Algebra: Inequalities – Some Problems with Solutions

Exercises – squares are non-negative.

2. Prove that for any non-negative \( a, b \),

\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

This result is AM-GM (see the next section) for the case \( n = 2 \).

**Solution.** Since \( a, b \) are non-negative, \( \sqrt{a}, \sqrt{b} \) both exist, and since a square is necessarily non-negative

\[
(\sqrt{a} - \sqrt{b})^2 \geq 0.
\]

Expanding and rearranging we get:

\[
a + b - 2\sqrt{a}\sqrt{b} \geq 0
\]

\[
\therefore a + b \geq 2\sqrt{ab}
\]

\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

Alternatively, by the Example 1(i) approach:

Assume \( a, b \geq 0 \), and consider \( \text{LHS} - \text{RHS} \) of the required conclusion,

\[
\text{LHS} - \text{RHS} = \frac{a + b}{2} - \sqrt{ab}
\]

\[
= \frac{1}{2}(a + b - 2\sqrt{ab})
\]

\[
= \frac{1}{2}(a + b - 2\sqrt{a} \cdot \sqrt{b})
\]

\[
= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2,
\]

noting that \( \sqrt{a}, \sqrt{b} \) both exist,

since \( a, b \geq 0 \)

\[
\geq 0,
\]

\[
\therefore \frac{a + b}{2} \geq \sqrt{ab}.
\]

3. For arbitrary \( a, b, c \in \mathbb{R} \), prove \( a^2 + b^2 + c^2 \geq ab + bc + ca \).

**Solution.** A standard start to solving inequalities is to rearrange them so that one side is zero. Observe the inequality is symmetric with respect to the variables \( a, b, c \). Finally, we note the standard technique of absorbing a mixed term by completion of the square, but also observe that we would have wanted each mixed term to have coefficient 2. This final observation suggests multiplication by 2. At this stage it should be clear how to attack this problem, and note that while we may work backwards to solve the problem, the proof should be given forwards, i.e. the required inequality should be the last line of the proof:

\[
(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0,
\]

since a sum of squares is non-negative

\[
a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2 \geq 0
\]

\[
2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca
\]

\[
a^2 + b^2 + c^2 \geq ab + bc + ca
\]
Alternatively, by the Example 1(i) approach:
Assume \( a, b, c \in \mathbb{R} \), and consider LHS – RHS of the required conclusion,

\[
\text{LHS} - \text{RHS} = a^2 + b^2 + c^2 - (ab + bc + ca)
\]

\[= \frac{1}{2}(a^2 + b^2 - 2ab) + \frac{1}{2}(a^2 + c^2 - 2ac) + \frac{1}{2}(b^2 + c^2 - 2bc)
\]

\[= \frac{1}{2}(a - b)^2 + \frac{1}{2}(a - c)^2 + \frac{1}{2}(b - c)^2 \geq 0, \quad \text{since squares are non-negative}
\]

\[\therefore a^2 + b^2 + c^2 \geq ab + bc + ca.
\]

4. Let \( a, b, c, d \in \mathbb{R} \). Prove that the numbers \( a - b^2, b - c^2, c - d^2, d - a^2 \) cannot all be larger than \( \frac{1}{4} \).

**Solution.** Assume for a contradiction that \( a, b, c, d \in \mathbb{R} \) such that all \( a - b^2, b - c^2, c - d^2, d - a^2 \) are larger than \( \frac{1}{4} \). Then

\[a - b^2 > \frac{1}{4}, \ b - c^2 > \frac{1}{4}, \ c - d^2 > \frac{1}{4}, \ d - a^2 > \frac{1}{4}
\]

\[\therefore a - b^2 + b - c^2 + c - d^2 + d - a^2 > 1, \quad \text{adding the previous inequalities}
\]

\[\therefore 0 > (a^2 - a + \frac{1}{4}) + (b^2 - b + \frac{1}{4}) + (c^2 - c + \frac{1}{4}) + (d^2 - d + \frac{1}{4})
\]

\[= (a - \frac{1}{2})^2 + (b - \frac{1}{2})^2 + (c - \frac{1}{2})^2 + (d - \frac{1}{2})^2,
\]

which is saying a sum of squares is negative, an impossibility.

Hence we have our contradiction, and thus in fact, the numbers \( a - b^2, b - c^2, c - d^2, d - a^2 \) cannot all be larger than \( \frac{1}{4} \).

**Exercises – AM-GM Examples.**

5. (1995 SMC) If \( 1 \leq n \in \mathbb{Z} \), prove that \( (n + 1)^n \geq 2^n n! \). When does equality hold?

