Mathematical Induction: Problems 1 to 4 with Solutions

1. Prove for any natural number $n$ that

(i) $1 + 3 + 5 + \ldots + 2n - 1 = n^2$;

**Solution.** Let $P(n) : 1 + 3 + 5 + \ldots + 2n - 1 = n^2$.

- First we prove $P(1)$.
  
  LHS of $P(1) = 1 = 1^2 = $ RHS of $P(1)$.

So $P(1)$ is true.

- Now we prove that for any natural number $k$ “if $P(k)$ is true then $P(k+1)$ is true.”

So assume $P(k)$ is true, i.e.

$$1 + 3 + 5 + \ldots + 2k - 1 = k^2.$$

Now try to deduce $P(k+1)$:

LHS of $P(k+1) = 1 + 3 + 5 + \ldots + 2k - 1 + 2(k+1) - 1$

$$= k^2 + 2k + 1, \text{ (by inductive assumption)}$$

$$= (k + 1)^2$$

$$= \text{ RHS of } P(k+1).$$

So $P(k+1)$ is true, if $P(k)$ is true.

- Hence, by induction $P(n)$ is true for all natural numbers $n$.

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$;

**Solution.** Let $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

- Firstly,

  LHS of $P(1) = 1^2 = 1$

  $$= \frac{1}{6}(1 + 1)(2.1 + 1) = \text{ RHS of } P(1).$$

So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number $k$, i.e.

  $$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k + 1)(2k + 1),$$

and deduce $P(k+1)$:

LHS of $P(k+1) = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2$

$$= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2, \text{ (by inductive assumption)}$$

$$= \frac{1}{6}(k + 1)(k(2k + 1) + 6(k + 1))$$

$$= \frac{1}{6}(k + 1)(2k^2 + 7k + 6)$$

$$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$$

$$= \frac{1}{6}(k + 1)(k + 1 + 1)(2(k + 1) + 1)$$

$$= \text{ RHS of } P(k+1).$$

So $P(k+1)$ is true, if $P(k)$ is true.

- Hence, by induction $P(n)$ is true for all natural numbers $n$. 
(iii) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$;

**Solution.** Let $P(n) : 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

- Firstly,

  
  LHS of $P(1) = 1^3 = 1$
  
  \[= \frac{1}{4}.1^2(1+1)^2 = \text{RHS of } P(1).\]

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number $k$, i.e.

  \[1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2.\]

  and deduce $P(k+1)$:

  \[
  \begin{align*}
  \text{LHS of } P(k+1) &= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
  &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3, \quad \text{(by inductive assumption)} \\
  &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\
  &= \frac{1}{4}(k+1)^2(2k^2 + 4k + 4) \\
  &= \frac{1}{4}(k+1)^2(2k + 2)^2 \\
  &= \frac{1}{4}(k+1)^2(k + 1 + 1)^2 \\
  &= \text{RHS of } P(k+1).
  \end{align*}
  \]

  So $P(k+1)$ is true, if $P(k)$ is true.

- Hence, by induction $P(n)$ is true for all natural numbers $n$.

(iv) $1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$;

**Solution.** Let $P(n) : 1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$;

- Firstly,

  
  LHS of $P(1) = 1^2 = 1$
  
  \[= \frac{1}{2}.1(6.1^2 - 3.1 - 1) = \text{RHS of } P(1).\]

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number $k$, i.e.

  \[1^2 + 4^2 + 7^2 + \cdots + (3k - 2)^2 = \frac{1}{2}k(6k^2 - 3k - 1)\]

  and deduce $P(k+1)$:

  \[
  \begin{align*}
  \text{LHS of } P(k+1) &= 1^2 + 4^2 + 7^2 + \cdots + (3k - 2)^2 + (3(k+1) - 2)^2 \\
  &= \frac{1}{2}k(6k^2 - 3k - 1) + 9k^2 + 6k + 1 \quad \text{(by inductive assumption)} \\
  &= \frac{1}{2}(6k^3 - 3k^2 - k + 18k^2 + 12k + 2) \\
  &= \frac{1}{2}(6k^3 + 15k^2 + 11k + 2) \\
  &= \frac{1}{2}(k + 1)(6k^2 + 9k + 2) \\
  &= \frac{1}{2}(k + 1)(6(k + 1)^2 - 12k - 6 + 9k + 2) \\
  &= \frac{1}{2}(k + 1)(6(k + 1)^2 - 3k - 4) \\
  &= \frac{1}{2}(k + 1)(6(k + 1)^2 - 3(k + 1) - 1) \\
  &= \text{RHS of } P(k+1).
  \end{align*}
  \]

  So $P(k+1)$ is true, if $P(k)$ is true.

