1 Introduction

At school when you met the topic of Inequalities you were interested in finding the set of solutions for which a given inequality, e.g. you might be asked:

For which $x \in \mathbb{R}$, is $x^2 \geq 3x - 2$.

One technique for solving such an inequality is rearrange it to have right hand side 0, and factorise the resulting left hand quadratic. It’s then straightforward to determine the sign of that left hand side expression, and hence find the solution of the inequality as interval(s) of $\mathbb{R}$:

$$x^2 \geq 3x - 2$$
$$x^2 - 3x + 2 \geq 0$$
$$(x - 2)(x - 1) \geq 0$$

Now, we observe that $(x - 2)(x - 1)$ is positive if both factors are negative or both factors are positive, and is zero when either factor is zero, i.e. the solution is

$$x \leq 1 \text{ or } x \geq 2.$$  

Of course, one needs to start this way to gain some familiarity with how inequations differ from equations.

However, our interest in these lectures is to prove certain Inequalities hold for all $x \in \mathbb{R}$. One technique for this might be to solve an inequality as above, and show that the solution interval is all of $\mathbb{R}$, but for the sorts of inequalities with which we will consider, often involving several variables, this is generally not a useful approach. Instead, we will build up an armoury of Standard Inequalities and use these to prove the results we are after. Before we do that, let’s start near the beginning.

2 Symbols and Elementary Rules

No doubt, you are very familiar with the symbols

$$\text{ } > \text{ } \geq \text{ } < \leq \text{ }$$

but you probably have not thought much about the rules they obey. Let us start with some properties of real numbers.

- A real number can only be one of positive, negative or 0. Put another way, for a real number $r$, one of $r$ or $-r$ is positive or else $r = 0$.

- The sum or product of two positive numbers is positive.

- Of course, for any real number $r$, $r + 0 = r$ and $r \cdot 0 = 0$. 
Now, recognise that \( a > b \) means that \( a - b \) is positive. Also \( a \geq b \) means that either \( a > b \) or \( a = b \). (Sometimes, it is useful to interpret \( a = b \) as: \( a - b = 0 \).) Of course, \( a < b \) means \( b > a \); and \( a \leq b \) means \( b \geq a \).

So now let's look at some rules that involve \( > \) and \( \geq \) (and \( < \) and \( \leq \)). In each rule \( a, b, c, d \) are real numbers. The proofs will seem obvious – notice in each case we have used just real number properties (the main ones we use are mentioned above.)

- If \( a > b \) then \( a + c > b + c \). \( \text{(Note that } c \text{ is allowed to be negative.)} \)
  
  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Now \( a - b = (a + c) - (b + c) \). So \( (a + c) - (b + c) \) is positive, i.e. \( a + c > b + c \).

- If \( a > b \) and \( c \) is positive then \( ac > bc \).
  
  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Also, let \( c \) be positive. Thus, \( (a - b)c = ac - bc \) is positive, i.e. \( ac > bc \).

- If \( a > b \) and \( c \) is negative then \( ac < bc \).
  
  **Proof.** Let \( a > b \) and \( c \) be negative, i.e. \( a - b \) and \( -c \) are positive. Thus, \( (a - b)(-c) = bc - ac \) is positive, i.e. \( bc > ac \) (or equivalently \( ac < bc \)).

- Always \( a^2 \geq 0 \). \( \text{(The minimum value property of a square.)} \)
  
  **Proof.** If \( a \) is positive then \( a \cdot a = a^2 \) is positive. If \( -a \) is positive then \( (-a) \cdot (-a) = a^2 \) is positive. If \( a = 0 \) then \( a \cdot a = a^2 \) is 0. Hence \( a^2 \) is positive or 0, i.e. \( a^2 \geq 0 \).

- If \( a > b \) and \( b > c \) then \( a > c \). \( \text{(Transitivity property)} \)
  
  **Proof.** Let \( a > b \) and \( b > c \), i.e. \( a - b \) and \( b - c \) are positive. Hence \( (a - b) + (b - c) \) is positive. But \( (a - b) + (b - c) = a - c \). Hence \( a - c \) is positive, i.e. \( a > c \).

- If \( a > b \) and \( c > d \) then \( a + c > b + d \).
  
