A Note on Existence and Uniqueness in Shape from Shading

Ryszard Kozerka
Department of Computer Science
University of Western Australia
Nedlands, WA 6009, Australia

Abstract

We discuss shape recovery of a smooth Lambertian surface illuminated by one (two, three) distant light-source(s). In the case of three (two) light sources, we show that the corresponding system of three (two) first-order non-linear partial differential equations has a unique (generically unique) solution. In the case of one overhead light-source, we discuss the existence and uniqueness of solutions to the corresponding eikonal equation. With regard to existence, we exhibit two classes of shading patterns for which no genuine shapes exist. In connection with uniqueness, we reveal circularly-symmetric eikonal equations for which there exist circularly-symmetric and non-circularly symmetric smooth solutions of special form.

1 Introduction

The shape-from-shading problem for a Lambertian surface was shown by Horn [7] to correspond to that of solving the first-order partial differential equation

\[ \frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2} \sqrt{u_x^2 + u_y^2} + 1} = E(x, y). \]  

(1.1)

Given \( 0 < E(x, y) \leq 1 \), the questions of the existence and uniqueness of solutions to (1.1) arise naturally. Existence corresponds to the problem of whether a given shading pattern with intensity between 0 and 1 is generated by a genuine Lambertian surface. Uniqueness corresponds to that of whether a shading pattern is due to one and only one Lambertian
shape. Some progress in solving these issues has been made in the case when \( p = (0, 0, -1) \). Then one can rewrite (1.1) as the eikonal equation

\[
\frac{1}{\bar{E}(x, y)} = \frac{u_x^2 + u_y^2}{E(x, y)}
\]

(1.2)

with \( \bar{E}(x, y) = E(x, y)^{-2} - 1 \). Brooks et al. [2, 3], Bruss [4], Deift and Sylvester [5], Oliensis [11], and Rouy and Tourin [13] contributed uniqueness results regarding this equation. Brooks et al. [1, 3], and Dupuis and Oliensis [6] established existence results. While all these results are far from being complete, they indicate, however, that uniqueness is rather exceptional and that existence is subject to many constraints.

In contrast with this, the shape of a Lambertian surface turns out to be uniquely determined by a triplet of images (three-source photometric stereo) obtained by illuminating a given scene from three different light-source directions. As shown by Horn [7] and Woodham [14], the system

\[
\frac{\hat{p}_{i1}u_x + \hat{p}_{i2}u_y - \hat{p}_{i3}}{\sqrt{\hat{p}_{i1}^2 + \hat{p}_{i2}^2 + \hat{p}_{i3}^2(u_x^2 + u_y^2) + 1}} = E_i(x, y)
\]

\( (i = 1, 2, 3) \) (1.3)

can be reduced to a system of the form \( u_x = F_1(x, y), u_y = F_2(x, y) \), where \( F_1 \) and \( F_2 \) are explicitly expressible in terms of \( E_1, E_2, E_3 \) and \( p = \hat{p}_1, q = \hat{p}_2, r = \hat{p}_3 \).

In the case of two-source photometric stereo (see Horn [7], Kozera [9, 10], Onn and Bruckstein [12], and Woodham [14]) one is also led to consideration of a system of the form

\[
\frac{\hat{p}_{i1}u_x + \hat{p}_{i2}u_y - \hat{p}_{i3}}{\sqrt{\hat{p}_{i1}^2 + \hat{p}_{i2}^2 + \hat{p}_{i3}^2(u_x^2 + u_y^2) + 1}} = E_i(x, y).
\]

\( (i = 1, 2) \) (1.4)

In this article, we overview recent existence and uniqueness result concerning (1.2 – 1.4). We first present a necessary and sufficient condition for the existence of a unique solution to (1.3). The main result concerning (1.4) will be that generically this system has, up to a constant, a unique solution. The entire discussion concerning (1.3) and (1.4) will make no appeal to any boundary conditions. With regard to (1.2), shading patterns will first be exhibited for which there is no corresponding object shape; next circularly-symmetric images will be presented admitting simultaneously circularly-symmetric and non-circularly symmetric solution surfaces such that the Euclidean norm of the gradient vanishes exactly at the center of the disc domain and diverges to infinity as the circumference of the domain is approached. A more detailed account of the work outlined in this note can be found in [1], [2], [3], [9], and [10].
2 Three-source photometric stereo

It is well known three-source photometric stereo leads to disambigous shape recovery (see [7] or [14]). We append this result by presenting a necessary and sufficient condition for the existence of a unique solution for a given triplet of images.

