Rank-nullity Theorem Applications

The following are some further examples of applications of the Rank-nullity Theorem:

**Theorem. (Rank-nullity)** Let $A$ be an $m \times n$ matrix. Then
\[
\text{rank}(A) + \text{null}(A) = n = \# \text{columns of } A.
\]

**Example 1.** Let $B$ be a $6 \times 4$ matrix of rank 4.
Then $\text{null}(B) = 4 - 4 = 0$. So $B\tilde{x} = 0 \iff \tilde{x} = 0$.

**Example 2.** Let $B$ be a $4 \times 6$ matrix of rank 4.
Then $\text{rank}(B) + \text{null}(B) = 6$. So $\text{null}(B) = 6 - 4 = 2$,
and hence the solution set of $B\tilde{x} = 0$ depends on 2 parameters.

**Example 3.** Let $A$ be an $m \times n$ matrix.
$A$ has $m$ rows $\implies \text{rank } A \leq m$.
$A$ has $n$ columns $\implies \text{column-rank}(A) = \text{rank}(A) \leq n$.
\[
\therefore \text{rank}(A) \leq \min(m, n).
\]
So if $m < n$ then $\text{rank}(A) < n$ and $\text{null}(A) = n - \text{rank}(A) > 0$.
This proves again that a homogeneous system with more unknowns ($n$) than equations ($m$)
always has a non-trivial solution.

**Example 4.** Given
\[
A = \begin{bmatrix}
0 & 2 & 4 \\
2 & 3 & 8 \\
-1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\text{ and } A \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
find a basis for $\text{rowspace}(A)$.

**Solution.** $\text{rank}(A) + \text{null}(A) = 3$ and $\text{null}(A) \geq 1$. So $\text{rank}(A) \leq 2$.
But first two rows are $\ell.i.$
Hence $\text{rank}(A) = 2$ and $\{(0 \ 2 \ 4), (2 \ 3 \ 8)\}$ is a basis for $\text{rowspace}(A)$.

**Remark 5.** Let $A$ be a (square) $n \times n$ matrix.
We proved that: $A$ is invertible $\iff \text{null}(A) = 0$.
Now we have from the Rank-nullity Theorem:
$A$ invertible $\iff \text{null}(A) = n$
$\iff \text{rank}(A) = n$
$\iff$ row vectors of $A$ are $\ell.i.$
$\iff$ column vectors of $A$ are $\ell.i.$