  **Proof.** Let \( a > b \) and \( c > d \), i.e. \( a - b \) and \( c - d \) are positive. Hence \( (a - b) + (c - d) \) is positive. But \( (a - b) + (c - d) = (a + c) - (b + d) \). Hence \( (a + c) - (b + d) \) is positive, i.e. \( a + c > b + d \).

- If \( 0 < a < b \) then \( \frac{1}{a} > \frac{1}{b} > 0 \).
  
  **Proof.** \( \text{Exercise.} \)

- If \( 0 < a < 1 \) and \( n \) is a natural number then \( 0 < a^n < 1 \).
  
  **Proof.** \( \text{Exercise.} \) \( \text{(Hint: use Mathematical Induction.)} \)

Observe that if we let \( a = x/y \), \( b = 1 \) and \( c = y \) then the second rule becomes:

If \( \frac{x}{y} > 1 \) and \( y \) is positive then \( x > y \).

Thus, we may prove that \( x > y \) by showing either

- \( x - y \) is positive; or

- \( \frac{x}{y} > 1 \) provided that \( y \) is positive.
Example 1. (i) If \(x, y\) are distinct positive numbers then
\[x^3 + y^3 > x^2y + xy^2.\]

**Proof.** We will show that \((x^3 + y^3) - (x^2y + xy^2)\) is positive. Now
\[
(x^3 + y^3) - (x^2y + xy^2) = x^3y - x^2y + y^3 - xy^2 = x^2(y - x) + y^2(y - x)
= (x^2 - y^2)(y - x)
= (x + y)(x - y)^2.
\]

Now, by our properties of real numbers and our rules, both \(x + y\) and \((x - y)^2\) are positive, and hence their product is positive, i.e. \(x^3 + y^3 > x^2y + xy^2\). \(\Box\)

(ii) If \(x > y > 0\) then
\[4x^3(x - y) > x^4 - y^4.\]

**Proof.** Since \(x > y > 0\) we have \(x > 0\) (using the transitivity property). Now \(x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)\) and each of \(x - y, x + y\) and \(x^2 + y^2\) is positive. (Check the details!) Hence \(x^4 - y^4\) is positive. We are now in a position to prove the result by showing that
\[
\frac{4x^3(x - y)}{x^4 - y^4} > 1.
\]

But,
\[
\frac{4x^3(x - y)}{x^4 - y^4} = \frac{4x^3(x - y)}{(x - y)(x^3 + x^2y + xy^2 + y^3)}
= \frac{4x^3}{x^3 + x^2y + xy^2 + y^3}\text{ since } x - y \neq 0
= \frac{4}{1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}}\text{ since } x \neq 0
> 1
\]

The last step is valid since \(0 < \frac{y}{x} < 1\). (Check all the skipped details!) Thus, we may deduce that \(4x^3(x - y) > x^4 - y^4\). \(\Box\)

3 Absolute values

Absolute values are often most easily treated from a geometric point of view. In particular, *the absolute value of a number measures its distance from 0*. We can extend this idea to interpret
\[|x - a|
\]
as the *distance of \(x\) from \(a\)*. Thus to solve
\[|x + 1| < 3\]
we may first rewrite it as
\[|x - (-1)| < 3\]
and interpret it as: *the distance of \(x\) from \(-1\) is less than 3* giving us \(-4 < x < 2\). (To see this, draw a number line.) Algebraically, we have the following definition for \(|x|\),
\[|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}\]

and note that for a positive real number \(a\) we have that
\[|x| < a \quad \text{if and only if } -a < x < a.\]

To gain some familiarity with manipulating absolute values, try the following exercises.
4 Triangle Inequality

Exercises.
1. Find the solution interval(s) for the following inequalities.
   (i) $|x + 7| > 3$
   (ii) $|2x - 7| < 2$
   (iii) $|x - 2| \geq |2x + 3|$
   (iv) $1 - x \geq |x - 1|$

You will find these exercises solved among some others in my Algebra – Inequalities: Problems with Some Solutions at: http://school.maths.uwa.edu.au/~gregg/Academy/2007/

4 Triangle Inequality

The name of this inequality comes from the geometric observation that the length of a side of triangle must lie between the difference and sum of the other two sides:

Theorem (Triangle Inequality). For any real numbers $x, y$,
   $$||x| - |y|| < |x + y| < |x| + |y|.$$  

5 Squares are never negative

We identified this property earlier, but it’s so important it bears repeating and putting it in its own section.