- Hence, by induction $P(n)$ is true for all natural numbers $n$.  

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(v) \(2^2 + 5^2 + 8^2 + \cdots + (3n - 1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)\).

**Solution.** Let \(P(n)\) : \(2^2 + 5^2 + 8^2 + \cdots + (3n - 1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)\).

- Firstly,
   \[
   \text{LHS of } P(1) = 2^2 = 4 \\
   = \frac{1}{2} \cdot 1(6 \cdot 1^2 + 3 \cdot 1 - 1) = \text{RHS of } P(1).
   \]
   So \(P(1)\) is true.

- Now assume \(P(k)\) is true, for some natural number \(k\), i.e.
   \[
   2^2 + 5^2 + 8^2 + \cdots + (3k - 1)^2 = \frac{1}{2}k(6k^2 + 3k - 1)
   \]
   and deduce \(P(k + 1)\). We could follow an approach similar to the previous exercise; instead, we will demonstrate another technique: that of expanding an expression in \(k\) in powers of \(k + 1\) by replacing \(k\) by \(k + 1 - 1\).

   \[
   \text{LHS of } P(k + 1) = 2^2 + 5^2 + 8^2 + \cdots + (3k - 1)^2 + (3k + 1 - 1)^2
   \]
   \[
   = \frac{1}{2}k(6k^2 + 3k - 1) + 9(k + 1)^2 - 6(k + 1) + 1, \text{ (by inductive assumption)}
   \]
   \[
   = \frac{1}{2}k(3k(2k + 1) - 1) + 9(k + 1)^2 - 6(k + 1) + 1
   \]
   \[
   = \frac{1}{2}k\left(3((k + 1) - 1)(2(k + 1) - 1) - 1\right) + 9(k + 1)^2 - 6(k + 1) + 1
   \]
   \[
   = \frac{1}{2}k\left(3(2(k + 1)^2 - 3(k + 1) + 1) - 1\right) + 9(k + 1)^2 - 6(k + 1) + 1
   \]
   \[
   = \frac{1}{2}((k + 1) - 1)(6(k + 1)^2 - 9(k + 1) + 2) + \frac{1}{2}(18(k + 1)^2 - 12(k + 1) + 2)
   \]
   \[
   = \frac{1}{2}\left((k + 1)(6(k + 1)^2 - 9(k + 1) + 2)
   \]
   \[
   - (6(k + 1)^2 - 9(k + 1) + 2)
   \]
   \[
   + (k + 1)(18(k + 1) - 12) + 2\right)
   \]
   \[
   = \frac{1}{2}\left((k + 1)(6(k + 1)^2 - 9(k + 1) + 2)
   \]
   \[
   - (k + 1)(6(k + 1) - 9) - 2
   \]
   \[
   + (k + 1)(18(k + 1) - 12) + 2\right)
   \]
   \[
   = \frac{1}{2}(k + 1)\left(6(k + 1)^2 - 9(k + 1) + 2
   \]
   \[
   - 6(k + 1) + 9
   \]
   \[
   - 18(k + 1) - 12\right)
   \]
   \[
   = \frac{1}{2}(k + 1)(6(k + 1)^2 + 3(k + 1) - 1)
   \]
   \[
   = \text{RHS of } P(k + 1).
   \]
   So \(P(k + 1)\) is true, if \(P(k)\) is true.

- Hence, by induction \(P(n)\) is true for all natural numbers \(n\).

(vi) \(1 \cdot 3 + 2 \cdot 4 + \cdots + n \cdot (n + 2) = \frac{1}{6}n(n + 1)(2n + 7)\).