Suppose that a Lambertian surface $S$, represented by the graph of a function $u$ of class $C^1$, is illuminated from three linearly independent directions, namely $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, $r = (r_1, r_2, r_3)$. The following result establishes uniqueness:

**Theorem 2.1** The first derivatives of $u$ can be expressed in terms of $E_1$, $E_2$, $E_3$, $p$, $q$, and $r$, where $\|p\| = \|q\| = \|r\| = 1$, in the form $u_x = f_1/f_3$ and $u_y = f_2/f_3$, where

\[
\begin{align*}
    f_1 &= (q_2 r_3 - q_3 r_2) E_1 + (p_3 r_2 - p_2 r_3) E_2 + (p_2 q_3 - p_3 q_2) E_3, \\
    f_2 &= (q_3 r_1 - q_1 r_3) E_1 + (p_1 r_3 - p_3 r_1) E_2 + (p_3 q_1 - p_1 q_3) E_3, \\
    f_3 &= (q_2 r_1 - q_1 r_2) E_1 + (p_1 r_2 - p_2 r_1) E_2 + (p_2 q_1 - p_1 q_2) E_3.  
\end{align*}
\]

Supplementing this is the following necessary and sufficient condition for three given functions $E_1$, $E_2$, and $E_3$ to be interpreted as images of a Lambertian surface illuminated from three linearly independent directions:

**Theorem 2.2** Suppose that $\Omega$ is a simply connected domain in $\mathbb{R}^2$, $E_1$, $E_2$, and $E_3$ are functions of class $C^1$ on $\Omega$ with values in $(0, 1)$, $p$, $q$, $r$ are three normalized linearly independent vectors and $f_1$, $f_2$, and $f_3$ are defined by (2.1). In order that there exist a solution $u$ of class $C^2$ to (1.3), it is necessary and sufficient that $(f_1/f_3)_y = (f_2/f_3)_x$.

3 Two-source photometric stereo

In this section we show that, generically, the shape of a smooth Lambertian surface is uniquely determined by a pair of two images.

Suppose that a Lambertian surface $S$, represented by the graph of a function $u$ of class $C^1$, is illuminated from two linearly independent directions, $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$. We then have the following:

**Theorem 3.1** The first derivatives of function $u$ can be expressed in terms of $E_1$, $E_2$, $p$, and $q$, where $\|p\| = \|q\| = 1$, in the form $u_x = e_1/e_3$ and $u_y = e_2/e_3$, where

\[
\begin{align*}
    e_1 &= (q_1 (p|q) - p_1) E_1 + (p_1 (p|q) - q_1) E_2 \\
    e_2 &= (q_2 (p|q) - p_2) E_1 + (p_2 (p|q) - q_2) E_2 \\
    e_3 &= (q_3 (p|q) - p_3) E_1 + (p_3 (p|q) - q_3) E_2.
\end{align*}
\]
\[ e_2 = (q_2\langle p|q \rangle - p_2)E_1 + (p_2\langle p|q \rangle - q_2)E_2 \\
+ (p_1q_3 - p_3q_1)\varepsilon \sqrt{\Lambda}, \\
\]
\[ e_3 = (p_3 - q_3\langle p|q \rangle)E_1 + (q_3 - p_3\langle p|q \rangle)E_2 \\
+ (p_1q_2 - p_2q_1)\varepsilon \sqrt{\Lambda}, \]  
(3.1)

and where \( \Lambda = \Lambda(x, y) = [1 - E_1^2(x, y) - E_2^2(x, y)] - \langle p|q \rangle[\langle p|q \rangle - 2E_1(x, y)E_2(x, y)] \), and \( \varepsilon = \varepsilon (x, y) \) is a function taking values \( \pm 1 \) so that \( f(x, y) = \varepsilon(x, y)\sqrt{\Lambda(x, y)} \) is a continuous function.