   The square of a real number is never negative, i.e.
   $$x^2 \geq 0,$$
   or more generally
   $$x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0,$$  

Exercises – squares are non-negative.
2. Prove that for any non-negative $a, b$,
   $$\frac{a + b}{2} \geq \sqrt{ab}.$$  

   This result is AM-GM (see the next section) for the case $n = 2$.
3. For arbitrary $a, b, c \in \mathbb{R}$, prove $a^2 + b^2 + c^2 \geq ab + bc + ca$.
4. Let $a, b, c, d \in \mathbb{R}$. Prove that the numbers $a - b^2$, $b - c^2$, $c - d^2$, $d - a^2$ cannot all be larger than $\frac{1}{4}$.

6 Arithmetic, Geometric and Harmonic Means

For a positive real number sequence $x_1, x_2, \ldots, x_n$, these means are defined by

   The Arithmetic Mean (AM) = $\frac{x_1 + x_2 + \cdots + x_n}{n}$

   The Geometric Mean (GM) = $\sqrt[n]{x_1 x_2 \cdots x_n}$

   The Harmonic Mean (HM) = $\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)^{-1} = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$
**Theorem (AM-GM-HM).** Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. Then

\[
\text{AM}(x_1, x_2, \ldots, x_n) \geq \text{GM}(x_1, x_2, \ldots, x_n) \geq \text{HM}(x_1, x_2, \ldots, x_n)
\]

i.e.

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}
\]

with equality \( \iff x_1 = x_2 = \cdots = x_n \).

**Proof.** The statement can be proved by induction. The case \( n = 1 \) is trivially true. The case \( n = 2 \) follows after starting with

\[
(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0.
\]

This gives \( \text{AM}(x_1, x_2) \geq \text{GM}(x_1, x_2) \), from which can be deduced \( \text{GM}(x_1, x_2) \geq \text{HM}(x_1, x_2) \).

Following Cauchy’s approach, we deduce the AM-GM inequality for \( n = 2^k \) from the cases \( n = 2 \) and \( n = k \). Similarly, we deduce the GM-HM inequality for \( n = 2^k \) from the cases \( n = 2 \) and \( n = k \).

At this stage, one has AM-GM-HM for all powers of 2. To deduce for general \( n \), first let \( \alpha = \text{AM}(x_1, x_2, \ldots, x_n) \). Then add in \((m - n)\) extra \( \alpha \)s, where \( m \) is a power of 2. Then

\[
\alpha = \text{AM}(x_1, x_2, \ldots, x_n) = \text{AM}(x_1, x_2, \ldots, x_n, \alpha, \ldots, \alpha)
\]

\[
\geq \text{GM}(x_1, x_2, \ldots, x_n, \alpha, \ldots, \alpha)
\]

\[
= \sqrt[n]{x_1 x_2 \cdots x_n \alpha^{m-n}}
\]

\[
\alpha^m \geq x_1 x_2 \cdots x_n \alpha^{m-n}
\]

\[
\alpha^n \geq x_1 x_2 \cdots x_n
\]

\[
\text{AM}(x_1, x_2, \ldots, x_n) = \alpha \geq \sqrt[n]{x_1 x_2 \cdots x_n} = \text{GM}(x_1, x_2, \ldots, x_n)
\]

The proof of the GM-HM inequality for general \( n \) is similar. Start with \( \alpha = \text{HM}(x_1, x_2, \ldots, x_n) \), again add in \((m - n)\) extra \( \alpha \)s, and deduce that \( \text{HM}(x_1, x_2, \ldots, x_n) \leq \text{GM}(x_1, x_2, \ldots, x_n) \).

**Exercises – AM-GM Examples.**

5. (1995 SMC) If \( 1 \leq n \in \mathbb{Z} \), prove that \((n + 1)^n \geq 2^n n!\). When does equality hold?

