On Complete Integrals and Uniqueness in Shape-from-Shading

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“Why is a contradiction more to be feared than a tautology”

L. Wittgenstein

ABSTRACT

In this paper, we analyze the problem of representing solutions of the first-order partial differential equation in terms of complete integrals. We show that some of the existing results referring to the representability problem are incomplete. Additionally, we analyze, in the context of complete integrals, the uniqueness problem for the shape recovery of a smooth Lambertian surface from an image obtained by illuminating this surface by an overhead, distant point light-source. Specifically, we revisit the uniqueness results already existing in the shape-from-shading literature that concern eikonal equations corresponding to the images of a Lambertian hemisphere and a Lambertian plane. We show that the latter results are incomplete and indicate how to fill the gaps in the corresponding proofs.
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1. INTRODUCTION

A monochrome photograph of a smooth object will typically exhibit brightness variation, or shading. Of interest to researchers in computer vision has been the problem of how object shape may be extracted from image shading. This shape-from-shading problem has been shown by Horn [12] to correspond to that of solving a first-order partial differential equation. Specifically, one seeks a function \( u \), representing surface depth in the direction of the z-axis, satisfying the image irradiance equation

\[
R(u_x, u_y) = E(x, y)
\]  

(1.1)
over some domain $\Omega \subset \mathbb{R}^2$. Here $R$ is a known function (the so-called \textit{reflectance map}) capturing the illumination and surface reflecting conditions, $E$ is an image formed by orthographic projection of light onto a plane parallel to the $xy$-plane, and $\Omega$ is the image domain. In this formulation, it is implicitly assumed that a small surface portion reflects light independently of its position in space. Thus, scene radiance is dependent only on lighting, surface lustre, and surface normal. By implication, light sources are infinitely far away, and internal surface reflections are disallowed.

Note that if function $u$ satisfies (1.1), then so too does $u + c$, where $c$ is a constant. In other words, the image of the surface formed by the graph of $u$ is preserved under a depth shift along the $z$-axis. It is therefore reasonable to assume that solutions to (1.1) are identified with classes of functions that satisfy (1.1) and differ by a constant.

An interesting case obtains when the reflectance map is specified so as to correspond to the situation in which a distant point light-source illuminates a \textit{Lambertian surface}. A small portion of such a surface acts as a \textit{perfect diffuser} appearing equally bright from all directions. According to Lambert’s law, if a small portion of a Lambertian surface with normal direction $n = (u_x, u_y, -1)$ is illuminated by a distant point light-source of unit power in direction $p = (p_1, p_2, p_3)$, then assuming that image irradiance is equal to scene radiance, the reflectance map is given by the cosine of the angle between these two directions. Thus, if $E(x, y)$ denotes the corresponding image, the image irradiance equation for the above situation takes the form

$$\langle p|n\rangle\|p\|^{-1}\|n\|^{-1} = E(x, y),$$

(1.2)

where $\langle \cdot | \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^3$, and $\| \cdot \|$ denotes the corresponding norm.

Given $0 < E(x, y) \leq 1$, the questions of the existence and uniqueness of solutions to (1.2) arise naturally. Existence corresponds to the problem of whether a given shading pattern with intensity between 0 and 1 is generated by a genuine Lambertian surface. Uniqueness corresponds to that of determining whether a shading pattern is due to a single choice of several Lambertian shapes. Some progress in elucidating of these issues
has been made under the assumption that a light source is situated overhead. In this case
\( p = (0, 0, -1) \), and with \( \mathcal{E}(x, y) = [E(x, y)]^{-2} - 1 \), one can rewrite the image irradiance
equation as the eikonal equation

\[
    u_x^2 + u_y^2 = \mathcal{E}(x, y).
\]

Brooks, Chojnacki, and Kozera [5, 6], Bruss [7], Deift and Sylvester [9], and Oliensis [19]
contributed important uniqueness results for this equation, whereas Brooks, Chojnacki,
and Kozera [4], Brooks and Chojnacki [6], Kimmel and Bruckstein [13], Rouy and Tourin
[21], Dupuis and Oliensis [11], and Kozera [15] established existence results. While all
these results are far from complete, they indicate that uniqueness is rather exceptional
and that existence is subject to many constraints.

Recently, rigorous uniqueness results concerning two-source and three-source photometric
stereo have been obtained. As it turns out, any Lambertian surface, illuminated
consecutively from two (three) linearly independent directions, can be generically uniquely
(uniquely) recovered from its two (three) images (see Kozera [16, 17], Onn and Bruckstein
[20], and Woodham [23]).

The main purpose of this paper is to analyze thoroughly a flawed Sneddon’s claim (see
[22, Section 7 pp. 61] about representability of any solution to a given first-order partial
differential equation in terms of either a complete or a general or a singular integral. The
literature on complete integrals is a bewildering collection of incomplete and erroneous
statements (see e.g. Dou [10] or [22]). A recent result by Chojnacki [8] sheds new light
on this topic and fills a gap in the literature. In order to convey an idea of real nature of
flaws stemming from Sneddon’s assertion we shall highlight (in the closing sections of this
paper) main points of Chojnacki’s result and present its consequences for eikonal equations
appearing in the analysis of the shape-from-shading problem.

We shall also critically inspect uniqueness results (Brooks [2, 3]) concerning the images
of a Lambertian hemisphere and a Lambertian plane, which resort to Sneddon’s erroneous
assertion and as such are invalid. Finally, we adopt a different approach so that the re-
results claimed in [2, 3], subject to minor reformulations, become valid. The discussion to
follow will be extended by a number of illustrative examples which mainly concern eikonal
equations commonly appearing in the study of shape-from-shading problem.

2. SOME REMARKS ON REPRESENTING SOLUTIONS OF FIRST-ORDER P.D.E.
   IN TERMS OF COMPLETE INTEGRAL AND ENVELOPES

   In this section we shall refer to the assertion of Sneddon about representability of
solutions to a first-order partial differential equation in terms of a complete integral and
envelopes (see [22, Section 7]). With the aid of examples, we highlight the main flaws in
Sneddon’s claim and rectify them in the subsequent section.

   In the closing part of this section, we revisit the proofs of the uniqueness results,
contained in [2, 3], which concern the images of a Lambertian unit hemisphere centered at
the origin and a Lambertian plane, both illuminated by an overhead, distant point-light
source. As the above proofs resort to an incomplete Sneddon’s assertion we show that a
clear contradiction results from the above mentioned uniqueness claims. Both uniqueness
results shall be re-established in the closing section of this paper.

2.1 Preliminaries

   We first recall the notion of a complete integral (see also Bell [1], [8], [10], and [22]).
For a given first-order partial differential equation

\[ F(x, y, u, u_x, u_y) = 0, \tag{2.1} \]

defined over an open region \( \Omega \subset \mathbb{R}^2 \), a function \( G(x, y, P_1, P_2) \) of class \( C^2 \) over \( \Omega \times V \n\)(where \( V \) is an open region of \( \mathbb{R}^2 \)) is called a complete integral of (2.1) if

(Ci) for each \( (P_1, P_2) \in V \), the function \( G \) is a \( C^2 \) solution to (2.1) on \( \Omega \),

(Cii) for each \( (x, y) \in \Omega \) and for each \( (P_1, P_2) \in V \), the rank of the matrix

\[
\begin{pmatrix}
G_{P_1} & G_{x_1P_1} & G_{x_2P_1} \\
G_{P_2} & G_{x_1P_2} & G_{x_2P_2}
\end{pmatrix}
\begin{pmatrix}
x, y \\
P_1 \\
P_2
\end{pmatrix}
\tag{2.2}
\]
equals two.

Condition (2.2) assures that parameters $P_1$ and $P_2$ are independent (see [10, Section 6]). For a given $C^1$ ($C^2$) function $\phi : \mathbb{R} \to \mathbb{R}$, we can form a one-parameter family

$$u_\phi(x, y; P_1) = G(x, y, P_1, \phi(P_1)),$$

and can generate, either locally or globally (if possible), its one-parameter envelope (called also the general integral), i.e. a function

$$u(x, y) = G(x, y, P_1(x, y), \phi(P_1(x, y))).$$

By eliminating parameter $P_1$ from the system of the form

$$u(x, y) = G(x, y, P_1, \phi(P_1)) \quad \text{and} \quad (G(x, y, P_1, \phi(P_1)))_{P_1} = 0,$$  \hspace{1cm} (2.3i)

we can then show that under certain conditions (see [8] and [10]), formula (2.3) defines a new $C^1$ ($C^2$) solution to (2.1). By choosing different functions $\phi$ we can obtain (still under certain conditions) many distinct one-parameter envelopes of $G$ and hence distinct solutions to (2.1). Furthermore, given a complete integral $G$, we can form, either locally or globally (if possible), its two-parameter envelope (called also the singular integral), i.e. a function

$$u(x, y) = G(x, y, P_1(x, y), P_2(x, y)).$$

By eliminating parameters $P_1$ and $P_2$ from the following system:

$$u(x, y) = G(x, y, P_1, P_2), \quad G_{P_1}(x, y, P_1, P_2) = 0, \quad \text{and} \quad G_{P_2}(x, y, P_1, P_2) = 0,$$  \hspace{1cm} (2.4i)

it may be shown that under certain conditions (see [8] and [10]), formula (2.4) defines a $C^1$ ($C^2$) solution to (2.1). For more detailed information about sufficient conditions that assure the local existence of a $C^1$ ($C^2$) solution to (2.1), expressed as a one-parameter or two-parameter envelope, the interested reader is referred to [8] and [10]. We shall also shortly treat this problem in Section 3.
REMARK 1. The general (singular) integral introduced above has a simple geometric interpretation. Given a one-parameter (two-parameter) family of surfaces (possibly a collection of graphs of functions) one can define (independently of the theory of partial differential equations) a one-parameter (two-parameter) envelope of the family in purely geometric terms. It can be shown, moreover, (see [1] and [22]) that, under certain conditions, any general (singular) integral (if it exists) touches each element of a family of surfaces represented by a complete integral along a *limiting curve (point)* called a *characteristic curve (point).*