**Solution.** Observe that \( n + 1 = (n - 1) + 2 = \cdots = 1 + n \). So by AM-GM we have

\[
\frac{n + 1}{2} \geq \sqrt{n \cdot 1}
\]

\[
\frac{(n - 1) + 2}{2} \geq \sqrt{(n - 1) \cdot 2}
\]

\[
\vdots
\]

\[
\frac{1 + n}{2} \geq \sqrt{1 \cdot n}
\]

\[\therefore \left( \frac{n + 1}{2} \right)^n \geq \sqrt{n! \cdot n!}, \quad \text{multiplying the previous n inequalities’ corresponding sides which are all positive}
\]

\[
\frac{(n + 1)^n}{2^n} \geq n!
\]

\[
(n + 1)^n \geq 2^n n!.
\]

If \( n \neq 1 \) then \( (n + 1)/2 > \sqrt{n \cdot 1} \). Thus equality occurs only if \( n = 1 \), when the inequality states: \( 2 = (1 + 1)^1 \geq 2^1 \cdot 1! \).

6. Prove that \( (a + b)(b + c)(c + a) \geq 8abc \) for all positive \( a, b, c \in \mathbb{R} \).
Solution. We use AM-GM on each pair of $a, b, c$.

\[
\frac{a + b}{2} \geq \sqrt{ab} \\
\frac{b + c}{2} \geq \sqrt{bc} \\
\frac{c + a}{2} \geq \sqrt{ca}
\]

\[
\frac{(a + b)(b + c)(c + a)}{8} \geq \sqrt{(abc)^2},
\]
multiplying the previous 3 inequalities’ corresponding sides which are all positive since $a, b, c > 0$.

7. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all $a, b, c \in \mathbb{R}$.

Solution.

\[
\frac{(a + b)^2}{2} \geq \sqrt{a^2b^2} \\
\frac{(b + c)^2}{2} \geq \sqrt{b^2c^2} \\
\frac{(c + a)^2}{2} \geq \sqrt{c^2a^2}
\]

\[
\therefore a^2 + b^2 + c^2 \geq |ab| + |bc| + |ca|,
\]
adding the previous 3 inequalities noting that $\sqrt{x^2} = |x|$ for any $x \in \mathbb{R}$.

8. Prove that $x(a - x) \leq a^2/4$ if $a, x \in \mathbb{R}, x > 0$.

Solution. Firstly, let us use squares are non-negative:

\[
\left(x - \frac{a}{2}\right)^2 \geq 0 \\
x^2 - ax + \frac{a^2}{4} \geq 0 \\
\frac{a^2}{4} \geq ax - x^2 = x(a - x)
\]

\[
x(a - x) \leq \frac{a^2}{4}
\]
and observe that we need no restriction on $a$ or $x$.

Alternatively, we use AM-GM. Firstly, assume $0 < x \leq a$. Then

\[
\frac{x + (a - x)}{2} \geq \sqrt{x(a - x)} \\
\frac{a}{2} \geq \sqrt{x(a - x)}
\]

\[
\therefore \frac{a^2}{4} = \left(\frac{a}{2}\right)^2 \geq x(a - x)
\]

\[
x(a - x) \leq \frac{a^2}{4}.
\]

To finish off we must show the inequality holds for $x > a$. But for $x > a$,

\[
a - x < 0
\]

\[
x(a - x) < 0 \leq \frac{a^2}{4}.
\]

So, in all cases for $x > 0$,

\[
x(a - x) \leq \frac{a^2}{4}.
\]
9. Prove that \(a + \frac{1}{a} \geq 2\), for all positive \(a \in \mathbb{R}\).

**Solution.**

\[
\frac{a + \frac{1}{a}}{2} \geq \sqrt{a \cdot \frac{1}{a}} = 1
\]

\[
a + \frac{1}{a} \geq 2.
\]

10. (1961 Swedish MO) For all positive \(x_1, x_2, \ldots, x_n \in \mathbb{R}\), prove that

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq n.
\]

**Solution.**

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq \sqrt[n]{\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdots \frac{x_n}{x_1}} = 1
\]

\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq n.
\]

**Exercises – Optimisation Application of AM-GM.**

14. Given \(a, b, c > 0\), find the minimum value of \(a + 2b + 7c\) such that \(a^2b^5c = 1\).

**Hint.** Let \(x_1 = x_2 = a^2, x_3 = \cdots = x_7 = \frac{2b}{5},\) and \(x_8 = 7c\), and note that minimum value is a fairly nasty looking surd, but it’s not difficult to obtain. We want the exact expression, but once you’ve found it you can find an approximate value with a calculator if you like ;‐).