3.1 The Case When \( \Lambda > 0 \)

When \( \Lambda \) is positive, the condition that the gradient \((u_x, u_y)\) be continuous implies that the function \( \varepsilon \) appearing in (3.1) takes on either the value 1 or the value \(-1\) over the entire image domain. As an immediate consequence, Theorem 3.1 implies that there are at most two solutions to (1.4). The following corollary formulates necessary and sufficient conditions for the existence of the exactly two solutions of class \( C^2 \) to the system (1.4).

**Corollary 3.1** Let \( E_1 \) and \( E_2 \) be functions of class \( C^1 \) over a simply connected region \( \Omega \) of \( \mathbb{R}^2 \) with values in \((0, 1] \) and let \( e_1, e_2, \) and \( e_3 \) be defined by (3.1). Suppose that \( \Lambda > 0 \) on \( \Omega \) and that, for each choice of sign, \( \sigma^\pm = (p_3 - q_3\langle p|q \rangle)E_1 + (q_3 - p_3\langle p|q \rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda} \) does not vanish over \( \Omega \). Then a necessary and sufficient condition for the existence of exactly two solutions of class \( C^2 \) to (1.4) is, for each choice of sign,

\[ (e_1/e_3)_y = (e_2/e_3)_x. \]  
(3.2)

For images \( E_1 \) and \( E_2 \) generated by a genuine Lambertian surface, that is, when (1.4) is satisfied for a certain function \( u \) of class \( C^2 \), exploitation of condition (3.2) leads to the following result:

**Theorem 3.2** Let \( p = (0, 0, -1) \) and \( q = (q_1, q_2, q_3) \) be such that \( q_1^2 + q_2^2 > 0 \) and \( \|q\| = 1 \), and let \( u \) be a function of class \( C^2 \) on a simply connected open subset \( \Omega \) of \( \mathbb{R}^2 \). Suppose that functions \( E_1 \) and \( E_2 \) are given by (1.4). Suppose, moreover, that \( \Lambda > 0 \) over \( \Omega \). In order that there exist a solution of class \( C^2 \) to (1.4) different from \( u \), it is necessary and sufficient that \( u \) satisfy

\[ q_1q_2(u_{yy} - u_{xx}) + (q_1^2 - q_2^2)u_{xy} = 0. \]  
(3.3)

Since generically \( u \) does not satisfy (3.3), it follows that \( u \) is the only solution to (1.4) in the \( C^2 \) class. In other words, the integrability condition disambiguates surface recovery for two image patterns.
3.2 The case when $\Lambda \equiv 0$

We now consider the case when, for given $E_1$ and $E_2$ defined over some domain $\Omega$, $\Lambda$ introduced in Theorem 3.1 vanishes.

An immediate consequence of Theorem 3.1 is that there is at most one solution to system (1.4). The next corollary formulates a necessary and sufficient condition for the existence of exactly one solution $u$ of class $C^2$ to the system (1.4).

**Corollary 3.2** Let $E_1$ and $E_2$ be functions of class $C^1$ over a simply connected region $\Omega$ of $\mathbb{R}^2$ with values in $(0,1]$. Suppose that $\Lambda \equiv 0$ on $\Omega$ and that $\sigma = (p_3 - q_3(p|q))E_1 + (q_3 - p_3(p|q))E_2$ does not vanish over $\Omega$. Then a necessary and sufficient condition for the existence of exactly one solution $u$ of class $C^2$ to (1.4) is $(g_1/g_3)_y = (g_2/g_3)_x$, where

\[
\begin{align*}
g_1 &= (q_1(p|q) - p_1)E_1 + (p_1(p|q) - q_1)E_2, \\
g_2 &= (q_2(p|q) - p_2)E_1 + (p_2(p|q) - q_2)E_2, \\
g_3 &= (p_3 - q_3(p|q))E_1 + (q_3 - p_3(p|q))E_2.
\end{align*}
\]

For a further discussion an interested reader is referred to [9] and [10].