REMARK 2. The notion of envelope (not necessarily of a one-parameter (two-parameter) family) can be introduced in a different and broader way (see e.g. [8]). Namely, assume that \( u \) is a \( C^1 (C^2) \) function. Given a family of \( C^2 \) functions \( G(x, y, P_1, P_2) \) defined over region \( \Omega \times \mathcal{P} \subset \mathbb{R}^2 \times \mathbb{R}^2 \), then if there exists a \( C^1 (C^2) \) function \( f : \Omega \rightarrow \mathcal{P} \) such that for any \((x, y) \in \Omega\) \[(x, y, u(x, y), u_x(x, y), u_y(x, y)) = (x, y, G(x, y, f_1(x, y), f_2(x, y)), G_x(x, y, f_1(x, y), f_2(x, y)), G_y(x, y, f_1(x, y), f_2(x, y)))\], then we say that \( u \) is a \( C^1 (C^2) \) envelope of \( G \). Furthermore, if \( f(\Omega) \) is a \( k \)-dimensional \((k = 0, 1, 2) \) \( C^1 (C^2) \) submanifold of \( \mathcal{P} \), then we say that \( u \) is a \( k \)-parameter envelope of \( G \).

So far, we have revisited a method of generating new solutions to (2.1) based on complete integrals. In other words, given an equation (2.1) with complete integral \( G \) we may generate locally (or globally) \( C^1 (C^2) \) solutions to (2.1) which turn out to be one-parameter (two-parameter) envelopes of \( G \) (if they exist). We shall now address a converse problem. Given a complete integral \( G \) and a solution \( u \) of a class \( C^1 (C^2) \) to (2.1), the problem is to represent \( u \) in terms of a one-parameter (two-parameter) envelope of a complete integral \( G \). There are many results in the literature addressing this issue, some of them being false. One such result reads (see [22, Section 7 pp. 61]):

**CII** “When, however, one complete integral has been obtained, every other solution, including every other complete integral, appears among the solution of type (2.3) and
Below we shall show (see Examples 2.2, 2.3, and 3.2) that the above assertion is invalid. We begin our analysis by presenting two examples. The first explicitly illustrates the use of one-parameter envelopes of (2.6) in generating new solutions to (2.5) (and as such constitutes a positive feedback to Sneddon’s statement CII). The second one indicates some difficulties that arise when CII claim is treated as a true statement. The examples to follow concern eikonal equations which correspond, in computer vision, to the genuine images of a Lambertian paraboloid and a Lambertian plane illuminated by an overhead, distant point light-source.

EXAMPLE 2.1. Consider the following image irradiance equation

\[
\frac{1}{\sqrt{1 + u_x^2(x, y) + u_y^2(x, y)}} = \frac{1}{\sqrt{1 + x^2 + y^2}}
\]

defined over some open region \( \Omega \subset \mathbb{R}^2 \setminus \{(0, 0)\} \). This equation captures the situation in which the Lambertian paraboloid \( v(x, y) = (x^2 + y^2)/2 \) is illuminated by an overhead, distant point light-source. It can equivalently be rewritten as

\[
u_x^2(x, y) + u_y^2(x, y) = x^2 + y^2.
\] (2.5)

Consider now the two-parameter family of functions

\[
G(x, y, \alpha, \beta) = (x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha) + \beta
\] (2.6)

of class \( C^2 \) (with respect to all variables), defined over \( \Omega \times \mathbb{R}^2 \). It is easy to see that for any \((\alpha, \beta) \in \mathbb{R}^2\), the function \( G(x, y, \alpha, \beta) \) is a \( C^2 \) solution to (2.5); in fact

\[
G_x(x, y, \alpha, \beta) = y(\cos^2 \alpha - \sin^2 \alpha) - 2x \cos \alpha \sin \alpha,
\]

\[
G_y(x, y, \alpha, \beta) = x(\cos^2 \alpha - \sin^2 \alpha) + 2y \sin \alpha \cos \alpha,
\]

\[
G_{xx}(x, y, \alpha, \beta) = -2x \sin \alpha \cos \alpha,
\]

\[
G_{yy}(x, y, \alpha, \beta) = 2y \sin \alpha \cos \alpha.
\]
and so
\[
G_x^2 + G_y^2 = y^2 (\cos^2 \alpha - \sin^2 \alpha)^2 + 4x^2 \cos^2 \alpha \sin^2 \alpha - 4xy \cos \alpha \sin \alpha (\cos^2 \alpha - \sin^2 \alpha) \\
+ x^2 (\cos^2 \alpha - \sin^2 \alpha)^2 + 4y^2 \sin^2 \alpha \cos^2 \alpha + 4xy \cos \alpha \sin \alpha (\cos^2 \alpha - \sin^2 \alpha) \\
= y^2 (\cos^4 \alpha + \sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha) + x^2 (\cos^4 \alpha + \sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha) \\
= (x^2 + y^2)(\cos^2 \alpha + \sin^2 \alpha)^2 \\
= x^2 + y^2.
\]

We now prove that condition (2.2) is fulfilled over \( \Omega \). Note that, since \( G_\beta \equiv 1 \), we have \( G_{\beta x} \equiv G_{\beta y} \equiv 0 \). Straightforward verification also shows that

\[
G_\alpha (x, y, \alpha, \beta) = (x \sin \alpha - y \cos \alpha)^2 - (x \cos \alpha + y \sin \alpha)^2.
\]

Hence

\[
G_{\alpha x}(x, y, \alpha, \beta) = 2x (\sin^2 \alpha - \cos^2 \alpha) - 4y \sin \alpha \cos \alpha = -2x \cos 2\alpha - 2y \sin 2\alpha,
\]

\[
G_{\alpha y}(x, y, \alpha, \beta) = 2y \cos 2\alpha - 2x \sin 2\alpha.
\]

Note that, as the rank of the matrix (2.2) is non-vanishing, it is enough to show that the rank of this matrix cannot be one. Assume the converse, which happens if and only if

\[-2x \cos 2\alpha - 2y \sin 2\alpha = 0 \quad \text{and} \quad 2y \cos 2\alpha - 2x \sin 2\alpha = 0.
\]

The last system can be rewritten as

\[
A(v) = \begin{pmatrix} -2x & -2y \\ 2y & -2x \end{pmatrix} \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where \( v = (\cos 2\alpha, \sin 2\alpha) \). Furthermore, both \( \det(A) = 4(x^2 + y^2) \) and \( (x, y) \neq (0, 0) \) imply that vector \( v = (0, 0) \) is a unique solution to (2.7). Hence \( \cos 2\alpha = \sin 2\alpha = 0 \), which contradicts identity \( \cos^2 2\alpha + \sin^2 2\alpha = 1 \). Thus, the rank of matrix (2.2) has to be equal to two and, accordingly, a function \( G \) introduced by (2.6) defines a complete integral of (2.5) over \( \Omega \times \mathbb{R}^2 \).

It is readily verified that the functions \( u_1(x, y) = xy, u_2(x, y) = -xy, u_3(x, y) = (y^2 - x^2)/2, u_4(x, y) = (x^2 - y^2)/2, u_5(x, y) = (x^2 + y^2)/2, \) and \( u_6(x, y) = -(x^2 + y^2)/2 \)
constitute different $C^2$ solutions to the equation (2.5). Note also that first four of them are explicitly generated by a complete integral (2.6) as

$$G(x, y, 0, 0) = u_1(x, y), \quad G(x, y, \pi/4, 0) = u_3(x, y),$$

$$G(x, y, \pi/2, 0) = u_2(x, y), \quad \text{and} \quad G(x, y, 3\pi/4, 0) = u_4(x, y).$$

Interestingly, it should also be additionally noted that formula (2.6) represents a two-parameter family of saddle-like surfaces obtained by revolving the graph of the function $u_1$ around the origin, or shifting it along the $z$-axis. From this geometric observation it is clear that $u_5$ and $u_6$ cannot be generated from $G$ by a mere fixing of parameters $\alpha$ and $\beta$.

