Recent Uniqueness Results in Shape from Shading

Ryszard Kozera

Department of Computer Science
The University of Western Australia
Nedlands, WA 6907
Australia
ryszard@cs.uwa.edu.au

Abstract. In this paper, we analyze, in the context of complete integrals and envelopes, the uniqueness problem for the shape recovery of a smooth Lambertian surface from an image obtained by illuminating this surface by an overhead, distant point light-source. Specifically, we revisit the uniqueness results of Brooks already existing in the shape-from-shading literature that concern eikonal equations corresponding to the images of a Lambertian hemisphere and a Lambertian plane. We show that the the latter results are incomplete and indicate how to fill the gaps in the corresponding proofs.

1 Introduction

The main purpose of this paper is to discuss briefly two topics. The first one is to show that Sneddon's claim ([9, Section 7 pp. 61]) about representability of any solution to a given first-order partial differential equation in terms of either a complete or a general or a singular integral is erroneous. The literature on complete integrals is a bewildering collection of incomplete and false statements (see e.g. Dou [6] or [9]). Recent results by Chojnacki [3] and Kozera [8] shed new light on this topic and fill a gap in the literature. The second goal of this paper is to critically inspect uniqueness results (see Brooks [1, 2]) concerning the images of a Lambertian hemisphere and a Lambertian plane, which resort to Sneddon's erroneous assertion and as such are invalid. Finally, we adopt a different approach so that the results claimed in [1, 2], subject to minor reformulations, become valid. For a more detailed analysis an interested reader is also referred to [3] or [8].

2 Some remarks on representing solutions of first-order P.D.E. in terms of complete integral and envelopes

In this section we shall refer to the erroneous assertion of Sneddon about representability of solutions to a first-order partial differential equation in terms of a complete integral and envelopes (see [9, Section 7]).
2.1 Preliminaries

We first recall the notion of a complete integral (see also [3] and [6]). For a given first-order partial differential equation

\[ F(x, y, u, u_x, u_y) = 0, \]  

(1)

defined over an open region \( \Omega \subset \mathbb{R}^2 \), a function \( G(x, y, P_1, P_2) \) of class \( C^2 \) over \( \Omega \times V \) (where \( V \) is an open region of \( \mathbb{R}^2 \)) is called a complete integral of (1) if

(Ci) for each \( (P_1, P_2) \in V \), the function \( G \) is a \( C^2 \) solution to (1) on \( \Omega \),

(Cii) for each \( (x, y) \in \Omega \) and for each \( (P_1, P_2) \in V \), the rank of the matrix

\[
\begin{pmatrix}
  G_{P_1} & G_{x_1}P_1 & G_{x_2}P_1 \\
  G_{P_2} & G_{x_1}P_2 & G_{x_2}P_2
\end{pmatrix}
\]  

(2)

equals two.

Condition (2) assures that parameters \( P_1 \) and \( P_2 \) are independent (see [6, Section 6]). Any graph of \( G(x, y, P_1^0, P_2^0) \) with both \( P_1^0 \) and \( P_2^0 \) fixed, is called zero-parameter envelope of \( G \). Moreover, for a given \( C^1 (C^2) \) function \( \phi : \mathbb{R} \to \mathbb{R} \), we can form a one-parameter subfamily of functions \( u_\phi(x, y, P_1) = G(x, y, P_1, \phi(P_1)) \), and then can generate, either locally or globally (if possible), its general integral (which graph is called a one-parameter envelope of \( G \)), i.e. a function

\[ u(x, y) = G(x, y, P_1(x, y), \phi(P_1(x, y))), \]  

(3)

by eliminating parameter \( P_1 \) from the system of the form

\[ u(x, y) = G(x, y, P_1, \phi(P_1)) \quad \text{and} \quad (G(x, y, P_1, \phi(P_1)))_{P_1} = 0. \]  

(4)

It can be then shown that under certain conditions (see [3] and [6]), formulae (3) and (4) define a new \( C^1 (C^2) \) solution to (1). By choosing different functions \( \phi \) we can thus obtain (still under certain conditions) many distinct general integrals of \( G \) and hence distinct solutions to (1). Furthermore, given a complete integral \( G \), we can form, either locally or globally (if possible), its singular integral (which graph is called a two-parameter envelope of \( G \)), i.e. a function

\[ u(x, y) = G(x, y, P_1(x, y), P_2(x, y)), \]  

(5)

by eliminating parameters \( P_1 \) and \( P_2 \) from the following system

\[ u(x, y) = G(x, y, P_1, P_2), \quad G_{P_1}(x, y, P_1, P_2) = 0, \quad \text{and} \quad G_{P_2}(x, y, P_1, P_2) = 0. \]  

(6)

As previously, one can also show that under certain conditions (see [3] and [6]), formulae (5) and (6) define a \( C^1 (C^2) \) solution to (1).