3.3 The case when $\Lambda \geq 0$

Assume that there exists at least one solution $u$ of class $C^2$ to (1.4) and that the functions $E_1$, $E_2$ appearing in these equations are continuous over $\Omega$. Suppose that the set $\{(x,y) \in \Omega : \Lambda = 0\}$ is a smooth curve $\Gamma$ such that $\Omega \setminus \Gamma = D_1 \cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$ (on which, of course, $\Lambda$ is positive). By Corollary 3.2, there exist at most two solutions $(u_1^1, u_1^2)$ to (1.4) of class $C^2$ over $D_1$ and at most two solutions $(u_2^1, u_2^2)$ to (1.4) of class $C^2$ over $D_2$, respectively. We do not exclude the possibility that $u_1^1 = u_1^2$ for either $i = 1$; or $i = 2$; or $i = 1$ and $i = 2$.

Clearly, the restriction of $u$ to $D_i$ coincides with either $u_1^1$ or $u_1^2$ for $i = 1, 2$. Suppose that for some $i$ and $j$ ($i, j = 1, 2$) and some constant $c$ \(\lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_1^i(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} u_2^j(x',y') = g^{ij}(x,y) - c\) exist for each $(x,y) \in \Gamma$. Set

\[
v_{ij}(x,y) = \begin{cases} 
  u_1^i(x,y) & \text{if } (x,y) \in D_1, \\
  g^{ij}(x,y) & \text{if } (x,y) \in \Gamma, \\
  u_2^j(x,y) + c & \text{if } (x,y) \in D_2 
\end{cases}
\]

and suppose that for each $(x,y) \in \Gamma$ the function $v_{ij}$ is of class $C^2$. Then $v_{ij}$ is a solution of class $C^2$ to (1.4) over $\Omega$ and in such a case we say that the functions $u_1^i$ and $u_2^j$ **bifurcate** along $\Gamma$ in the $C^2$ class. It is clear that, up to a constant, one can define in this way at most four solutions of class $C^2$ to (1.4) over $\Omega$. The next result establishes necessary and sufficient conditions for such a bifurcation to take place.
Corollary 3.3 Let \( p = (0, 0, -1) \) and let \( q = (q_1, q_2, q_3) \) be such that \( q_1^2 + q_2^2 > 0 \) and \( \| q \| = 1 \). For a given pair of continuous functions \( E_1 \) and \( E_2 \) defining system (1.4), suppose that the set \( \{(x, y) \in \Omega : \Lambda = 0 \} \) is a smooth curve \( \Gamma \) such that \( \Omega \setminus \Gamma = D_1 \cup D_2 \), where \( D_1 \) and \( D_2 \) are disjoint open subsets of \( \Omega \). Assume that there exist two different solutions of class \( C^2 \) to (1.4) over \( D_1 \) and two different solutions of class \( C^2 \) to (1.4) over \( D_2 \), respectively. Let \( u \) be a solution over \( D_1 \) and \( v \) be a solution over \( D_2 \) such that \( h(x, y) = \lim_{(x', y') \to (x, y) \in \Gamma} u(x', y') = \lim_{(x', y') \to (x, y) \in \Gamma} v(x', y') + c \) for some choice of constant \( c \). Assume, moreover, that the pair \((u, v)\) defines in (3.4) a \( C^1 \) class function \( z \) over \( \Omega \). If \( q_1^2 - q_2^2 \neq 0 \), then the function \( z \) is of class \( C^2 \) over \( \Omega \) if and only if, for each \((x, y) \in \Gamma\),

\[
\begin{align*}
\lim_{(x', y') \to (x, y) \in \Gamma} u_{xx}(x', y') &= \lim_{(x', y') \to (x, y) \in \Gamma} v_{xx}(x', y'), \\
\lim_{(x', y') \to (x, y) \in \Gamma} u_{yy}(x', y') &= \lim_{(x', y') \to (x, y) \in \Gamma} v_{yy}(x', y').
\end{align*}
\]