We now derive $u_5$ (and similarly $u_6$) by generating an appropriate one-parameter envelope of the complete integral (2.6). To this end, let $\beta(\alpha) \equiv 0$. We shall eliminate $\alpha = \alpha(x, y)$ from the system

$$u_5(x, y) = G(x, y, \alpha, 0), \quad G_\alpha(x, y, \alpha, 0) = 0$$

so that the corresponding one-parameter envelope of (2.6) coincides with $u_5$. The last system can be rewritten as

$$(x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha) = (x^2 + y^2)/2, \quad (2.8)$$

$$(x \sin \alpha - y \cos \alpha)^2 - (x \cos \alpha + y \sin \alpha)^2 = 0. \quad (2.9)$$

In order to find at least one solution to (2.8) and (2.9), it is enough to find a function $\alpha$ such that

$$x \cos \alpha + y \sin \alpha = \sqrt{(x^2 + y^2)/2} \quad \text{and} \quad y \cos \alpha - x \sin \alpha = \sqrt{(x^2 + y^2)/2}.$$

The last two equations can be reformulated in matrix equation as:

$$
\begin{pmatrix}
y \\
-x
\end{pmatrix}
\begin{pmatrix}
sin \alpha \\
\cos \alpha
\end{pmatrix} =
\begin{pmatrix}
\sqrt{(x^2 + y^2)/2} \\
\sqrt{(x^2 + y^2)/2}
\end{pmatrix}.
$$

Since $x^2 + y^2 \neq 0$, we finally obtain

$$
\cos \alpha = \frac{x + y}{\sqrt{2x^2 + y^2}} \quad \text{and} \quad \sin \alpha = \frac{y - x}{\sqrt{2x^2 + y^2}}.
$$
Thus, \( \alpha(x, y) = \arctan \left( \frac{y-x}{y+x} \right) \) for \( x \neq -y \) \( (\alpha(x, y) = \arccot \left( \frac{x+y}{y-x} \right) \) for \( x \neq y \). In consequence, for any \( U \subset \Omega \) which does not intersect the line \( x = -y \) \( (x = y) \), we can generate \( u_5 \) as a one-parameter envelope of the complete integral (2.6). Note that the function \( u_6 \) can be obtained in a similar manner. \( \square \)

**REMARK 3.** It will be shown later (see Example 2.3) that it is possible to introduce a different complete integral to (2.5) (similar to that constructed in (2.19)) “containing” a function \( u_5 \), for which none of its one-parameter (two-parameter) envelopes coincides with \( u_6 \). Note also that the last example emphasizes the local character of the notion of the envelope.

**EXAMPLE 2.2.** Consider the following image irradiance equation (corresponding to the image of a Lambertian plane \( u(x, y) = ax + by + c \), with \( a^2 + b^2 = 1 \), illuminated by an overhead distant point light-source direction)

\[
\frac{1}{\sqrt{1 + u_x^2(x, y) + u_y^2(x, y)}} = \frac{1}{\sqrt{2}}
\]

defined over some region \( \Omega \subset U_1 \), where \( U_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \). This equation can be rewritten in the equivalent eikonal form

\[ u_x^2(x, y) + u_y^2(x, y) = 1. \]  
(2.10)

Consider now a two-parameter \( C^2 \) family of cones

\[ G(x, y, a, b) = \sqrt{(x-a)^2 + (y-b)^2} \]  
(2.11)

defined over \( \Omega \times V \), where \( V \subset (\mathbb{R}^2 \setminus \tilde{U}_1) \). It is easy to observe that, for any fixed \((a^*, b^*) \in V\), a function \( G(x, y, a^*, b^*) \) is a \( C^2 \) solution to (2.10). Moreover,

\[
G_a(x, y, a, b) = \frac{a-x}{\sqrt{(x-a)^2 + (y-b)^2}}, \quad G_b(x, y, a, b) = \frac{b-y}{\sqrt{(x-a)^2 + (y-b)^2}},
\]

\[
G_{ax}(x, y, a, b) = \frac{-(y-b)^2}{((x-a)^2 + (y-b)^2)^{3/2}}, \quad G_{bx}(x, y, a, b) = \frac{(x-a)(y-b)}{((x-a)^2 + (y-b)^2)^{3/2}},
\]

\[
G_{ay}(x, y, a, b) = \frac{(x-a)(y-b)}{((x-a)^2 + (y-b)^2)^{3/2}}, \quad G_{by}(x, y, a, b) = \frac{-x-a^2}{((x-a)^2 + (y-b)^2)^{3/2}}.
\]
We may now readily infer that the rank of the matrix (2.2) is equal to two. Assume the contrary. Then all 2-minors of matrix (2.2) have two vanish. Inspection shows, however, that minors $M_{12}$ and $M_{13}$ vanish if and only if $(x, y) = (a, b)$. This clearly cannot happen as for each $(x, y, a, b) \in \Omega \times V$ we have $(x, y) \neq (a, b)$. Thus, formula (2.11) defines a complete integral for the eikonal equation (2.10). Note that, geometrically, this complete integral defines a two-parameter family of cones with apexes off the set $U_1$.

We shall show now that the function $v(x, y) = x + 2$, being a $C^2$ solution to (2.10), cannot be represented as a one-parameter envelope of (2.11) expressed in the form $v(x, y) = G(x, y, a(x, y), b(a(x, y)))$. Suppose the contrary. Then, for some $a \to b(a)$, and for some $(x, y) \to a(x, y)$, we have

\[
(x - a)^2 + (y - b(a))^2 = (x + 2)^2,
\]

(2.12)

\[
(x - a) + (y - b(a))b'(a) = 0.
\]

(2.13)

Differentiating (2.12) with respect to $y$, we get

\[
[(x - a) + (y - b(a))b'(a)] \frac{da}{dy} + b(a) - y = 0.
\]

Hence, in view of (2.13), $y = b(a)$ and further, still by (2.13), $x = a$. Finally, $y = b(x)$, which is absurd. On the other hand, note that the graph of the function $v$ is a one-parameter $C^2$ envelope of $G(x, y, a(b), b)$ with $a(b) \equiv -2$; that is, the envelope of

\[
G(x, y, -2, b) = \sqrt{(x + 2)^2 + (y - b)^2},
\]

where $(a(b), b) \in (\mathbb{R}^2 \setminus \bar{U}_1)$. Upon analyzing the above case we come to the following critical conclusion:

*the definition of a one-parameter envelope, specified by (2.3), needs to be treated in a symmetric manner if Sneddon's assertion CII is to be correct.*

Analogously, the function $v_1(x, y) = x$ cannot be represented as the one-parameter envelope of a subfamily of (2.11) expressed in the form $G(x, y, a, b(a))$. On the other hand,
if \( v_1 \) happens to be the one-parameter envelope of \( G(x, y, a(b), b) \), we obtain
\[
(x - a(b))^2 + (y - b)^2 = x^2, \tag{2.14}
\]
\[(x - a(b))a'(b) + (y - b) = 0. \tag{2.15}\]

By differentiating (2.14) with respect to \( y \), we get
\[
[(x - a(b))a'(b) + (y - b)] \frac{\partial b}{\partial y} + b - y = 0.
\]
Hence, in view of (2.15), \( b(x, y) = y \), and further, by (2.14), we have \( (x - a(b))^2 = x^2 \).
The latter is only possible for \( a(b) = 0 \) or \( a(b) = 2x \). The first case is impossible as \( (a(b), b) \not\in (\mathbb{R}^2 \setminus \tilde{U}_1) \). So is the second, as then \( 2x = a(y) \), which is absurd. A straightforward verification of conditions (2.4i) shows that \( v_1 \) cannot be either represented as a singular integral of (2.11). This result is expected in view of the simple geometric observation that none of the elements of the family of surfaces (2.11) (defined over \( \Omega \times V \)) can touch the graph of the function \( v_1 \) along a potential characteristic curve (point).

Thus we arrive at the conclusion that the assertion CII cannot be universally true.

The intrigued reader might already have been tempted to seek an amendment to Sneddon’s claim. This issue will be discussed more thoroughly in Section 3 of this article (see also [8] or [10]). Nevertheless, it is worth noting now that a natural improvement should at least contain a case in which an envelope has a non-uniform parametric form. In order to explain the sense of this non-uniformity the reader is referred to Example 3.2, where \( u \) is represented (even locally) as
\[
 u(x, y) = \begin{cases} 
 G(x, y, a(x, y), b(a(x, y))) & y > 0 \& x \leq 0, \\
 G(x, y, a(b(x, y)), b(x, y)) & y < 0 \& x \geq 0.
\end{cases}
\]
Such a non-uniform case is clearly not considered in Sneddon’s assertion CII.

We shall close Example 2.2 with one more critical observation. Care must be taken with regard to local and global meanings of the representability of the solution in terms of a complete integral. Consider a \( C^2 \) function \( z(x, y) = 1 - \sqrt{x^2 + y^2} \) defined over \( \tilde{U}_\varepsilon \setminus \{0, 0\} \),
where \( \tilde{U}_\varepsilon = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 - \varepsilon \} \) and \( \varepsilon \), satisfying inequalities \( 0 < \varepsilon < 1 \), is fixed. Assume that \( (a, b) \in \mathbb{R}^2 \setminus \tilde{U}_\varepsilon \). Then it is easy to verify that formula (2.11) defines a new complete integral of (2.10) over \( (\tilde{U}_\varepsilon \setminus \{(0, 0)\}) \times (\mathbb{R}^2 \setminus \tilde{U}_\varepsilon) \subset \mathbb{R}^4 \). To find functions \( a(x, y) \) and \( b(x, y) \) such that

\[
1 - \sqrt{x^2 + y^2} = \sqrt{(x - a)^2 + (y - b)^2},
\]

note that

\[
\frac{x}{\sqrt{x^2 + y^2}} = \frac{x - a}{\sqrt{(x - a)^2 + (y - b)^2}}, \quad \frac{y}{\sqrt{x^2 + y^2}} = \frac{y - b}{\sqrt{(x - a)^2 + (y - b)^2}}.
\]