So far, we have revisited a method of generating new solutions to (1) based on complete integrals. In other words, given an equation (1) with complete integral
we may generate locally (or globally) $C^1$ ($C^2$) solutions to (1) which turn out to be general (singular) integrals of $G$ (if they exist). We shall now address a converse problem. Given a complete integral $G$ and a solution $u$ of a class $C^1$ ($C^2$) to (1), the problem is to represent $u$ in terms of a general (singular) integral of a complete integral $G$. There are many results in the literature addressing this issue, some of them being in fact false. One such result reads (see [9, Section 7 pp. 61]):

(CII) "When, however, one complete integral has been obtained, every other solution, including every other complete integral, appears among the solution of type (3) and (5) corresponding to the complete integral we have found."

2.2 Counterexample to Sneddon’s claim

In this subsection we shall show that the above assertion is invalid. The example to follow indicates fundamental difficulties that arise when CII claim is treated as a true statement.

EXAMPLE 1. Consider the following image irradiance equation (corresponding to the image of a Lambertian plane $u(x, y) = ax + by + c$, with $a^2 + b^2 = 1$, illuminated by an overhead distant point light-source direction)

$$\frac{1}{\sqrt{1 + u_x^2(x, y) + u_y^2(x, y)}} = \frac{1}{\sqrt{2}}$$

(7)

defined over some region $\Omega \subset U_1$, where $U_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. This equation can be rewritten in the equivalent eikonal form

$$u_x^2(x, y) + u_y^2(x, y) = 1.$$  

(8)

Consider now a two-parameter $C^2$ family of cones

$$G(x, y, a, b) = \sqrt{(x - a)^2 + (y - b)^2}$$

(9)
defined over $\Omega \times V$, where $V \subset (\mathbb{R}^2 \setminus \bar{U}_1)$. It is easy to observe that, for any fixed $(a^*, b^*) \in V$, a function $G(x, y, a^*, b^*)$ is a $C^2$ solution to (8). Moreover, a simple verification shows that the rank of the matrix (2) equals two. Thus, formula (9) defines a complete integral for the eikonal equation (8).

We shall show now that the function $v(x, y) = x + 2$, being a $C^2$ solution to (8), cannot be represented as a general integral of (9) expressed in the form $v(x, y) = G(x, y, a(x, y), b(a(x, y)))$. Suppose the contrary. Then, for some $a \to b(a)$, and for some $(x, y) \to a(x, y)$, we have

$$(x - a)^2 + (y - b(a))^2 = (x + 2)^2,$$

(10)

$$(x - a) + (y - b(a))b'(a) = 0.$$  

(11)

Differentiating (10) with respect to $y$, we get

$$[(x - a) + (y - b(a))b'(a)] \frac{da}{dy} + b(a) - y = 0.$$
Hence, in view of (11), \( y = b(a) \) and further, still by (11), \( x = a \). Finally, \( y = b(x) \), which is absurd. On the other hand, note that the graph of the function \( v \) is a one-parameter \( C^2 \) envelope of \( G(x, y, a(b), b) \) with \( a(b) \equiv -2 \). Upon analyzing the above case we come to the following critical conclusion:

*If Sneddon’s assertion CII is to be correct, then the definition of a general integral, specified by (3) and (4), needs to be treated at least in a symmetric manner.*

Analogously, the function \( v_1(x, y) = x \) (being a solution to (8)) cannot be represented as the general integral of any subfamily of (9) expressed in the form \( G(x, y, a(b), b) \). On the other hand, if \( v_1 \) happens to be a general integral of \( G(x, y, a(b), b) \), we obtain

\[
(x - a(b))^2 + (y - b)^2 = x^2, \tag{12}
\]
\[
(x - a(b))a'(b) + (y - b) = 0. \tag{13}
\]

By differentiating (12) with respect to \( y \), we get

\[
[(x - a(b))a'(b) + (y - b)] \frac{\partial b}{\partial y} + b - y = 0.
\]

Hence, in view of (13), \( b(x, y) = y \), and further, by (12), we have \( (x - a(b))^2 = x^2 \). The latter is only possible for \( a(b) = 0 \) or \( a(b) = 2x \). The first case is impossible as \( (a(b), b) \notin (\mathbb{R}^2 \setminus \hat{U}_1) \). So is the second, as then \( 2x = a(y) \), which is absurd.