If \( q_1^2 - q_2^2 = 0 \), then the function \( z \) is of class \( C^2 \) over \( \Omega \) if and only if, for each \((x, y) \in \Gamma\), either

\[
\lim_{(x', y') \to (x, y) \in \Gamma} u_{xx}(x', y') = \lim_{(x', y') \to (x, y) \in \Gamma} v_{xx}(x', y'),
\]

or

\[
\begin{align*}
\lim_{(x', y') \to (x, y) \in \Gamma} u_{xy}(x', y') &= \lim_{(x', y') \to (x, y) \in \Gamma} v_{xy}(x', y'), \\
\lim_{(x', y') \to (x, y) \in \Gamma} u_{yy}(x', y') &= \lim_{(x', y') \to (x, y) \in \Gamma} v_{yy}(x', y').
\end{align*}
\]

4 One image shape-from-shading

This section concerns solvability of (1.2). In the first subsection, we reveal two classes of images for which no physically-realisable solution shape exist. The second subsection presents a (non-)uniqueness result.

4.1 Images without solution

Let \( R \) be either a positive number or \(+\infty\). Let \( f \) be a non-negative continuous function on the interval \([0, R]\) vanishing exactly at zero. Consider equation (1.2) with \( E(x, y) = f(\sqrt{x^2 + y^2}) \) given over \( D(R) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\} \). With this special form of image, the class of circularly-symmetric solutions is of the form \( \pm U + k \), where \( U(x, y) = \)
\[ \int_0^R \sqrt{f(\sigma)} \, d\sigma. \] A condition on \( f \) guaranteeing that all solutions to the corresponding eikonal equation are unbounded may readily be formulated. Clearly, in the class of circularly-symmetric solutions, this sufficient condition is

\[ \int_0^R \sqrt{f(\sigma)} \, d\sigma = +\infty. \]  

(4.1)

It is less evident, though true, that the same condition is sufficient in the general case. In fact, we have the following:

**Theorem 4.1** Let \( f \) be a non-negative continuous function on \([0, R]\) vanishing exactly at zero and satisfying (4.1). Then there is no bounded \( C^1 \) solution in \( D(R) \) to (1.2) with \( E \) given by \( E(x, y) = f(\sqrt{x^2 + y^2}) \).

Interestingly, condition (4.1) is not only sufficient but also necessary for the unboundedness of all solutions to the equation in question. We have the following:

**Theorem 4.2** Let \( f \) be a non-negative continuous function in \([0, R]\) vanishing exactly at zero and satisfying \( \int_0^R \sqrt{f(\sigma)} \, d\sigma < \infty \). Then every solution in \( D(R) \) to (1.2) with \( E(x, y) = f(\sqrt{x^2 + y^2}) \) is bounded.

Observe that whether the integral \( \int_0^R \sqrt{f(\sigma)} \, d\sigma \) is finite or infinite depends exclusively on the behaviour of \( f \) near \( R \). The integral will be infinite if, for example, \( f(r) \) diverges to infinity sufficiently rapidly as \( r \) tends to \( R \). This means that, in the context of real images of Lambertian surfaces illuminated by an overhead point-source, a circularly-symmetric image cannot be derived from a genuine shape if it gets dark too quickly as the image boundary is approached.