Furthermore

\[
\frac{x - a}{1 - \sqrt{x^2 + y^2}} = -\frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{y - b}{1 - \sqrt{x^2 + y^2}} = -\frac{y}{\sqrt{x^2 + y^2}}.
\]

Hence

\[
\begin{align*}
a(x, y) &= x + \frac{x}{\sqrt{x^2 + y^2}}(1 - \sqrt{x^2 + y^2}) = \frac{x}{\sqrt{x^2 + y^2}}, \\
b(x, y) &= y + \frac{y}{\sqrt{x^2 + y^2}}(1 - \sqrt{x^2 + y^2}) = \frac{y}{\sqrt{x^2 + y^2}}.
\end{align*}
\]

It is easy to see now that \( z(x, y) = G(x, y, a(x, y), b(x, y)) \) and that \( a^2(x, y) + b^2(x, y) = 1 \). The last equation implies that

\[
a(b(x, y)) = \pm \sqrt{1 - b^2(x, y)} \quad \text{or} \quad b(a(x, y)) = \pm \sqrt{1 - a^2(x, y)}.
\]

Accordingly, assuming now that \( z \) is a one-parameter envelope of (2.11) of the form \( G(x, y, a, b(a)) \) (or \( G(x, y, a(b), b) \)), it is clear that none of the above relations can be globally or even locally true (in any neighbourhood of \( a = \pm 1 \) (or \( b \pm 1 \))). Moreover, functions \( a(b) \) and \( b(a) \) fail to be \( C^1 \) functions at \( b = \pm 1 \) and \( a = \pm 1 \), respectively. In other words, we may conclude that a \( C^2 \) function \( z(x, y) = 1 - \sqrt{x^2 + y^2} \) can be represented neither globally, as a general integral of (2.11), nor locally in any neighbourhood of points \( a(x, y) = \pm 1 \) or \( b(x, y) = \pm 1 \) (see also Example 3.3). Note, however, that in a broader sense of definition of an envelope (see Remark 2) a function \( z \) is a \( C^2 \) envelope of \( G \) (either locally or globally) as there exists a pair of \( C^2 \) functions \( (f_1(x, y), f_2(x, y)) = (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2}) \) such that for any \( (x, y) \in \tilde{U}_\varepsilon \setminus \{(0, 0)\} \)
\[ z(x, y) = G(x, y, a(x, y), b(x, y)), \] where \((a(x, y), b(x, y)) = (f_1(x, y), f_2(x, y)) \in \mathbb{R}^2 \setminus \bar{U}_e. \]

\[ \square \]

Summing up observations made so far, it is clear that assertion made in CII is invalid. The last example explicitly shows that some of the solutions may also be represented in terms of a symmetric uniform parametrisation \(G(x, y, a(b(x, y)), b(x, y))\), whereas the others can be expressed in terms of a non-uniform parametrisation gathering cases \(G(x, y, a(b(x, y)), b(x, y))\) and \(G(x, y, a(x, y), b(a(x, y)))\) together (these situations are not covered by CII). In Subsection 3.1 (see also [8]), we shall further discuss the above issues.

2.2. Revisiting Uniqueness Proof for an Image of a Lambertian Hemisphere

We shall now revisit uniqueness proofs (see [2, 3]) relating to the images of a Lambertian hemisphere and a Lambertian plane, which adopt Sneddon’s erroneous assertion. We show that a confusing contradiction is reached as a consequence. Additionally, we shall also emphasize some important aspects concerning the notion of complete integral and envelopes. A uniqueness proof, appearing in [2, 3], which concerns the image of a Lambertian hemisphere takes the following approach:

(a) generate a complete integral to (2.16),

(b) generate all of its one-parameter and two-parameter envelopes (introduced in (2.3) and (2.4)) and show that they are not smooth over \(\Omega_{(x,y)} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}\),

(c) using (b) combined with CII, claim a uniqueness result for (2.16) over \(\Omega_{(x,y)}\).

Suppose that a Lambertian northern unit hemisphere \(S\) centered at the origin, represented by the graph of the function \(u_+(x, y) = \sqrt{1 - x^2 - y^2} + C\), where \(C\) is an arbitrary constant, is illuminated by a distant, overhead point light-source. We have the following
image irradiance equation

\[
\frac{1}{\sqrt{z_x^2(x,y) + z_y^2(x,y) + 1}} = \sqrt{1 - x^2 - y^2}
\] (2.16)

defined over a unit disc \( \Omega_{(x,y)} \). This can be rewritten into an equivalent eikonal form

\[
z_x^2(x,y) + z_y^2(x,y) = \frac{x^2 + y^2}{1 - x^2 - y^2}.
\]

Note that the function \( u_- (x, y) = -\sqrt{1 - x^2 - y^2} + C \) also satisfies (2.16) and hence constitutes another solution to equation (2.16). The last equation rewritten in polar coordinates takes the form

\[
z_r^2(r, \theta) + \frac{1}{r^2} z_\theta^2(r, \theta) = \frac{r^2}{1 - r^2}
\] (2.17)

over \( \Omega_{(r,\theta)} = \{(r, \theta) \in \mathbb{R}^2 : 0 < r < 1, \ 0 \leq \theta < 2\pi\} \). Consider now the following two-parameter system of solutions to (2.17) (see Luneburg [18, Section 27]; also reappearing in [12, Subsection 11.10])

\[
z(r, \theta; k, M) = k\theta + g(r, k) + M,
\]

where the function \( g(r, k) \) satisfies

\[
\left( \frac{\partial g}{\partial r}(r, \theta) \right)^2 + \frac{k^2}{r^2} = \frac{r^2}{1 - r^2},
\]

that is, \( g \) is up to a constant, given by

\[
g(r, k) = \pm \int_{r_0}^{r} \sqrt{\frac{s^2}{1 - s^2} - \frac{k^2}{s^2}} ds.
\]

According to Brooks’ claims, we hence obtain the complete integral

\[
z(r, \theta; k, M) = k\theta \pm \int_{r_0}^{r} \sqrt{\frac{s^2}{1 - s^2} - \frac{k^2}{s^2}} ds + M
\] (2.18)

having, among others, the following properties:

(i) \( z(r, \theta; 0, M) = \pm \int_{r_0}^{r} \sqrt{\frac{s^2}{1 - s^2}} ds + M = \pm \sqrt{1 - r^2} + C \), which corresponds to our hemisphere solutions \( u_- \) and \( u_+ \).
(ii) $g(r, k)$ is only defined for
\[ \frac{r^2}{1-r^2} \geq \frac{k^2}{r^2}. \]

Thus we require that
\[ r_k = \sqrt{\frac{|k| \sqrt{k^2 + 4k^2} - k^2}{2}} \leq r < 1, \quad r_k < r_0 < 1, \]

and so when $k \neq 0$, $z(r, \theta; k, M)$ is defined only over the annulus $\Omega^{r_k}_{(r, \theta)} = \{(r, \theta) \in \mathbb{R}^2 : r_k < r < 1, \quad 0 \leq \theta < 2\pi\}$.

Note, however, that in order to be able to claim that formula (2.18) defines a complete integral we need to be more rigorous with what we understand by a complete integral.

First, it is easy to check that any member of the two-paramenter family of functions $z(r, \theta; k, M)$ is a $C^2$ function over $(\Omega^{r_k}_{(r, \theta)}) \backslash ([0, 1] \times \{0\}) \times \mathbb{R}^2$ which, for any fixed $(k^*, M^*)$, satisfies (2.17). Note, moreover, that parameters $k$ and $M$ are independent as $\text{rank}(A) = 2$, where
\[
A = \begin{pmatrix}
z_k & z_{r_k} & z_{\theta k} \\
z_M & z_r & z_{\theta M}
\end{pmatrix} = \begin{pmatrix}
\theta + k \int_{r_0}^r \frac{1}{s} \sqrt{\frac{1}{(1-s^2)^2} - \frac{k^2}{s^2}} ds + k \frac{1}{r} \left( \frac{r^2 - k^2}{r^2} \right)^{-1/2} 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

The fulfillment of conditions (Ci) and (Cii) is still not sufficient to claim that (2.18) defines a complete integral to (2.17). Note first that, for any fixed $(k^*, M^*)$, we should obtain a uniquely defined function $z(r, \theta; k^*, M^*)$ that is a solution to (2.17). This is not the case for the family of functions defined by (2.18). Clearly, for any fixed $(x, y, k^*, M^*)$, formula (2.18) gives two values. To eliminate this ambiguity (which will turn out to be a decisive factor in reaching a contradiction), we need to decide which sign should be chosen. For the sake of convenience, we choose the plus sign and hence obtain
\[
z(r, \theta; k, M) = k\theta + \int_{r_0}^r \sqrt{\frac{s^2}{1-s^2} - \frac{k^2}{s^2}} ds + M \tag{2.19}
\]
(the opposite case can be treated analogously). We see that, part of the condition (i) is not satisfied as $u_-$ cannot now be represented by (2.19) for some $(k, M)$. This “lost solution”, however, should easily be obtained from either a one-parameter or a two-parameter envelope of (2.19) (provided Sneddon’s claim is correct). As the assertion CII (see Examples
2.2, 2.3, and 3.2) is not true, we may expect that \( u_\cdot \) cannot be retrieved from the complete integral (2.19) as either a one-parameter or a two-parameter envelope. This will stand in obvious contradiction with \( CII \).