A straightforward verification of conditions (6) shows that \( v_1 \) cannot be either represented as a singular integral of (9).

Thus we arrive at the conclusion that the assertion CII cannot be universally true. \( \square \)

For a more extensive analysis including correction of Sneddon’s assertion CII an interested reader is referred to [3] and [8].

3 Erroneous application of complete integrals in computer vision

In this section, we revisit the proofs of the uniqueness results, contained in [1, 2] which concern the images of a Lambertian unit hemisphere centered at the origin and of a Lambertian plane, both illuminated by an overhead, distant point-light source. As the above Brooks’ both proofs resort to an incomplete Sneddon’s assertion we show that a clear contradiction results from the above mentioned uniqueness claims. We begin by quoting the precise statements of the uniqueness results presented in [1, 2].

*Consider an image of the Lambertian unit hemisphere centered at the origin and illuminated from an overhead distant point-light source direction. Then the functions \( u(x, y) = \pm \sqrt{1 - x^2 - y^2} + C \) are the only solutions to the corresponding image irradiance equation (14) defined over \( \Omega(x, y) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \).*
Consider an image of the Lambertian plane \( u(x, y) = ax + by + C \) illuminated from an overhead distant point light-source direction. Then any solution to the corresponding image irradiance equation (21) is a ruled surface (see Klingenberg [7, Definition 3.7.4]).

3.1 Revisiting Uniqueness Proof for an Image of a Lambertian Hemisphere

We shall first briefly revisit the proof of claim BI which takes Sneddon’s assertion CII for granted and involves the following pattern:

(a) first, to generate a complete integral to (14),

(b) next, to generate all of its general and singular integrals (introduced by (4) and (6)) and to show that they are not smooth over \( \Omega_{(x,y)} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \),

(c) finally, by using (b) combined with assertion CII, to claim a uniqueness result for (14) over \( \Omega_{(x,y)} \).

Suppose that a Lambertian northern unit hemisphere \( S \) centered at the origin (represented by the graph of the function \( u_4(x, y) = \sqrt{1 - x^2 - y^2} + C \), where \( C \) is an arbitrary constant) is illuminated by a distant, overhead point light-source. We have the following image irradiance equation

\[
\frac{1}{\sqrt{z_x^2(x, y) + z_y^2(x, y) + 1}} = \sqrt{1 - x^2 - y^2}, \tag{14}
\]

defined over a unit disc \( \Omega_{(x,y)} \). This can be rewritten into an equivalent eikonal form \( z_x^2(x, y) + z_y^2(x, y) = (x^2 + y^2)(1 - x^2 - y^2)^{-1} \). Note that the function \( u_4(x, y) = -\sqrt{1 - x^2 - y^2} + C \) constitutes another solution to equation (14). The last equation rewritten in polar coordinates takes the form

\[
z_r^2(r, \theta) + \frac{1}{r^2} z_\theta^2(r, \theta) = \frac{r^2}{1 - r^2}, \tag{15}
\]

defined over \( \Omega_{(r, \theta)} = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < 1, \ 0 \leq \theta < 2\pi\} \). Consider now the following two-parameter system of solutions \( z(r, \theta; k, M) = k\theta + g(r, k) + M \) to (15), where the function \( g(r, k) \) satisfies

\[
\left( \frac{\partial g}{\partial r}(r, \theta) \right)^2 + \frac{k^2}{r^2} = \frac{r^2}{1 - r^2},
\]

that is, \( g \) is up to a constant, given by \( g(r, k) = \pm \int_{r_0}^r \left( \frac{s^2}{1-s^2} \right)^{-1/2} ds \). According to Brooks’ claims, we hence obtain the complete integral

\[
z(r, \theta; k, M) = k\theta \pm \int_{r_0}^r \sqrt{\frac{s^2}{1-s^2} - \frac{k^2}{s^2}} ds + M \tag{16}
\]

having, among others, the following properties:
(i) \[ z(r, \theta; 0, M) = \pm \int_{r_0}^{r} (s^2/(1 - s^2))^{1/2} ds + M = \pm \sqrt{1 - r^2} + C, \] which corresponds to our hemisphere solutions \( u_\pm \) and \( u_+ \),

(ii) \( g(r, k) \) is only defined for \( (r^2/(1 - r^2)) \geq (k^2/r^2) \). Thus we require that \( r_h = \left( (|k|(k^2 + 4)^{1/2} - k^2)/2 \right)^{1/2} \leq r < 1, \ r_h < r_0 < 1 \), and so when \( k \neq 0 \), \( z(r, \theta; k, M) \) is defined only over the annulus \( \Omega_{(r, \theta)}^* = \{(r, \theta) \in \mathbb{R}^2 : r_h < r < 1, \ 0 \leq \theta < 2\pi \} \).