6. Prove that \((a + b)(b + c)(c + a) \geq 8abc\) for all positive \( a, b, c \in \mathbb{R} \).

7. Prove that \(a^2 + b^2 + c^2 \geq ab + bc + ca\) for all \( a, b, c \in \mathbb{R} \).

8. Prove that \( x(a - x) \leq a^2/4\) if \( a, x \in \mathbb{R}, x > 0 \).

9. Prove that \( a + 1/a \geq 2\), for all positive \( a \in \mathbb{R} \).

10. (1961 Swedish MO) For all positive \( x_1, x_2, \ldots, x_n \in \mathbb{R} \), prove that

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq n.
\]

**Exercises – AM-HM Examples.**

11. (1998 Irish MO) Prove that if \( 0 < a, b, c \in \mathbb{R} \) then

\[
\frac{1}{a + b + c} \leq \frac{2}{a + b} + \frac{2}{b + c} + \frac{2}{c + a} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.
\]
12. (1976 British MO) Prove that if $0 < a, b, c \in \mathbb{R}$ then
\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}.
\]

13. For positive $x_1, x_2, x_3, x_4 \in \mathbb{R}$, prove that
\[
\frac{x_1 + x_3}{x_1 + x_2} + \frac{x_2 + x_4}{x_2 + x_3} + \frac{x_3 + x_1}{x_3 + x_4} + \frac{x_4 + x_2}{x_4 + x_1} \geq 4.
\]

7 Optimisation applications

Later you will probably use calculus almost exclusively, when you need to find the maximum or minimum of a function, but you shouldn’t forget that you can often use inequalities techniques for this purpose, and such solutions are often exquisitely short and elegant!

Example 2. (i) Find the minimum value of $x^2 + 8x + 23$ and the value(s) of $x$ for which this minimum is attained.

Solution. We complete the square:
\[
x^2 + 8x + 23 = (x + 4)^2 + 7 \geq 7,
\]
since the square $(x + 4)^2 \geq 0$.

Thus the expression is bounded below by $7 = 0 + 7$, and since at $x = -4$ we have $(x + 4)^2 = 0$, in fact the lower bound is attained, i.e. the expression has a minimum value 7 that is attained at $x = -4$.

(ii) (Adapted from AIMO 2008) Find the maximum value of $E$ satisfying
\[
A + B + C + D + E = 0 \quad (1)
\]
\[
A^2 + B^2 + C^2 + D^2 + E^2 = 80. \quad (2)
\]

Solution. By AM-GM, for $n = 2$, we have
\[
\frac{A^2 + B^2}{2} \geq AB, \quad \frac{A^2 + C^2}{2} \geq AC, \quad \ldots, \quad \frac{C^2 + D^2}{2} \geq CD,
\]
with these all becoming equalities if $A = B = C = D$. We will use this in step (4) below. Isolating $E$ in (1) we have
\[
E = -(A + B + C + D) \quad (3)
\]
\[
E^2 = (A + B + C + D)^2
\]
\[
= A^2 + B^2 + C^2 + D^2 + 2AB + 2AC + \cdots + 2CD
\]
\[
\leq A^2 + B^2 + C^2 + D^2 + (A^2 + B^2) + (A^2 + C^2) + \cdots + (C^2 + D^2) \quad (4)
\]
\[
= 4(A^2 + B^2 + C^2 + D^2), \quad \text{since from } 2AB, 2AC \text{ and } 2AD \text{ we obtain } 3A^2 \text{ and by symmetry there are as many } A^2 \text{'s as } B^2 \text{'s, } C^2 \text{'s and } D^2 \text{'s}
\]
\[
= 4(80 - E^2), \quad \text{using } (2)
\]
\[
\therefore 5E^2 \leq 4 \cdot 80
\]
\[
E^2 \leq 4 \cdot 16
\]
\[
E \leq 8, \quad \text{with equality if } A = B = C = D = -2.
\]

Therefore, the maximum value of $E$ is 8, attained when $A = B = C = D = -2$.

Later, in your mathematics career you will learn how to do the following problem by Lagrange multipliers, but AM-GM is quicker!
Exercises – Optimisation Application of AM-GM.

14. Given $a, b, c > 0$, find the minimum value of $a + 2b + 7c$ such that $a^2b^5c = 1$.

*Hint.* Let $x_1 = x_2 = \frac{a}{2}$, $x_3 = \cdots = x_7 = \frac{2b}{3}$, and $x_8 = 7c$, and note that minimum value is a fairly nasty looking surd, but it’s not difficult to obtain. We want the exact expression, but once you’ve found it you can find an approximate value with a calculator if you like ;-).