In order to claim that (2.19) defines a complete integral of (2.17), still, one more aspect has to be taken into account. Note that any properly defined complete integral needs to be a two-parameter family of functions defined over the same subregion \( U_{r\theta} \subset \Omega_{(r,\theta)} \setminus ([0,1) \times \{0\}) \). As a result, we need to determine an open set \( V_{kM} \subset \mathbb{R}^2 \) such that for any \((k, M) \in V_{kM} \) each function \( z(r, \theta; k, M) \) is defined over the same open subset \( U_{r\theta} \subset \mathbb{R}^2 \). As was claimed in (ii), the function

\[
g(r, k) = \int_{r_0}^r \sqrt{\frac{s^2}{1-s^2} - \frac{k^2}{s^2}} ds,
\]

is only defined over the annulus \( \Omega_{(r,\theta)}^{r_k} \). It may readily be seen that, for any \((k, M) \in \mathbb{R}^2 \), \( 0 < r_k < 1 \), \( \lim_{k \to \pm \infty} r_k = 1 \), and \( \lim_{k \to 0} r_k = 0 \). Hence, as the annulus \( \Omega_{(r,\theta)}^{r_k} \) depends on the choice of \( k \) (its internal radius changes with \( k \) running over \( \mathbb{R} \)), we need to find an open subset \( V_{kM} \subset \mathbb{R}^2 \) such that \( g(r, k) \) is defined over the same \( U_{r\theta} \). Let \( \varepsilon \) be a positive number. A simple inspection yields that, given \( 0 < \varepsilon < 4 \), we have

\[
-\sqrt{\frac{\varepsilon}{4-2\sqrt{\varepsilon}}} < k < \sqrt{\frac{\varepsilon}{4-2\sqrt{\varepsilon}}} \quad \text{if and only if} \quad r_k = \sqrt{\frac{|k|\sqrt{k^2 + 4} - k^2}{2}} < \sqrt{\frac{\varepsilon}{2}}.
\]

Now we are ready to claim that \( z \) defined by (2.19) over a region \((\Omega_{(r,\theta)}^{\delta(\varepsilon)} \setminus ([0,1) \times \{0\})) \times V_{kM}^{\sigma(\varepsilon)} \subset \mathbb{R}^4 \), where \( \delta(\varepsilon) = \sqrt{\frac{\varepsilon}{2}} \), \( \sigma(\varepsilon) = \sqrt{\frac{\varepsilon}{4-2\sqrt{\varepsilon}}} \), and \( V_{kM}^{\sigma(\varepsilon)} = \{(k, M) \in \mathbb{R}^2 : -\sigma(\varepsilon) < k < -\sigma(\varepsilon)\} \) defines a complete integral to (2.17) (note that \( r_0 \) is any fixed number such that \( \delta(\varepsilon) < r_0 < 1 \)). Following Sneddon's representation claim \( CII \) we can now obtain one-parameter envelopes of (2.19) (if they exist; see Section 3) by eliminating \( k \) from the system of the equations:

\[
f(r, \theta, z, k) = k\theta + g(r, k) + M(k) - z = 0, \tag{2.20}
\]

\[
\frac{\partial f}{\partial k}(r, \theta, z, k) = 0. \tag{2.21}
\]

Assume temporarily that such a one-parameter envelope exists. In view of (2.3), we obtain

\[
h(r, \theta) = f(r, \theta, z(r, \theta), k(r, \theta)) = 0
\]

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for some smooth real function \( M(k) \). The chain rule applied to the last equation yields

\[
0 = \frac{\partial h}{\partial \theta}(r, \theta) = \frac{\partial f}{\partial \theta}(r, \theta, z, k) + \frac{\partial f}{\partial r}(r, \theta, z, k) \frac{\partial r}{\partial \theta}(r, \theta) + \frac{\partial f}{\partial z}(r, \theta, r, k) \frac{\partial z}{\partial \theta}(r, \theta) + \frac{\partial f}{\partial k}(r, \theta, r, k) \frac{\partial k}{\partial \theta}(r, \theta).
\]

In view of (2.20) and (2.21), we have \( f_z \equiv -1, \ f_\theta = k, \) and \( f_k \equiv 0. \) Accordingly,

\[
\frac{\partial z}{\partial \theta}(r, \theta) = k(r, \theta). \tag{2.22}
\]

Thus, the angular rate of change of depth of any one-parameter envelope of (2.19) at \((r, \theta) \in \Omega^{(\epsilon)}(r, \theta) \setminus ([0,1] \times \{0\})\) is equal to the value of \( k \) at point \((r, \theta)\).

From now on we shall drop a further course of Brooks' uniqueness proof, which in conclusion claims that all envelopes of the type (2.3i) fail to be smooth functions over \( \Omega^{(\epsilon)}(r, \theta) \) (as not being periodic functions) and thus, by applying assertion \( CII \), infers uniqueness.

We shall now show that (similarly to Example 2.2) a complete integral (2.19) and any of its envelopes of type (2.3i) or (2.4i) cannot generate all possible solutions to the equation (2.17).

Note that, for uniqueness consideration a specific class of functions has to be \textit{a priori} specified (we implicitly assume that the chosen class is that of \( C^2 \) functions). The latter turns out to be an important factor in generating any uniqueness result and so cannot be omitted. We shall address this problem in Section 4 by recalling that uniqueness results for the image of a Lambertian hemisphere differ substantially for \( C^2 \) and \( C^1 \) classes (see [9], [5], or [6]) and as such need a delicate mathematical treatment.

To reach a contradiction, note first that the third case of Sneddon's claim \( CII \), referring to the singular integral, does not need to be considered here. Analyzing the existence of singular integrals, one can, however, easily verify that the system (2.4i) is never satisfied as \( (\partial z/\partial M) = 1 \) (parameters \((k, M)\) are treated here as independent). Thus, we can infer that there is no singular integral to (2.17). It will be shown in Section 3 that only constant functions are singular solutions for an arbitrary eikonal equation. Recall that the "complete integral" (2.18) had to be reshaped so as to become a meaningful function as introduced in (2.19) (the latter does not contain \( u_- \)). If Sneddon's statement were to be
correct, we should be able to represent (at least locally) the function \( u_- \) as a one-parameter envelope of some family \( z(r, \theta; k, M(k)) \) (there is no singular integral here). This, however, never happens as (2.22) combined with \( u_- = 0 \) yields

\[
\frac{\partial u_-}{\partial \theta}(r, \theta) = k(r, \theta) = 0.
\]

Furthermore, in order to obtain the function \( u_- \) either locally or globally, a parameter \( k \) has to vanish everywhere in which case we find that

\[
z(r, \theta; 0, M(0)) = u_+(r, \theta) + C,
\]

which is obviously different than \( u_- \), a contradiction. Note, moreover, that \( k(r, \theta) \equiv 0 \) cannot define a differentiable function \( M = M(k) \) in some neighbourhood of \( k_0 = 0 \). This fact is another source of contradiction. In the final effort to save this uniqueness proof for the image of a Lambertian hemisphere, we might try to treat Sneddon’s statement \( CII \) in a broader sense. Namely, we could include symmetric case of one-parameter envelopes of the family \( G(x, y, \phi(P_2), P_2) \) (see Example 2.2). An easy verification shows, however, that then we also have \( k(M) \equiv 0 \). Hence, we cannot represent (even locally) \( u_- \) as an envelope of

\[
z(r, \theta; k(M), M) = \theta k(M) + \int_{r_0}^r \sqrt{\frac{s^2}{1 - s^2} - \frac{k(M)^2}{s^2}} ds + M
\]

and thus we obtain the same type of contradiction as before. Consequently, it is clear that complete integral (2.19) together with its general and singular integrals do not generate (either locally or globally) all solutions to the equation (2.17) corresponding to the image of the Lambertian hemisphere.

Interestingly, assertion (c) was proved correctly for the set of \( C^2 \) functions by Deift and Sylvester [9]. Note also that locally or globally (over any annulus \( r_k < r < 1 \) with the interval \( [0, 1) \times \{0\} \) deleted) formula (2.19) defines a wide range of solutions to (2.17). Furthermore, it should be emphasized, that the main drawbacks of the “uniqueness proof” of [2, 3] rely on resorting to an erroneous assertion \( CII \) as well as on applying complete integrals in a global manner. It is clear from the outset that a two-parameter family of
functions introduced in (2.19) and defined over a common domain \(\Omega_\delta^{(\varepsilon)}\setminus([0,1] \times \{0\})\) (and not continuous over \(\Omega_{(r,\theta)}^{\delta(\varepsilon)}\) for any \(k \neq 0\)) can never generate a smooth periodic envelope \(u(r, \theta) = u(r, \theta + 2\pi)\) defined over the entire region \(\Omega^{(\varepsilon)}_{(r,\theta)}\). □

**EXAMPLE 2.3.** In this example we re-examine Sneddon’s statement CII (applied to the particular choice of complete integrals of type (2.19)) by revisiting Examples 2.1 and 2.2.