Note, however, that the fulfillment of conditions \( Ci \) and \( Cii \) (which are satisfied here) is still not sufficient to claim that (16) defines a complete integral to (15). Observe first that, for any fixed \((k^*, M^*)\), we should obtain a uniquely defined function \( z(r, \theta; k^*, M^*) \) that is a solution to (15). This is not the case for the family of functions defined by (16). Clearly, for any fixed \((x, y, k^*, M^*)\), formula (16) yields two values. In order to eliminate this ambiguity, we need to determine which sign should be chosen. For the sake of convenience, we choose the plus sign and hence obtain

\[ z(r, \theta; k, M) = k\theta + \int_{r_0}^{r} \sqrt{\frac{s^2}{1 - s^2} - \frac{k^2}{s^2}} ds + M \quad (17) \]

(the opposite case can be treated analogously). Consequently, we see here that, part of the condition (i) is not satisfied as \( u_- \) cannot be now represented by (17), for some \((k, M)\). This “lost solution”, however, should easily be obtained from either a general or a singular integral of (17) (provided Sneddon’s claim is correct). As the assertion \( CII \) is not true (see Example 1), we may hope that \( u_- \) cannot be retrieved from the complete integral (17) as either a general or a singular integral. This will stand in obvious contradiction with \( CII \). Following Sneddon’s representation claim \( CII \) we can now obtain a general integral of (17) by eliminating \( k \) from the system of the equations:

\[ f(r, \theta; z, k) = k\theta + g(r, k) + M(k) - z = 0, \quad (18) \]

\[ \frac{\partial f}{\partial k}(r, \theta; z, k) = 0. \quad (19) \]

Assume temporarily that such a general integral exists (if it does not exist then \( u_- \) cannot be expressed as a general integral). In view of (4), we obtain \( h(r, \theta) = f(r, \theta; z(r, \theta), k(r, \theta)) = 0 \) for some smooth real function \( M(k) \). The chain rule applied to the last equation yields

\[ 0 = \frac{\partial h}{\partial \theta}(r, \theta) = \frac{\partial f}{\partial \theta}(\kappa) + \frac{\partial f}{\partial r}(\kappa) \frac{\partial r}{\partial \theta}(r, \theta) + \frac{\partial f}{\partial z}(\kappa) \frac{\partial z}{\partial \theta}(r, \theta) + \frac{\partial f}{\partial k}(\kappa) \frac{\partial k}{\partial \theta}(r, \theta), \]

where \( \kappa = (r, \theta, z, k) \). By (18) and (19), we have \( f_z \equiv -1, f_\theta = k, \) and \( f_k \equiv 0 \). Accordingly,

\[ \frac{\partial z}{\partial \theta}(r, \theta) = k(r, \theta). \quad (20) \]

Thus, the angular rate of change of depth of any general integral of (17) at \((r, \theta) \in \Omega_{(r, \theta)}^* \setminus ([0, 1] \times \{0\})\) is equal to the value of \( k \) at point \((r, \theta)\).
From now on we shall drop a further course of Brooks’ uniqueness proof, which in conclusion claims that all general integrals of the type (4) fail to be smooth functions over $\Omega_{(r,\theta)}^{(c)}$ (as not being periodic functions) and thus, by applying assertion $CII$, infers uniqueness. We shall now show that (similarly to Example 1) a complete integral (17) and any of its general and singular integrals of type (4) and (6), respectively, cannot generate all possible solutions to the equation (15).

Note that, for uniqueness consideration a specific class of functions has to be a priori specified (we implicitly assume that the chosen class is that of $C^2$ function).