8 Generalising AM-GM-HM

The following theorem generalises the AM-GM-HM Theorem. Firstly, the Quadratic Mean (QM) is the 2-Power Mean:

$$QM(x_1, x_2, \ldots, x_n) = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$  

In general, the $k$-Power Mean (PM$_k$), $k \in \mathbb{R}$, is given by

$$PM_k(x_1, x_2, \ldots, x_n) = \begin{cases} \sqrt[k]{\frac{x_1^k + x_2^k + \cdots + x_n^k}{n}}, & \text{if } k \neq 0 \\ \sqrt[n]{x_1 \cdot x_2 \cdots x_n}, & \text{if } k = 0. \end{cases}$$

With this definition, the AM is the 1-Power Mean, the GM is the 0-Power Mean, and the HM is the $-1$-Power Mean.

**Theorem** (Power Mean (Hölder Mean)). Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then

$$k \geq \ell \implies PM_k(x_1, x_2, \ldots, x_n) \geq PM_\ell(x_1, x_2, \ldots, x_n),$$

with equality $\iff x_1 = x_2 = \cdots = x_n$.

In particular,

$$\cdots \geq QM(x_1, \ldots, x_n) \geq AM(x_1, \ldots, x_n) \geq GM(x_1, \ldots, x_n) \geq HM(x_1, \ldots, x_n).$$

Exercises – QM-AM-HM and Power Mean.

15. Prove the QM-AM part of the above theorem for the case $n = 2$.

16. Show that, if $a, b > 0$ and $a + b = 1$ then

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$  

17. (1976 Vietnam MO) For positive $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_1 + x_2 + \cdots + x_n = 1$ and nonnegative $k \in \mathbb{Z}$, prove that

$$\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k} \geq n^{k+1}.$$
9 The Cauchy-Schwarz Inequality

**Theorem** (Cauchy-Schwarz Inequality). For \(a_1, a_2, \ldots, a_n \in \mathbb{R}\) and \(b_1, b_2, \ldots, b_n \in \mathbb{R}\),
\[
(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)
\]
with equality if and only if
\[
a_1 : b_1 = a_2 : b_2 = \cdots = a_n : b_n.
\]

The Cauchy-Schwarz Inequality is most easily remembered in terms of vectors:
\[
\|a\| \|b\| \geq |a \cdot b|
\]
i.e.
\[
s \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2
\]
where the \(a_i, b_i \in \mathbb{R}\) for all \(i\), with equality if and only if \(a \parallel b\).

**Proof.** Firstly, we give a proof without using vectors.
Since \((a_i x + b_i)^2 \geq 0\),
\[
\sum_{i=1}^{n} (a_i x + b_i)^2 \geq 0
\]
\[
\left( \sum_{i=1}^{n} a_i^2 \right)x^2 + 2 \left( \sum_{i=1}^{n} a_i b_i \right)x + \left( \sum_{i=1}^{n} b_i^2 \right) \geq 0
\]
The lefthand side of the last inequality is a quadratic polynomial in \(x\). Since it is nonnegative its graph either touches the \(x\)-axis at one point (i.e. the polynomial has *exactly* one zero) or is entirely above the \(x\)-axis (i.e. the polynomial has no real zeros). Consequently, the polynomial's discriminant is *non-positive*:
\[
4 \left( \sum_{i=1}^{n} a_i b_i \right)^2 - 4 \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \leq 0
\]
\[
\left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \geq \sum_{i=1}^{n} a_i b_i.
\]
Equality occurs when the polynomial has *exactly* one zero, which is to say that there is an \(x\) such that \(a_i x + b_i = 0\) for all \(i\), which is equivalent to saying the ratios \(a_i : b_i\) are equal for all \(i\).

\(\square\) In terms of vectors, we have the identity
\[
a \cdot b = \|a\| \|b\| \cos \theta,
\]
where \(\theta\) is the angle between the ‘tails’ of the vectors. Squaring and using \(|\cos \theta| \leq 1\), we have
\[
\|a\|^2 \|b\|^2 \geq \|a\|^2 \|b\|^2 \cos^2 \theta
\]
\[
= |a \cdot b|^2.
\]
Equality occurs if and only if \(\cos \theta = 1 \iff a \parallel b\).
Exercises – Cauchy-Schwarz.