(i) The equation (2.5) rewritten in the polar coordinate system takes the following equivalent form

\[
z_r^2(r, \theta) + \frac{1}{r^2}z_\theta^2(r, \theta) = r^2. \tag{2.23}
\]

We now introduce a new complete integral of (2.23) (different from that specified in (2.6) but similar to the one defined in (2.19) which contains \(u_5\) for \(k = 0\)) by putting

\[
z(r, \theta; k, M) = k\theta + \int_{r_0}^{r} \sqrt{s^2 - \frac{k^2}{s^2}} ds + M \tag{2.24}
\]

over any subregion of \(\Omega_{(r,\theta)}^{\sqrt{\varepsilon}} \times V_{kM}^{\varepsilon} \subset \mathbb{R}^4\), where \(\Omega_{(r,\theta)}^{\sqrt{\varepsilon}} = \{(r, \theta) \in \mathbb{R}^2 : \sqrt{\varepsilon} < r < 1, \ 0 < \theta < 2\pi\}\), \(V_{kM}^{\varepsilon} = \{ (k, M) \in \mathbb{R}^2 : -\varepsilon < k < \varepsilon \}\), and \(0 < \varepsilon < 1\) and \(r_0\) is fixed such that \(\sqrt{\varepsilon} < r_0 < 1\). Clearly, by repeating the analysis from Subsection 2.2, it is easy to see that a \(C^2\) function \(u_6(r, \theta) = -\frac{r^2}{2} + C\) cannot be expressed (either locally or globally) as one-parameter (two-parameter) envelope of (2.24) (for any constant \(C\)).

(ii) Similarly, consider now the transformed eikonal equation (2.10) written as

\[
z_r^2(r, \theta) + \frac{1}{r^2}z_\theta^2(r, \theta) = 1. \tag{2.25}
\]

An easy verification shows that the two-parameter family of functions

\[
z(r, \theta; k, M) = k\theta + \int_{r_0}^{r} \sqrt{1 - \frac{k^2}{s^2}} ds + M \tag{2.26}
\]

defines a complete integral to (2.25) over any subregion of \(U \subset \Omega_{(r,\theta)}^{\varepsilon} \times V_{kM}^{\varepsilon} \subset \mathbb{R}^4\), where \(\Omega_{(r,\theta)}^{\varepsilon} = \{(r, \theta) \in \mathbb{R}^2 : \varepsilon < r < 1, \ 0 < \theta < 2\pi\}\), \(V_{kM}^{\varepsilon} = \{ (k, M) \in \mathbb{R}^2 : -\varepsilon < k < \varepsilon \}\),
and \(0 < \varepsilon < 1\) and \(r_0\) is fixed such that \(\varepsilon < r_0 < 1\). Note that, for \(k = 0\), formula (2.26) determines a family of cones with apexes centered at \((0,0,M - r_0)\) and thus defines a complete integral different from the one introduced in (2.11). Similarly, by repeating the argument from Subsection 2.2, we can easily show that a \(C^2\) function \(u(r, \theta) = -r + C\) (reversals of cones), where \(C\) is an arbitrary constant, cannot be represented (either locally or globally) as one-parameter (two-parameter) envelope of (2.26). Moreover, consider the non-circularly symmetric function \(z(r, \theta) = -r \cos \theta + C\) (where \(C\) is an arbitrary constant) with

\[
z_r(r, \theta) = -\cos \theta \quad \text{and} \quad z_\theta(r, \theta) = r \sin \theta, \tag{2.27}
\]

defined over any subdomain \(U_1\) of \(\Omega_1 = \{ (r, \theta) : 0 < \theta < \frac{\pi}{2} \}\). Assume that \(z\) is a one-parameter envelope of (2.26) satisfying

\[
f_1(r, \theta, z, k) = k\theta + g(r, k) + M(k) - z = 0, \tag{2.28}
\]

\[
\frac{\partial f_1}{\partial k}(r, \theta, z, k) = 0. \tag{2.29}
\]

The chain rule applied twice to (2.28) in conjunction with (2.29) yields

\[
z_\theta(r, \theta) = k(r, \theta) \quad \text{and} \quad z_r(r, \theta) = \sqrt{1 - k^2/r^2}.
\]

In view of (2.27), we have that

\[
\cos \theta = -\sqrt{1 - (r^2 \sin^2 \theta)/r^2} = -|\cos \theta|.
\]

The last equation is clearly impossible over \(U \cap U_1\), a contradiction. Analogously, we can show that \(z\) cannot be a one-parameter envelope satisfying

\[
f_2(r, \theta, z, M) = \theta k(M) + g(r, k(M)) + M - z = 0,
\]

\[
\frac{\partial f_2}{\partial M}(r, \theta, z, M) = 0.
\]

Clearly, the equation in question has no singular solutions. Thus, the function \(z\) cannot be obtained as a one-parameter (two-parameter) envelope of (2.26) (either locally or globally) over \(U \cap \Omega_1\). \(\square\)
REMARK 4. As Example 2.2 shows, given a complete integral \( G \), it is possible to represent \( v + C \) as an envelope of \( G \) for some choice of the constant \( C \), and it is not possible to represent \( v + C \) as an envelope of \( G \) for some other choices of \( C \). This, is not a problem in shape-from-shading analysis as we are interested only in relative depth. Examples 2.3(ii) show, however, that by using Sneddon's assertion we can still lose a vast number of solutions including translated, reversed, or differently shaped surfaces.

2.3. Revisiting Uniqueness Proof for an Image of a Lambertian Plane

In this closing subsection we shall briefly refer to another uniqueness assertion appearing in [2, 3] that concerns the image of a Lambertian plane illuminated from an overhead, distant point light-source direction. In this case the corresponding image irradiance equation

\[
\frac{1}{\sqrt{u_x^2(x, y) + u_y^2(x, y) + 1}} = \frac{1}{\sqrt{a^2 + b^2 + 1}}
\]

can be transformed into the equivalent eikonal equation

\[
u_x^2(x, y) + u_y^2(x, y) = c,
\]

where \( a^2 + b^2 = c \). One can easily show that the family of planes

\[
u(x, y) = ax + by + C
\]

together with the family of cones

\[
v(x, y, a_1, b_1) = \sqrt{c}\sqrt{(x - a_1)^2 + (y - b_1)^2} + C
\]

constitute \( C^2 \) solutions to (2.30) over any open \( U \subset \mathbb{R}^2 \), where \( C \) is an arbitrary constant and \((a_1, b_1) \notin U \). To obtain all other solutions to (2.30) a similar approach to that applied for the image of a Lambertian hemisphere is adopted in [2, 3]. Namely, by initially choosing a particular complete integral of (2.30) (of the type (2.31)) and by using assertion CII the following final conclusion is reached:
All solutions to (2.30) are ruled surfaces (see next definition or Klingenberg [14, Definition 3.7.4]).

DEFINITION 1.4. A smooth surface $S$ in $\mathbb{R}^3$ is called a ruled surface if it can be represented in the parametric form

$$X(t, s) = \alpha(s) + tw(s),$$

where $s \to \alpha(s)$ and $s \to w(s)$ are smooth mappings from $\mathbb{R}$ onto $\mathbb{R}^3$. Here $w(s)$ is a vector field along a curve $\alpha(s)$. The curves $s = \text{const}$ are lines in $\mathbb{R}^3$ and are called generators of $X$. If, in addition, the unit normal vector field $n(t, s)$ to the surface $S$ is constant along generators, then $S$ is called developable.

As in the previous case, given the invalidity of Sneddon’s claim, we cannot assume the validity the above-stated uniqueness result.

It should be noted, however, that, for $C^2$ surfaces, the above assertion, subject to minor reformulations, can be proved without any recourse to the theory of complete integrals (see Section 4, Proposition 4.1).

3. REPRESENTABILITY OF SOLUTIONS OF THE FIRST-ORDER P.D.E IN TERMS OF COMPLETE INTEGRAL

In this section, we tackle the problem of finding sufficient conditions for representing any solution to (2.1) as an appropriate envelope of a complete integral (see Remark 2). Some of the existing results point out difficulties in generating such statements (e.g. [24, Chapter 1, Paragraph 4]), whereas others, contain erroneous assertions (e.g. Sneddon [22]). Recently, a paper by Chojnacki [8] has rectified the flawed statements and filled an existing gap in the literature. We shall use Chojnacki’s results and rediscuss their meaning in the context of the equation (2.1). Finally, in concluding this section, we shall highlight the main flaws of Sneddon’s false assertion $CII$ and supplement the entire analysis by a number of illustrative examples.
3.1. Representation of Solutions in Terms of Complete Integral

We begin by quoting the main result of Chojnacki (when \( n = 2 \)):

**THEOREM 3.1.** Let \( G(x, y, P_1, P_2) \) be a complete integral of the equation (2.1) defined over some region \( N \subset M \times \mathcal{P} \), where \( M, \mathcal{P} \subset \mathbb{R}^2 \), and let \( n^0 = (x^0, y^0, P^0_1, P^0_2) \in \mathbb{R}^4 \) and \((p^0, q^0) \in \mathbb{R}^2 \) be such that

\[
(dF)(x^0, y^0, u^0, p^0, q^0) \neq 0, \quad (3.1)
\]

\[
(x^0, y^0, G(n^0), G_x(n^0), G_y(n^0)) = (x^0, y^0, u^0, p^0, q^0). \quad (3.2)
\]

If \( u \) is a \( C^k \) \((k = 1, 2)\) solution to (2.1) on an open neighbourhood of \( \Omega \subset M \) of \((x^0, y^0)\), with

\[
(x^0, y^0, u(x^0, y^0), u_x(x^0, y^0), u_y(x^0, y^0)) = (x^0, y^0, u^0, p^0, q^0) \quad (3.3)
\]

then there exists an open neighbourhood \( U \subset \Omega \) of \((x^0, y^0)\) and a \( C^{k-1} \) function \( f = (f_1, f_2) \) from \( U \) into \( \mathcal{P} \) such that

\[
f(x^0, y^0) = (P^0_1, P^0_2) \quad (3.4)
\]

and, for each \((x, y) \in U\)

\[
(x, y, u(x, y), u_x(x, y), u_y(x, y)) = (x, y, G(x, y, f_1(x, y), f_2(x, y)), G_x(x, y, f_1(x, y), f_2(x, y)), G_y(x, y, f_1(x, y), f_2(x, y))). \quad (3.5)
\]

Moreover, if, for some open neighbourhood \( V \subset M \) of \((x^0, y^0)\), a continuous function \( g \) from \( V \) into \( \mathcal{P} \) satisfies (3.4) and (3.5), then \( g \) and \( f \) coincide on an open neighbourhood \( V' \subset V \cap U \) of \((x^0, y^0)\).