**Solution.** Let \(P(n)\) : \(1 \cdot 3 + 2 \cdot 4 + \cdots + n \cdot (n + 2) = \frac{1}{6}n(n + 1)(2n + 7)\).

- Firstly,
   \[
   \text{LHS of } P(1) = 1 \cdot 3 = 3
   \]
   \[
   = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9 \\
   = \frac{1}{6} \cdot 1(1 + 1)(2 \cdot 1 + 7) = \text{RHS of } P(1).
   \]
So \( P(1) \) is true.

\[ 1 \cdot 3 + 2 \cdot 4 + \cdots + k \cdot (k + 2) = \frac{1}{6}k(k + 1)(2k + 7). \]

\( P(k) \) is true, for some natural number \( k \), i.e.

[148x748]• Now assume \( P(k) \) is true, for some natural number \( k \), i.e.

\[ 1 \cdot 3 + 2 \cdot 4 + \cdots + k \cdot (k + 2) = \frac{1}{6}k(k + 1)(2k + 7). \]

and deduce \( P(k + 1) \):

\[ \text{LHS of } P(k + 1) = 1 \cdot 3 + 2 \cdot 4 + \cdots + k \cdot (k + 2) + (k + 1) \cdot (k + 1 + 2) \]

\[ = \left( \text{LHS of } P(k) \right) + (k + 1)(k + 3) \]

\[ = \left( \text{RHS of } P(k) \right) + (k + 1)(k + 3), \text{(by inductive assumption)} \]

\[ = \frac{1}{6}k(k + 1)(2k + 7) + (k + 1)(k + 3) \]

\[ = \frac{1}{6}(k + 1)(2k^2 + 13k + 18) \]

\[ = \frac{1}{6}(k + 1)(k + 2)(2k + 9) \]

\[ = \frac{1}{6}(k + 1)(k + 1 + 1)(2k + 1 + 7) \]

\[ = \text{RHS of } P(k + 1). \]

So \( P(k + 1) \) is true, if \( P(k) \) is true.

• Hence, by induction \( P(n) \) is true for all natural numbers \( n \).

2. Prove that for any natural number \( n \),

\[ 2(\sqrt{n} + 1 - 1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}. \]

Solution. Let \( P(n) : 2(\sqrt{n} + 1 - 1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} \). Now \( P(n) \) should be thought of as two simultaneous inequalities, namely:

\[ \text{LHS}(n) < \text{M}(n) \text{ and } \text{M}(n) < \text{RHS}(n), \]

where

\[ \text{LHS}(n) := 2(\sqrt{n} + 1 - 1), \]

\[ \text{M}(n) := 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \text{ and } \]

\[ \text{RHS}(n) := 2\sqrt{n}. \]

(M is mnemonic for “middle.”)

• Firstly,

\[ \text{LHS}(1) = 2(\sqrt{2} - 1) = \frac{2(\sqrt{2} - 1)(\sqrt{2} + 1)}{\sqrt{2} + 1} = \frac{2}{\sqrt{2} + 1} < \frac{2}{1 + 1} = 1 = \text{M}(1), \]

and \( \text{M}(1) = 1 < 2 = 2\sqrt{1} = \text{RHS}(1) \).

So \( P(1) \) is true.

• Now assume \( P(k) \) is true, for some natural number \( k \), i.e.

\[ \text{M}(k) > \text{LHS}(k) \text{ and } \text{M}(k) < \text{RHS}(k), \]

...
and deduce \( P(k+1) \):

\[
M(k + 1) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}
\]

\[
= M(k) + \frac{1}{\sqrt{k+1}}
\]

\[
> \text{LHS}(k) + \frac{1}{\sqrt{k+1}}, \text{ (by inductive assumption)}
\]

\[
= 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}
\]

\[
= 2(\sqrt{k+2} - 1) - 2(\sqrt{k+2} - \sqrt{k+1}) + \frac{1}{\sqrt{k+1}}
\]

\[
= 2(\sqrt{k+2} - 1) - \frac{2(\sqrt{k+2} - \sqrt{k+1})(\sqrt{k+2} + \sqrt{k+1})}{\sqrt{k+2} + \sqrt{k+1}} + \frac{1}{\sqrt{k+1}}
\]

\[
= 2(\sqrt{k+2} - 1) - \frac{2}{\sqrt{k+2} + \sqrt{k+1}} + \frac{1}{\sqrt{k+1}}
\]

\[
> 2(\sqrt{k+2} - 1) - \frac{2}{\sqrt{k+1} + \sqrt{k+1}} + \frac{1}{\sqrt{k+1}}
\]

\[
= 2(\sqrt{k+2} - 1) - \frac{2}{2\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = 2(\sqrt{k+2} - 1) = \text{LHS}(k+1),
\]

