First Year students often find the concept of linear independence hard to grasp. This is not surprising, since the definition was arrived at after much hind-sight had been developed. So, the definition is somewhat unnatural for the novice. To make the definition a little more palatable we introduce the following term.

For a set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \} \) define the\(^1 \) “linear independence equation” to be

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}.
\]

Now we may express the definition for linear independence in the following way:

A set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \} \) is linearly independent \( \iff \) the linear independence equation has a unique solution (namely, the trivial solution).

This says, that to test whether a set of vectors is linearly independent we should first write down “the” corresponding linear independence equation and then solve it. We then decide the set of vectors is linearly independent if we find only one solution; otherwise we decide the set of vectors is linearly dependent. Let us apply the definition to a proof type problem.

**Problem.** Given \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is linearly independent, is the set \( \{ \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_1 \} \) linearly independent?

**Solution.** We are given that \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is linearly independent so it’s linear independence equation,

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{0}
\]

has only the solution \( \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0. \)

Is the set \( \{ \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_1 \} \) linearly independent? We write down it’s linear independence equation, using variables that don’t conflict with any already chosen:

\[
\alpha(\mathbf{u}_1 + \mathbf{u}_2) + \beta(\mathbf{u}_2 + \mathbf{u}_3) + \gamma(\mathbf{u}_3 + \mathbf{u}_1) = \mathbf{0}
\]

which on rearrangement gives

\[
(\alpha + \gamma)\mathbf{u}_1 + (\alpha + \beta)\mathbf{u}_2 + (\beta + \gamma)\mathbf{u}_3 = \mathbf{0}.
\]

Comparing this equation with \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \)’s linear independence equation, which has just the trivial solution, we have:

\[
\alpha + \gamma = 0, \quad \alpha + \beta = 0, \quad \beta + \gamma = 0
\]

which itself is a system of equations (in the variables \( \alpha, \beta, \gamma \)). Representing this system as an augmented matrix and reducing to RE we have:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix}
\]

From the final RE matrix we see the system for \( \alpha, \beta, \gamma \) has full rank (i.e. there are as many steps (echelons) as there can possibly be). So the solution is unique (and that solution must be the trivial one: \( \alpha = 0, \quad \beta = 0, \quad \gamma = 0 \)).

Hence the set \( \{ \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_1 \} \) is linearly independent.

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\(^1\)The word *the* is perhaps improperly used here, since any list of \( n \) distinct non-conflicting variables can be used in place of the vector coefficients, \( \alpha_1, \alpha_2, \ldots, \alpha_n \).