The existence of transformed images \( E \) for which there is no solution whatsoever to (1.2) is guaranteed by the following result (see [1] and [8]):

**Theorem 4.3** Let \( \Omega \) be a bounded open connected subset of the \( \mathbb{R}^2 \) with boundary \( \partial \Omega \) being a piecewise \( C^1 \) curve of length \( \ell_{\partial \Omega} \). Let \((x_0, y_0)\) be a point in \( \Omega \) and \( r \) be a positive number such that the closed disc \( \overline{D}(x_0, y_0, r) \) of radius \( r \) centered at \((x_0, y_0)\) is contained in \( \Omega \). Suppose \( E \) is a non-negative continuous function on the closure of \( \Omega \), positive in \( \Omega \), such that

\[ 4r\sqrt{\varepsilon_1} > \ell_{\partial \Omega}\sqrt{\varepsilon_2}, \]  

(4.2)

where \( \varepsilon_1 = \min\{E(x, y) : (x, y) \in \overline{D}(x_0, y_0, r)\} \) and \( \varepsilon_2 = \max\{E(x, y) : (x, y) \in \partial\Omega\} \). Then there is no \( C^1 \) solution to (1.2) in \( \Omega \).

Note that the theorem is of local character: if \( \Omega \) is a subset of a domain \( \Delta \) and \( E \) is a non-
negative function on $\Delta$ whose restriction to $\Omega$ satisfies (4.2) for some choice of $D(x_0, y_0, r)$ in $\Omega$, then, obviously, there is no $C^1$ solution to (1.2) in $\Delta$. Reformulated in terms of Lambertian shading, this locality property can be expressed as saying that no genuine image can admit too dark a spot on too bright a background, assuming that the background does not contain a point having unit brightness. The precise balance between the qualifications “too dark” and “too bright” is, of course, given by condition (4.2).

4.2 Ambiguous shading patterns

If $u$ is a solution of (1.2), then so too is any member of the family $\pm u + k$, where $k$ is an arbitrary constant. Thus, the image of the surface represented by $u$ will be preserved under either a depth-shift of $u$ along the $z$-axis, a reversal (or inversion) of $u$ with respect to the $xy$-plane, or a combination of these transformations. These surfaces may clearly be said to possess a common shape. Of interest is the situation of essential uniqueness in which a family of the type specified above constitutes, within some class of functions, the complete set of solutions to an equation of the form given in (1). Uniqueness of this kind has been demonstrated in the case where $E(x, y) = (x^2 + y^2)(1 - x^2 - y^2)^{-1}$. Deit and Sylvester [5], proved that $\pm (1 - x^2 - y^2)^{1/2} + k$ are the only $C^2$ solutions to this equation over the unit disc $D(1)$. All of these solutions are hemispherical in shape. In an effort to obtain a more general result, Bruss [4] asserted the following: if $D(R)$ is the disc in the $xy$-plane with radius $R$ centered at the origin, and $f$ is a continuous function on $[0, R)$ of class $C^2$ over $(0, R)$ satisfying the following conditions:

(i) $f(0) = 0$ and $f(r) > 0$ for $0 < r < R$,

(ii) $\lim_{r \to 0} f'(r) = 0$, $\lim_{r \to 0} f''(r)$ exists and is positive,

(iii) $\lim_{r \to R} f(r) = +\infty$,

then all solutions of class $C^2$ to (1.2) in $D(R)$ with $E(x, y) = f(\sqrt{x^2 + y^2})$ take the form $\pm \int_0^{\sqrt{x^2 + y^2}} \sqrt{f(\sigma)} d\sigma + k$, and so are circularly symmetric with common shape. This result of Bruss is amongst the most important in the field. It turns out, however, that the above assertion is invalid. In [2, 3] we have been able to reveal a class of functions $f$, possessing the above properties, for which the corresponding eikonal equations have a bounded, nonspherically-symmetric solution of class $C^\infty$. Our approach is constructive in nature in that
we develop non-circularly-symmetric solutions from pieces of surface glued together to form $C^2$ wholes. The construction of solutions is divided into several steps. The graph of any of the non-circularly-symmetric solutions in question takes the form of a saddle having four regions of monotonicity spread out over four quadrants in the $xy$-plane determined by the lines $y = \pm x$. First, we construct a portion of a typical solution over the quadrant containing the positive $x$-halfaxis; the three remaining portions can easily be generated from this one. Next, we specify a class of functions $f$ for which the portions over all four quadrants can be smoothly pasted together and describe the corresponding process of gluing. Finally, we analyze the differentiability properties of the solutions obtained.

References


