geq \text{HM}(x_1, x_2, \ldots, x_n)
\]

i.e.

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}
\]

with equality \( \iff \) \( x_1 = x_2 = \cdots = x_n \).

Exercises – AM-GM Examples.

5. (1995 SMC) If \( 1 \leq n \in \mathbb{Z} \), prove that \( (n + 1)^n \geq 2^n n! \). When does equality hold?

6. Prove that \( (a + b)(b + c)(c + a) \geq 8abc \) for all positive \( a, b, c \in \mathbb{R} \).

7. Prove that \( a^2 + b^2 + c^2 \geq ab + bc + ca \) for all \( a, b, c \in \mathbb{R} \).

8. Prove that \( x(a - x) \leq a^2/4 \) if \( a, x \in \mathbb{R}, x > 0 \).

9. Prove that \( a + 1/a \geq 2 \), for all positive \( a \in \mathbb{R} \).

10. (1961 Swedish MO) For all positive \( x_1, x_2, \ldots, x_n \in \mathbb{R} \), prove that

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq n.
\]
Exercises – AM-HM Examples.

11. (1998 Irish MO) Prove that if $0 < a, b, c \in \mathbb{R}$ then
\[
\frac{1}{a + b + c} \leq \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.
\]

12. (1976 British MO) Prove that if $0 < a, b, c \in \mathbb{R}$ then
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}.
\]

13. For positive $x_1, x_2, x_3, x_4 \in \mathbb{R}$, prove that
\[
\frac{x_1 + x_3}{x_1 + x_2} + \frac{x_2 + x_4}{x_2 + x_3} + \frac{x_3 + x_1}{x_3 + x_4} + \frac{x_4 + x_2}{x_4 + x_1} \geq 4.
\]

5 The Cauchy-Schwarz Inequality

Theorem 3 (Cauchy-Schwarz Inequality). For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$,
\[
(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)
\]
with equality if and only if $a_1 : b_1 = a_2 : b_2 = \cdots = a_n : b_n$.

The Cauchy-Schwarz Inequality is most easily remembered in terms of vectors:
\[
\|a\| \cdot \|b\| \geq |a \cdot b|^2
\]

i.e.
\[
\sum_i a_i^2 \cdot \sum_i b_i^2 \geq \left( \sum_i a_i b_i \right)^2
\]

where the $a_i, b_i \in \mathbb{R}$ for all $i$, with equality if and only if $a \parallel b$.

Exercises – Cauchy-Schwarz.

14. Prove that for $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and positive $h_1, h_2, \ldots, h_n \in \mathbb{R}$, where $n \in \mathbb{N}$,
\[
\sum_{i=1}^{n} \frac{a_i^2}{h_i} \geq \frac{(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} h_i}.
\]

15. If $0 < a, b, c, d \in \mathbb{R}$, prove that
\[
\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a + b + c + d}.
\]

16. For all $a, b, c \in \mathbb{R}$, prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

17. If $0 < a, b, c, d \in \mathbb{R}$ such that $(a^2 + b^2)^3 = c^2 + d^2$, prove that
\[
\frac{a^3}{c} + \frac{c^3}{d} \geq 1.
\]

18. (1990 USSR MO) Let $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that
\[
\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \cdots + \frac{a_n^2}{a_n + a_1} \geq \frac{1}{2}.
\]