**REMARK 5.** Note that the above theorem is of local character and that, in the case of the eikonal equation

\[
u^2_x(x, y) + u^2_y(x, y) = \mathcal{E}(x, y), \quad (3.6)
\]

it cannot be applied in any neighbourhood of the singular point \((x_0, y_0)\) \( (i.e. \) where \( \mathcal{E} \) attains its minimum value such that \( \mathcal{E}(x_0, y_0) = 0 \) as condition (3.1) is then not fulfilled.
It is easy to observe that Theorem 3.1 reduces the problem of finding $C^2$ solutions to (2.1) to that of solving a system of quasi-linear equations (3.9) (specified below). To see this, note that if $u$ is a $C^2$ solution to (2.1) satisfying (3.5) for some $C^1$ function then clearly

$$u(x, y) = G(x, y, f_1(x, y), f_2(x, y))$$  \hspace{1cm} (3.7)

and

$$u_x(x, y) = G_x(x, y, f_1(x, y), f_2(x, y)), \quad u_y(x, y) = G_y(x, y, f_1(x, y), f_2(x, y)).$$ \hspace{1cm} (3.8)

Differentiating (3.7) and combining it with (3.8) yields

$$\frac{\partial G}{\partial P_1}(x, y, f_1(x, y), f_2(x, y)) \frac{\partial f_1}{\partial x}(x, y) + \frac{\partial G}{\partial P_2}(x, y, f_1(x, y), f_2(x, y)) \frac{\partial f_2}{\partial x}(x, y) = 0,$$ \hspace{1cm} (3.9)

$$\frac{\partial G}{\partial P_1}(x, y, f_1(x, y), f_2(x, y)) \frac{\partial f_1}{\partial y}(x, y) + \frac{\partial G}{\partial P_2}(x, y, f_1(x, y), f_2(x, y)) \frac{\partial f_2}{\partial y}(x, y) = 0.$$

Conversely, if $f$ is a $C^k$ ($k = 1, 2$) function satisfying (3.9), then $u$ given by (3.7) is a $C^k$ solution to (2.1). There are cases when (3.9) can be easily handled. For example, when the function $f$ has a constant rank at each point $(x, y) \in U$ \textit{(i.e.} rank$(df(x, y)) = k_1$ where $k_1 = 0, 1, 2$). Note also that if rank$(df) = const$, then $f(U)$ is a $k_1$-dimensional $C^k$ submanifold of $\mathcal{P}$ (the graph of $u$ is an envelope of the $k_1$-parameter family of surfaces $(x, y, G(x, y, f_1(x, y), f_2(x, y)))$. To validate the above remark (as well as to understand better the incompleteness of Sneddon’s assertion $CII$) we shall briefly discuss three possible cases (for a more extended and detailed analysis the interested reader is referred to [8]). Retaining the notation from the above theorem, assume that $u$ is a $C^2$ solution to (2.1) on an open neighbourhood of $(x_0, y_0) \in U$ such that (3.4) and (3.5) hold for some $C^1$ function $f : U \rightarrow \mathcal{P}$.

\textit{Case 1.} Assume that, for all $(x, y) \in U$, rank$(df(x, y)) = 0$ ($k_1 = 0$). Then, clearly, $f$ is a constant function and so $f(x, y) = (f_1(x, y), f_2(x, y)) = (P_1^0, P_2^0)$ implies

$$u(x, y) = G(x, y, P_1^0, P_2^0).$$ \hspace{1cm} (3.10)
We thus obtain a solution (0-parameter envelope), of class $C^2$, which is one of the elements of the initial family of functions constituting the complete integral.

Case 2. Assume that, for all $(x, y) \in U$, rank$(df(x, y)) = 1$ ($k_1 = 1$). By shrinking $U$ if necessary, we may assume that $(\partial f_1/\partial x)(x, y) \neq 0$ or $(\partial f_1/\partial y)(x, y) \neq 0$ for all $(x, y) \in U$.

An inspection of the standard proof of the rank theorem reveals that there exist open neighbourhoods $U \supset V \ni (x_0, y_0)$, $V_P \ni (P_0^1, P_0^2) = f(x_0, y_0)$, and open neighbourhoods $Q \subset \mathbb{R}^2$, $Q_P \subset \mathbb{R}^2$, and $C^1$ diffeomorphisms $\phi : Q \to V$, $\psi : V_P \to Q_P$ such that $f = \psi \circ g \circ \phi$, where a function $g$ is defined as $g(x, y) = (x, 0)$. Hence

$$f(x, y) = (\psi_1(\phi_1(x, y), 0), \psi_2(\phi_1(x, y), 0))$$

and thus $f_1(x, y) = \psi_1(\phi_1(x, y), 0)$. Note that

$$\frac{\partial f_1}{\partial x}(x, y) = \frac{\partial \psi_1}{\partial \phi_1}(\phi_1(x, y), 0) \frac{\partial \phi_1}{\partial x}(x, y),$$

$$\frac{\partial f_1}{\partial y}(x, y) = \frac{\partial \psi_1}{\partial \phi_1}(\phi_1(x, y), 0) \frac{\partial \phi_1}{\partial y}(x, y).$$

It is now clear that $(\partial \psi_1/\partial \phi_1)(\phi_1(x, y), 0) \neq 0$, since the last two equations would otherwise assure $(\partial f_1/\partial x)(x, y) = (\partial f_1/\partial y)(x, y) = 0$. Now by the implicit function theorem (applied to the equation $f_1(x, y) = \psi_1(\phi_1(x, y), 0)$), there exists a $C^1$ function $\tilde{\phi}_1$ such that $\phi_1 = \tilde{\phi}_1(f_1)$. Thus, we finally obtain

$$f(x, y) = (f_1(x, y), \psi_2(\tilde{\phi}_1(f_1(x, y)), 0)). \quad (3.11)$$

By (3.11), the function $f = (f_1, f_2)$ can be represented as $f = (f_1, f_2(f_1))$, and hence

$$u(x, y) = G(x, y, P_1(x, y), P_2(x, y)) = G(x, y, f_1(x, y), \Phi(f_1(x, y))). \quad (3.12)$$

The chain rule applied to (3.12) yields

$$u_x(x, y) = G_x(x, y, P_1(x, y), P_2(x, y))$$

$$+ G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial x}(x, y) + G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial x}(x, y),$$

$$u_y(x, y) = G_y(x, y, P_1(x, y), P_2(x, y))$$

$$+ G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial y}(x, y) + G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial y}(x, y),$$

$$u_{xx}(x, y) = G_{xx}(x, y, P_1(x, y), P_2(x, y))$$

$$+ 2G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial x}(x, y) + 2G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial x}(x, y),$$

$$u_{yy}(x, y) = G_{yy}(x, y, P_1(x, y), P_2(x, y))$$

$$+ 2G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial y}(x, y) + 2G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial y}(x, y),$$

$$u_{xy}(x, y) = G_{xy}(x, y, P_1(x, y), P_2(x, y))$$

$$+ G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial x}(x, y) + G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial x}(x, y),$$

$$u_{yx}(x, y) = G_{yx}(x, y, P_1(x, y), P_2(x, y))$$

$$+ G_{P_1}(x, y, P_1, P_2) \frac{\partial P_1}{\partial y}(x, y) + G_{P_2}(x, y, P_1, P_2) \Phi'(P_1(x, y)) \frac{\partial P_1}{\partial y}(x, y).$$
and furthermore, as \( u_x = G_x \) and \( u_y = G_y \), we have

\[
\frac{\partial P_1}{\partial x}(x, y)(G_{P_1}(x, y, P_1, P_2) + G_{P_2}(x, y, P_1, P_2)\Phi'(P_1(x, y))) = 0,
\]

\[
\frac{\partial P_1}{\partial y}(x, y)(G_{P_1}(x, y, P_1, P_2) + G_{P_2}(x, y, P_1, P_2)\Phi'(P_1(x, y))) = 0.
\]

As both \( P_{1x}(x, y) \) and \( P_{1y}(x, y) \) cannot vanish, we finally obtain

\[
0 = G_{P_1}(x, y, P_1, P_2) + G_{P_2}(x, y, P_1, P_2)\Phi'(P_1(x, y)) = (G(x, y, P_1(x, y), \Phi(P_1(x, y))))P_1.
\]

Consequently \( u \) satisfies one-parameter envelope equations (2.3i)

\[
(G(x, y, P_1(x, y), \Phi(P_1(x, y))))P_1 = 0 \quad \text{and} \quad u(x, y) = G(x, y, P_1(x, y), \Phi(P_1(x, y))).
\]

We thus arrive at the one-parameter envelope of \( G \) introduced in (2.3) and defined by the system (2.3i) (also considered by Sneddon in CII).