To reach a contradiction, note first that the third case of Sneddon’s claim $CIII$, referring to the singular integral, does not need to be considered here. Analyzing the existence of singular integrals, one can, however, easily verify that the system (6) is never satisfied as $(\partial z/\partial M) \equiv 1$ (parameters $(k, M)$ are treated here as independent). Thus, we can infer that there is no singular integral to (15). Recall that the “complete integral” (16) had to be re-shaped so as to become a meaningful function as introduced in (17) (the latter does not contain $u_\perp$). If Sneddon’s statement were to be correct, we should be able to represent (at least locally) the function $u_\perp$ as a general integral of some subfamily $z(r; \theta; k, M(k))$ (there is no singular integral here). This, however, never happens as (20) combined with $u_{\perp,\theta}(r, \theta) = 0$ yields

$$\frac{\partial u_{\perp}}{\partial \theta}(r, \theta) = k(r, \theta) = 0.$$  

Furthermore, in order to obtain the function $u_\perp$ either locally or globally, a parameter $k$ has to vanish everywhere in which case we find that $z(r, \theta; 0, M(0)) = u_{\perp}(r, \theta) + C$, which is obviously different than $u_{\perp}$, a contradiction. In a final effort to save this uniqueness proof for the image of a Lambertian hemisphere, we might try to treat Sneddon’s statement $CII$ in a broader sense. Namely, we could include symmetric case of general integral of the family $G(x, y, \phi(P_2), P_2)$ (see also Example 1). An easy verification shows, however, that then we also have $k(M) \equiv 0$. Hence, we cannot represent (even locally) the graph of $u_\perp$ as an envelope of

$$z(r, \theta; k(M), M) = \theta k(M) + \int_0^r \sqrt{\frac{s^2}{1 - s^2} - \frac{k(M)^2}{s^2}} ds + M$$

and thus we obtain the same type of contradiction as before. Consequently, it is clear that complete integral (17) together with its general and singular integrals do not generate (either locally or globally) all solutions to the equation (15) corresponding to the image of the Lambertian hemisphere. \(\square\)

Interestingly, assertion (c) was proved correctly for the set of $C^2$ functions by Deift and Sylvester [5].

### 3.2 Revisiting Uniqueness Proof for an Image of a Lambertian Plane

In this closing subsection we shall briefly refer to the uniqueness assertion $BII$ appearing in [1, 2] that concerns the image of a Lambertian plane illuminated from
an overhead, distant point light-source direction. In this case the corresponding image irradiance equation

\[
\frac{1}{\sqrt{u_x^2(x, y) + u_y^2(x, y)}} = \frac{1}{\sqrt{a^2 + b^2}}
\]

can be transformed into the equivalent eikonal equation

\[
u_x^2(x, y) + \nu_y^2(x, y) = c,
\]

where \( a^2 + b^2 = c \). One can easily show that the family of planes

\[
u(x, y) = ax + by + C
\]

together with the family of cones \( \nu(x, y, a_1, b_1) = \sqrt{c\sqrt{(x-a_1)^2 + (y-b_1)^2} + C} \) constitute \( C^2 \) solutions to (21) over any open \( U \subseteq \mathbb{R}^2 \), where \( C \) is an arbitrary constant and \((a_1, b_1) \notin U \). In order to obtain all other solutions to (21) a similar approach to that applied for the image of a Lambertian hemisphere is adopted in [1, 2]. Namely, by initially choosing a particular complete integral of (21) (of the type (22)) and by using assertion CII the following final conclusion is reached:

all solutions to (21) are ruled surfaces (see [7, Definition 3.7.4]).

As in the previous case, given the invalidity of Sneddon’s claim, we cannot assume the validity the proof for the above stated uniqueness result. It should be noted, however, that, for \( C^2 \) surfaces, the above assertion, subject to minor reformulations, can be proved without any recourse to the theory of complete integrals and envelopes (see Section 4, Proposition 1).

4 Uniqueness for Images of Lambertian Hemisphere and Plane

A natural question arises as to whether incomplete uniqueness assertions from [1, 2] are true. Uniqueness for equation (14) has been demonstrated by Deift and Sylvester [5], who showed that \( \pm(1 - x^2 - y^2)^{1/2} + k \) are the only \( C^2 \) solutions to this equation over the unit disc \( D(1) = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \). As for the image of the Lambertian plane, one can prove the following:

PROPOSITION 1. The graph of any \( C^2 \) solution to the eikonal equation (21) defined over some region \( \Omega \subseteq \mathbb{R}^2 \), with \( c \geq 0 \), is a developable surface (and so its Gaussian curvature vanishes).

Proof. The validity of the proposition is obviously true for \( c \) vanishing (the only solution to (21) is a constant function).