18. Prove that for $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and positive $h_1, h_2, \ldots, h_n \in \mathbb{R}$, where $n \in \mathbb{N}$,

$$
\sum_{i=1}^{n} \frac{a_i^2}{h_i} \geq \left( \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} h_i} \right)^2.
$$

19. If $0 < a, b, c, d \in \mathbb{R}$, prove that

$$
\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.
$$

20. For all $a, b, c, d \in \mathbb{R}$, prove that $a^2 + b^2 + c^2 \geq a b + b c + c a$.

21. If $0 < a, b, c, d \in \mathbb{R}$ such that $(a^2 + b^2)^3 = c^2 + d^2$, prove that

$$
\frac{a^3}{c} + \frac{c^3}{d} \geq 1.
$$

22. (1990 USSR MO) Let $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$
\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \cdots + \frac{a_n^2}{a_1 + a_n} \geq \frac{1}{2}.
$$

10 Rearrangements

**Theorem** (Rearrangement Inequality). Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathbb{R}$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, where $n \in \mathbb{N}$ and let $z_1, z_2, \ldots, z_n$ be any permutation (rearrangement) of $y_1, y_2, \ldots, y_n$. Then

$$
x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 z_1 + x_2 z_2 + \cdots + x_n z_n \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.
$$

**Remark.**

(i) Suppose we have two number sequences (terms $x_i$ and $y_j$, respectively) of length $n$. Then the Rearrangement Inequality says that of the expressions one can form that are sums of $n$ product pairs $x_i y_j$, the minimum value is achieved when the largest $x_i$ is paired with the smallest $y_j$, the second-largest $x_i$ is paired with the second-smallest $y_j$, and so on; and the maximum value is achieved when the largest $x_i$ is paired with the largest $y_j$, the second-largest $x_i$ is paired with the second-largest $y_j$, etc. Any other choice of pairings gives a value that lies between these minimum and maximum values.

(ii) Note that there is no assumption about the sign of the $x_i$ or $y_i$. A clue to why this is can be obtained from a glance at how the “partial proof” of the theorem starts.

**Partial proof of Rearrangement Inequality.** The following is a ‘start’ giving the general idea. Suppose $x_1 \leq x_2 \leq x_3$ and $y_1 \leq y_2 \leq y_3$. Then

$$
(x_3 - x_2)(y_3 - y_2) \geq 0
$$

$$
x_2 y_2 + x_3 y_3 \geq x_2 y_3 + x_3 y_2
$$

$$
x_1 y_1 + x_2 y_2 + x_3 y_3 \geq x_1 y_1 + x_2 y_3 + x_3 y_2
$$

Proceeding in this way leads to a general proof. □
Exercises – rearrangements.

23. For all \(a, b, c \in \mathbb{R}\) prove \(a^2 + b^2 + c^2 \geq ab + bc + ac\).

24. (1935 Eötvös Competition) Let \(y_1, y_2, \ldots, y_n\) be any permutation of the positive real numbers \(x_1, x_2, \ldots, x_n\). Prove that

\[
\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n} \geq n.
\]

25. (1976 British MO) For positive \(a, b, c \in \mathbb{R}\), prove that

\[
\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}.
\]

26. (2002 Canadian MO) For positive \(x, y, z \in \mathbb{R}\), prove that

\[
\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.
\]

The Chebyshev Inequality

The following theorem essentially extends the Rearrangement Inequality, and we show that it follows from the Rearrangement Inequality.

**Theorem** (Chebyshev Inequality). If \(a_1 \leq a_2 \leq \cdots \leq a_n\) and \(b_1 \leq b_2 \leq \cdots \leq b_n\) then

\[
\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \geq \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n} \geq \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}.
\]

**Proof.** Assume \(a_1 \leq a_2 \leq \cdots \leq a_n\) and \(b_1 \leq b_2 \leq \cdots \leq b_n\). Then by the Rearrangement Inequality, we have the following \(n\) inequalities:

\[
\begin{align*}
    a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\
    a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\geq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\
    a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\geq a_1 b_2 + a_2 b_3 + \cdots + a_n b_1 \\
    a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\geq a_1 b_3 + a_2 b_4 + \cdots + a_n b_2 \\
    &\vdots \\
    a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &\geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1.
\end{align*}
\]

Now, adding these \(n\) inequalities, followed by dividing through by \(n^2\) gives the result:

\[
\begin{align*}
    n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) &\geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \\
    &\geq n(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) \\
    &\vdots
\end{align*}
\]

Exercises – Chebyshev Inequality.

27. (2002 Tournament of the Towns) Let \(x, y, z \in \mathbb{R}\) such that \(0 < x, y, z < \pi/2\). Prove that

\[
\frac{x \cos x + y \cos y + z \cos z}{x + y + z} \leq \frac{\cos x + \cos y + \cos z}{3}.
\]