Maths Olympiad question – further general comments

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- If you read the dangerous bend with respect to how $\land$ (meet) and $\lor$ (join) operations could be defined for sequences, you will also have read that distribution rules for these operations (as they have been defined there) do not hold in general. Yet, in the proof of the previous item, you will notice that effectively a distribution rule was o.k. here. This might appear to be a contradiction. It isn’t! . . .

When we say

$$P \text{ does not hold in general}$$

you should imagine there are brackets as so:

$$P \text{ does not (hold in general)}$$

as opposed to

$$P \text{ (does not hold) in general.}$$

That is, (with the first bracketing) $P$ may hold sometimes but it doesn’t hold all the time – equivalently, we could say there is at least one counterexample to $P$ holding. (with the second bracketing we would mean: $P$ never holds).

Exercises.

1. Give a counter-example to show for sequences $A, B, C$ that, for the given definitions of $\land$ and $\lor$, the distribution rules

$$\begin{align*}
(A \land B) \lor C &= (A \lor C) \land (B \lor C) \\
(A \lor B) \land C &= (A \land C) \lor (B \land C)
\end{align*}$$

do not hold in general.

Hint: focus on the differences between sets and sequences.

2. Interpret your sequences $A, B, C$ (of Exercise 1.) as sets and replace $\land$ by $\cap$ and $\lor$ by $\cup$. Demonstrate that with these changes the distribution rules now hold (for this one example).

3. Explain why the distribution rule is valid in the proof of the previous item. What special properties do the sequences have to make it work?

Hint: focus on the differences between sets and sequences.

- In proving two sets are equal, sometimes direct enumeration is impractical, e.g. the two sets may be either quite large or infinite. In such cases, we may use the fact that

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$
i.e. we may show \( A = B \) by:

Choose an **arbitrary** element \( a \in A \)
Show \( a \in B \)
From the above can deduce \( a \in B \) for all \( a \in A \), i.e. \( A \subseteq B \).
Then . . . choose an **arbitrary** element \( b \in B \)
Show \( b \in A \)
Deduce \( b \in A \) for all \( b \in B \), i.e. \( B \subseteq A \).

Observe that we did essentially this when proving that, for \( x,y \) with the right properties,
\[ S(x) \cup S(y) = \mathbb{N}. \]

First we showed \( S(\alpha) \subseteq \mathbb{N} \), where \( \alpha \) has the right properties, so that \( S(x) \subseteq \mathbb{N} \) and \( S(y) \subseteq \mathbb{N} \) and hence \( S(x) \cup S(y) \subseteq \mathbb{N} \).

Next, we took an arbitrary \( n \in \mathbb{N} \) and showed that either \( n \in S(x) \) or \( n \in S(y) \), i.e. \( n \in S(x) \cup S(y) \), and hence we showed that \( \mathbb{N} \subseteq S(x) \cup S(y) \).

It may seem that we are cheating here . . . the argument suggests we take just one element of \( A \) and show that it is in \( B \) and then deduce that every element of \( A \) is in \( B \). The crucial word here is **arbitrary** – the argument is really a proforma: we could replace our choice of \( a \) with any other element of \( A \) and the same argument shows \( a \in B \).

- When **two** sets are large and **finite**, we may use the following fact as an alternative to the above approach, in proving the two sets are equal.

\[
A = B \iff A \subseteq B \text{ and } \#A = \#B
\]

i.e. we may show \( A = B \) by:

Choose an **arbitrary** element \( a \in A \)
Show \( a \in B \)
Deduce \( a \in B \) for all \( a \in A \), i.e. \( A \subseteq B \).
Then . . . show \( \#A = \#B \).

Observe that we could have used this approach to prove that, for \( x,y \) with the right properties,
\[
S(x) \cup S(y) = \mathbb{N}.
\]

We showed \( S(\alpha) \subseteq \{1,2,\ldots,n-1\} \), when \( \alpha \) had the right properties, so that \( S(x) \cup S(y) \subseteq \{1,2,\ldots,n-1\} \).

We also showed that \( S(x) \cap S(y) = \emptyset \) and \( \#S(x) + \#S(y) = n-1 \), so that
\[
\#(S(x) \cup S(y)) = n-1.
\]
Since \( \# \{1,2,\ldots,n-1\} = n-1 \), we may deduce
\[
S(x) \cup S(y) = \{1,2,\ldots,n-1\}.
\]

Then, by **induction** we can deduce
\[
S(x) \cup S(y) = \mathbb{N}.
\]

Notice, we have deduced two **infinite** sets are equal using a method valid only for **finite** sets, by carefully avoiding infinite cardinals!
In developing the proof to Q5 of the set of Mathematics Olympiad problems a lot of seemingly unnecessary notation was used and the proof ran to several pages. In defence of this, here is a quote from *The Problems of Mathematics* by Ian Stewart:

\[ \ldots \text{Someone once stated a theorem about prime numbers, claiming that it could never be proved because there was no good notation for primes. Carl Friedrich Gauss proved it from a standing start in five minutes, saying (somewhat sourly) \textquote{what he needs is notions, not notations}. Calculations are merely a means to an end. If a theorem is proved by an enormous calculation, that result is not properly understood until the reasons the calculation works can be isolated and seen to appear as natural and inevitable. Not all ideas are mathematics; but all good mathematics must contain an idea.} \]

The notation \( \land \) and \( \lor \) is not necessary for understanding the proof of Q5 – it was merely a vehicle used to emphasise the differences between the notions of set and sequence. Notation, in general, is introduced in order to express ideas in a succinct way. Just recapping, the key steps in the proof of Beatty’s theorem were:

- For all \( n \in \mathbb{N} \),
  \[ \#S_n(x) + \#S_n(y) = \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor = n - 1. \]

- If \( \alpha \) is \( x \) or \( y \) then
  \[ \#S_{n+1}(\alpha) = \begin{cases} 
  \#S_n(\alpha), & \text{if } n \notin S(\alpha) \\
  \#S_n(\alpha) + 1, & \text{if } n \in S(\alpha). 
\end{cases} \]

- For all \( n \in \mathbb{N} \),
  \[ \#S_{n+1}(x) + \#S_{n+1}(y) = \#S_n(x) + \#S_n(y) + 1. \]

Hopefully, having written it this way you will also see the conclusion (Beatty’s Theorem) as both natural and inevitable.

**Final pointer:** When proving a statement of the form: *If conditions then conclusion*, your proof will usually start with

\[ \text{Let conditions} \]

and end with

\[ \text{Therefore conclusion.} \]

You should identify in your proof where each condition is used. If there are some conditions that haven’t been used you should check that they really are unnecessary (i.e. that they were red herrings); a good way to do this is to construct examples that satisfy the conditions you have used but do not satisfy the supposed red herring conditions and check the conclusion still holds; if it doesn’t then there is something wrong with your proof – it’s possible you only need to identify a “red herring condition” at one of the steps.