**Solution.** Assume \(a, b, c > 0\) and \(a^2b^5c = 1\). Then

\[
\frac{a + 2b + 7c}{8} = \frac{2 \cdot \frac{a}{2} + 5 \cdot \frac{2b}{5} + 7c}{8}
\]

\[
\geq \sqrt[8]{\left(\frac{a}{2}\right)^2 \cdot \left(\frac{2b}{5}\right)^2 \cdot 7c}
\]

\[
= \sqrt[8]{\frac{2^3 \cdot 7}{5^5} \cdot a^2b^5c}
\]

\[
= \sqrt[8]{\frac{2^3 \cdot 7}{5^5}}, \quad \text{since } a^2b^5c = 1
\]

\[
\therefore a + 2b + 7c \geq 8 \cdot \sqrt[8]{\frac{2^3 \cdot 7}{5^5}}.
\]

Let’s call the nasty looking surd on the last line, \(B\).

Then what we have shown, so far, is that \(a + 2b + 7c\) is bounded below by \(B\).

We do not know as yet whether \(a + 2b + 7c\) can actually be \(B\).

We applied AM-GM to

\[
x_1 = x_2 = \frac{a}{2}, \quad x_3 = x_4 = \cdots = x_7 = \frac{2b}{5}, \quad x_8 = 7c.
\]

The last part of AM-GM says that equality of \(\text{AM}(x_1, \ldots, x_8)\) and \(\text{GM}(x_1, \ldots, x_8)\) occurs if and only if all the \(x_i\) are equal, which means \(B\) is attained exactly when

\[
\frac{a}{2} = \frac{2b}{5} = 7c.
\]
Thus now we can say that

\[ \min(a + 2b + 7c) = B = \sqrt[5]{\frac{2^2 \cdot 7}{5^5}} \approx 4.839, \]

and this occurs when \( \frac{a}{2} = \frac{2b}{5} = \frac{7c}{8} \), i.e. when \( a \approx 1.2, b \approx 1.44, c \approx 0.087 \).

**Exercises – QM-AM-HM and Power Mean.**

15. Prove the QM-AM part of the above theorem for the case \( n = 2 \).

**Solution.** We are required to prove:

If \( a, b \geq 0 \) then

\[ \sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a + b}{2}, \]

with equality \( \iff a = b \).

\( \hat{\check{\square}} \) Note, the following argument was discovered by working backwards, but then written forwards.

Assume that \( a, b \geq 0 \). Then

\[
(a - b)^2 \geq 0 \\
a^2 + b^2 - 2ab \geq 0, \quad \text{(expanding the LHS)}
\]

\[
2a^2 + 2b^2 \geq a^2 + b^2 + 2ab, \quad \text{(adding } a^2 + b^2 + 2ab \text{ to both sides)}
\]

\[
\frac{a^2 + b^2}{2} \geq \frac{a^2 + b^2 + 2ab}{4} \]

\[
= \left( \frac{a + b}{2} \right)^2
\]

\[
\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a + b}{2},
\]

since both sides are non-negative and we have applied \( f(u) = \sqrt{u} \) to both sides, noting that \( f \) is an increasing (and hence inequality-direction-preserving) function for \( u \geq 0 \).

Also, \( a, b \geq 0 \implies |a + b| = a + b \)

(recall: \( \sqrt{u^2} = |u| \)).

The statement about equality follows from observing that

\( (a - b)^2 = 0 \iff a = b, \)

and seeing that each ‘\( \geq \)’ in the above argument then becomes an ‘\( = \)’.

**Exercise.** (Extra one given in class). Prove AM-HM from AM-GM for \( n = 2 \).

**Solution.** We are allowed to assume AM-GM for \( n = 2 \). So we have,

\[
\text{for all } a, b \geq 0, \frac{a + b}{2} \geq \sqrt{ab}.
\]

For AM-HM, the premise is \( a, b > 0 \). So assume \( a, b > 0 \). Then

\[
\frac{a + b}{2} \geq \sqrt{ab}, \quad \text{by AM-GM.} \tag{1}
\]
Also, $\frac{1}{a} \cdot \frac{1}{b} > 0$. So, applying AM-GM to $\frac{1}{a}, \frac{1}{b}$ we have

$$\frac{\frac{1}{a} + \frac{1}{b}}{2} \geq \sqrt{\frac{1}{a} \cdot \frac{1}{b}}$$

$$= \frac{1}{\sqrt{ab}}$$

∴ $$\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab},$$

since both sides are positive

∴ $$\frac{a + b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}},$$

combining with (1)

∴ AM($a, b$) = $\frac{a + b}{2} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} = \text{HM}(a, b)$. 