Also \ldots \quad M(k + 1) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}

\[
= M(k) + \frac{1}{\sqrt{k+1}}
\]

\[
< \text{RHS}(k) + \frac{1}{\sqrt{k+1}}, \text{ (by inductive assumption)}
\]

\[
= 2\sqrt{k} + \frac{1}{\sqrt{k+1}}
\]

\[
= 2\sqrt{k+1} - 2(\sqrt{k+1} - \sqrt{k}) + \frac{1}{\sqrt{k+1}}
\]

\[
= 2\sqrt{k+1} - \frac{2(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} + \frac{1}{\sqrt{k+1}}
\]

\[
= 2\sqrt{k+1} - \frac{2}{\sqrt{k+1} + \sqrt{k}} + \frac{1}{\sqrt{k+1}}
\]

\[
< 2\sqrt{k+1} - \frac{2}{\sqrt{k+1} + \sqrt{k+1}} + \frac{1}{\sqrt{k+1}}
\]

\[
= 2\sqrt{k+1} - \frac{2}{2\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = 2\sqrt{k+1} = \text{RHS}(k+1).
\]

i.e. \( M(k+1) > M(k + 1) \) and \( M(k+1) > \text{RHS}(k+1) \).

So \( P(k+1) \) is true, if \( P(k) \) is true.

- Hence, by induction \( P(n) \) is true for all natural numbers \( n \).
3. Prove $3^n > 2^n$ for all natural numbers $n$.

**Solution.** Let $P(n) : 3^n > 2^n$.

- Firstly,
  \[
  \text{LHS of } P(1) = 3^1 = 3 \\
  > 2 = 2^1 = \text{RHS of } P(1).
  \]
  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number $k$, i.e.
  \[3^k > 2^k\]
  and deduce $P(k + 1)$:

  \[
  \text{LHS of } P(k + 1) = 3^{k+1} \\
  = 3^k \cdot 3 \\
  > 2^k \cdot 3, \text{ (by inductive assumption)} \\
  > 2^k \cdot 2 \\
  = 2^{k+1} \\
  = \text{RHS of } P(k + 1).
  \]

  i.e. LHS of $P(k + 1) > \text{RHS of } P(k + 1)$.
  
  So $P(k + 1)$ is true, if $P(k)$ is true.

- Hence, by induction $3^n > 2^n$ for all natural numbers $n$.

4. Prove Bernoulli’s Inequality which states:

If $x \geq -1$ then $(1 + x)^n \geq 1 + nx$ for all natural numbers $n$.

**Solution.** Let $P(n) : (1 + x)^n \geq 1 + nx$, if $x \geq -1$.

- Firstly,
  \[
  \text{LHS of } P(1) = (1 + x)^1 = 1 + x \\
  = 1 + 1.x = \text{RHS of } P(1).
  \]
  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number $k$, i.e.
  \[(1 + x)^k \geq 1 + kx, \text{ if } x \geq -1\]
  and deduce $P(k + 1)$:

  \[
  \text{LHS of } P(k + 1) = (1 + x)^{k+1} \\
  = (1 + x)^k.(1 + x) \\
  = (\text{LHS of } P(k)).(1 + x) \\
  \geq (\text{RHS of } P(k)).(1 + x), \text{ (by inductive assumption...} 1 + x \geq 0 \text{ since } x \geq -1) \\
  = (1 + kx)(1 + x) \\
  = 1 + (k + 1)x + kx^2 \\
  \geq 1 + (k + 1)x, \text{ (since } k > 0, x^2 \geq 0, \text{ so that } kx^2 \geq 0) \\
  = \text{RHS of } P(k + 1).
  \]
  So $P(k + 1)$ is true, if $P(k)$ is true.

- Hence, by induction $P(n)$ is true for all natural numbers $n$. 