Assume now that \( c > 0 \). Suppose that \( u \) is a solution of class \( C^2 \) to (21) over some region \( U \). For each \( s \in (-s_0, s_0) \), let

\[
t \to \left( \bar{x}(t, s), \bar{y}(t, s), \bar{u}(t, s), \bar{\nu}(t, s), \bar{\nu}(t, s), \bar{\nu}(t, s) \right)
\]
be the solution of the characteristic system of equations associated with (21)

\[
\begin{align*}
(i) \quad \frac{d\tilde{x}}{dt}(t, s) &= 2\tilde{p}(t, s), \\
(ii) \quad \frac{d\tilde{y}}{dt}(t, s) &= 2\tilde{q}(t, s), \\
(iii) \quad \frac{d\tilde{u}}{dt}(t, s) &= 2c, \\
(iv) \quad \frac{d\tilde{v}}{dt}(t, s) &= 0, \\
v) \quad \frac{d\tilde{q}}{dt}(t, s) &= 0,
\end{align*}
\]

that satisfies the initial conditions

\[
\begin{align*}
(i) \quad \tilde{x}(0, s) &= x_0(s), \\
(ii) \quad \tilde{y}(0, s) &= y_0(s), \\
(iii) \quad \tilde{u}(0, s) &= u_0(s), \\
(iv) \quad \tilde{v}(0, s) &= v_0(s), \\
v) \quad \tilde{q}(0, s) &= q_0(s),
\end{align*}
\]

where \( u_0(s) = u(x_0(s), y_0(s)) \), \( p_0(s) = u_x(x_0(s), y_0(s)) \), \( q_0(s) = u_y(x_0(s), y_0(s)) \), and is defined on a maximal interval. It is readily verified that (23) \((i, ii)\) combined first with (23) \((iv, v)\), \((24)\) \((iv, v)\), and then with (24) \((i, ii)\) yield

\[
\tilde{x}(t, s) = 2tq_0(s) + x_0(s) \quad \tilde{y}(t, s) = 2tp_0(s) + y_0(s).
\]

Clearly, (23)\((iii)\) with (24)\((iii)\) imply

\[
\tilde{u}(t, s) = 2ct + u_0(s).
\]

By the fundamental property of solutions to characteristic system (see e.g. Courant and Hilbert [4, Chapter 2, Paragraph 3] we have \( \tilde{u}(t, s) = u(x(t, s), y(t, s)) \). Thus, by (25) and (26), a \( C^2 \) surface \( S_u \), being a graph of \( u \), can be represented in the following parametric form

\[
\begin{pmatrix}
\tilde{x}(t, s) \\
\tilde{y}(t, s) \\
\tilde{u}(t, s)
\end{pmatrix}
= \begin{pmatrix}
x_0(s) \\
y_0(s) \\
u_0(s)
\end{pmatrix}
+ t \begin{pmatrix}
2p_0(s) \\
2q_0(s) \\
2c
\end{pmatrix}.
\]

Thus \( S_u \) is a ruled surface. Observe now that by differentiating both sides of (21) with respect to \( x \) and then with respect to \( y \), we obtain the following system of equations

\[
- u_yu_{xy} = u_xu_{xx}, \quad u_xu_{yx} = -u_yu_{yy},
\]

Multiplying the first equation by the second one and taking into account that \( u \) is a \( C^2 \) function we deduce that \( u \) satisfies the following equation

\[
u_xu_y(u_{xx}u_{yy} - u_{xy}^2) = 0.
\]
Let \((x', y')\) be an arbitrary point in \(\Omega\). We now show that equation (28) implies that the Gaussian curvature of the graph \(S_u\) of \(u\) at point \((x', y', u(x', y'))\) expressed as
\[
K_u(x', y') = \frac{u_{xx}(x', y')u_{yy}(x', y') - u_{xy}^2(x', y')}{(1 + u_x^2(x', y') + u_y^2(x', y'))^2}
\]
vanishes. To this, assume that \(u_x(x', y')u_y(x', y') \neq 0\). Then clearly (28) yields that \(K_u(x', y') = 0\). If, on the other hand, \(u_x(x', y')u_y(x', y') = 0\), then as \(c > 0\) only one derivative can vanish (say \(u_x(x', y') = 0\)). Hence by simple inspection of (27) we deduce that \(u_{xy}(x', y') = u_{yy}(x', y') = 0\), and thus we also have \(K_u(x', y') = 0\). As surface \(S_u\) is ruled and its Gaussian curvature everywhere vanishes [7, Proposition 3.7.5] assures that \(S_u\) is developable. \(\Box\)

5 Acknowledgements

This research was conducted under Alexander von Humboldt Research Fellowship during author’s stay at Technical University of Berlin and at Warsaw University.

References