A symmetric result can also be obtained under the assumption that \( (\partial f_2/\partial x)(x, y) \neq 0 \) or \( (\partial f_2/\partial y)(x, y) \neq 0 \). Then obviously

\[
f(x, y) = (\psi_1(\tilde{\phi}_2(0, f_2(x, y))), f_2(x, y)).
\]

which leads to the local representation \( f = (f_1(f_2), f_1) \) and next to

\[
u(x, y) = G(x, y, P_1(x, y), P_2(x, y)) = G(x, y, \psi(f_2(x, y)), f_2(x, y)).
\]

Analogously, we shall arrive at the one-parameter envelope of \( G \) (not explicitly present in Sneddon’s assertion CII) which also satisfies

\[
(G(x, y, \psi(P_2(x, y)), P_2(x, y)))P_2 = 0 \quad \text{and} \quad u(x, y) = G(x, y, \psi(P_2(x, y)), P_2(x, y)).
\]

Note that \( f(V) \) is a one dimensional \( C^1 \) submanifold of \( \mathcal{P} \) which can be parametrised as either \( (t, \Phi(t)) \) or \( (\Psi(t), t) \). Note also that, as \( \text{rank}(df) \equiv 1 \) over \( U \) (by shrinking if necessary), the rank theorem implies that, over \( U \), we have a “uniform” parametrisation of \( (f_1(x, y), f_2(x, y)) \) i.e. either as \( (t, \Phi(t)) \) or as \( (\Psi(t), t) \). Example 3.3 (presented at the end of this section) underlines two important factors. First, we see that \( \text{rank}(df(x, y)) = 1 \)
for each \((x, y) \in \Omega\) does not assure a \textit{global uniform parametrisation}. It may happen that there is a “flip” in parametrisation. Secondly, we see that, even locally, it is possible to represent \((f_1(x, y), f_2(x, y))\) by using either both symmetric forms \((t, \Phi(t))\) and \((\Psi(t), t)\), or only one of them.

\textbf{Case 3.} Assume that, for all \((x, y) \in U\), \(\text{rank}(df(x, y)) = 2\) \((k_1 = 2)\). Clearly, \(f(U)\) is then a two dimensional \(C^1\) submanifold (an open subset) of \(\mathcal{P}\) and hence can be parametrised by two independent parameters. Interestingly, in this case a function \(f\) (provided conditions (3.4) and (3.5) hold) is determined uniquely by the system (3.9) and so too is \(u\) (see [8, Proposition 1]). Note that (3.9) combined with \(\text{rank}(df) = 2\) implies immediately that

\[
\frac{\partial G}{\partial P_1}(x, y, f_1(x, y), f_2(x, y)) = 0 \quad \text{and} \quad \frac{\partial G}{\partial P_2}(x, y, f_1(x, y), f_2(x, y)) = 0.
\]

The last two equations together with (3.5) characterise any singular integral (being a two-parameter envelope of \(G\)) specified in (2.4i) (such a class of solutions was considered by Sneddon’s assertion CII). Finally, it should be noted that one can show (see [8], [10], or [22]) that any singular integral \(u\) (if it exists) can be characterised exclusively in terms of equation (2.1) (without appeal to any complete integral of (2.1)). This can be achieved by solving the following system of equations:

\[
F(x, y, u, p, q) = 0, \quad \frac{\partial F}{\partial p}(x, y, u, p, q) = 0, \quad \text{and} \quad \frac{\partial F}{\partial q}(x, y, u, p, q) = 0. \tag{3.13}
\]

Note that system (3.13) is not equivalent to the system formed by (3.9), (3.4), and (3.5), which always yields a unique singular integral (if it exists). This observation is obvious upon noting that the system (3.13) imposes less constraints on finding singular integrals than the other one. Namely, an “\textit{initial germ}”, that is conditions (3.2), (3.3), and (3.4), may not be specified. \(\square\)

\textbf{Remark 6.} Conclusions resulting from the last case have important consequences for the eikonal equation. Namely, a non-existence of singular integrals is easily assured. Indeed, suppose that there exists a complete integral \(G\) of (3.6) for which we can generate
a singular integral to (3.6). Then, in view of (3.13), any singular integral \( u \) satisfies
\[ u_x(x, y) = u_y(x, y) = 0 \]
and hence is a constant function. Accordingly, the right-hand side of (3.6) vanishes everywhere (note that the condition (3.1) is here nowhere satisfied). Consequently, a complete integral \( G \) satisfies
\[ u_x(x, y) = G_x(x, y, P_1(x, y), P_2(x, y)) \equiv 0 \quad \text{and} \quad u_y(x, y) = G_y(x, y, P_1(x, y), P_2(x, y)) \equiv 0, \]
and so
\[ G_xP_1(x, y, P_1(x, y), P_2(x, y)) \equiv 0, \quad G_yP_1(x, y, P_1(x, y), P_2(x, y)) \equiv 0, \]
\[ G_xP_2(x, y, P_1(x, y), P_2(x, y)) \equiv 0, \quad \text{and} \quad G_yP_2(x, y, P_1(x, y), P_2(x, y)) \equiv 0. \]
The last system of equations has no solutions for otherwise the rank of matrix (2.2) would be less than two, a contradiction.

3.2. Final Remarks and Conclusions

We now briefly analyze the main flaws in Sneddon’s assertion CII, and illustrate them geometrically and analytically in Example 3.2.

(3i) It may happen that (even locally), for any open neighbourhood of \((x^0, y^0)\), the rank(df) at a point \((x, y) \in U\) varies as \((x, y)\) runs over \( U \) and so \( f(U) \) is not a \( C^k \) \((k = 1, 2)\) submanifold of \( \mathcal{P} \). This leads to a flaw of Sneddon’s statement CII which covers only the case with constant rank(df) over \( U \) (we assume that case (2.3) is treated symmetrically). There can be solutions for which rank(df) varies and none of Sneddon’s representations is valid (see [8] or here Example 3.2). Interestingly enough, the case of changing, locally, rank(df) from 2 to 1 or from 2 to 0 is impossible (see [8, Proposition 2]). Thus the rank(df) (if non-constant) can only vary here between 0 to 1 as \((x, y)\) runs over \( U \). Clearly in such a case we may have a non-uniform parameterisation (as mentioned in the Case 2 of Subsection 3.1, a “flip” is also possible for rank(df) \( \equiv 1\); see [18] or Example 3.3) for an envelope \( u \). For example, for the function \( f \) for which
\[
\begin{align*}
u(x, y) = \begin{cases} 
G(x, y, P_1(x, y), \Phi(P_1(x, y))) & (x, y) \in U_1 \quad \text{rank(df}(x, y)) = 1, \\
G(x, y, \Psi(P_2(x, y)), P_2(x, y)) & (x, y) \in U_2 \quad \text{rank(df}(x, y)) = 1,
\end{cases}
\end{align*}
\]
where $U = U_1 \cup U_2 \cup \Gamma$, $U_1$ and $U_2$ are two open disjoint subsets of $U$, and $\Gamma$ is a curve over which $\text{rank}(df) = 0$, none of Sneddon’s envelope representations apply in any neighbourhood of any point belonging to the curve $\Gamma$ (see Examples 3.2 and 3.3).

(3ii) As the Theorem 1.3 is of local character care must be taken with complete integrals applied globally (see Example 2.1). Note also that for an eikonal equation condition (3.1) is not satisfied at any singular point (see Remark 5). In such a case, Theorem 3.1 is of no applicability whatsoever. Thus for an eikonal equation, any analysis of representability of solutions to equation (3.6) in terms of complete integrals, in any neighbourhood of a singular point, requires a separate treatment (see Example 3.2).

(3iii) For an arbitrary eikonal equation, Remark 6 assures that there are no singular integrals (so they do not have to be considered while generating envelopes from any complete integral of (3.6)).

(3iv) It may also happen that, given a complete integral $G$ of (2.1), an “initial germ” (for generating locally a new solution $u$ to (2.1) as an envelope of $G$) may not satisfy conditions (3.2), (3.3), and (3.4). This is closely related to the range of choices of parameters $(P_1, P_2)$ which may be too restrictive to define a family of functions for which each graph “touches a graph of $u$” so that conditions (3.2), (3.3), and (3.4) are fulfilled (see also Remark 1). This can be another source of losing solutions (see Example 3.2).

(3v) Note finally, that as the analysis of Case 1, 2, and 3 from Subsection 3.1 reveals, a general definition of an envelope (introduced in Remark 2) subsumes the three categories of envelopes considered by Sneddon. Additionally, Examples 2.2, 3.2, and 3.3 show, however, that this more general definition has an essentially broader meaning that the definition adopted by Sneddon. Furthermore, note that, as Theorem 3.1 assures, this more general definition of an envelope together with conditions (3.1), (3.2), and (3.3) form a constrained set of sufficient conditions enough to generate a positive result about representability of any solution to a given first-order partial differential equation in terms of complete integral and envelopes.
In the next example we illustrate the above-mentioned difficulties, arising while trying to represent any solution of (2.1) in terms of a given complete integral and envelopes.

**EXAMPLE 3.2.**

(3i) We shall first show that the rank(df) can vary when \((x, y)\) runs over \(U\) (locally or globally). Consider the following family of functions

\[
G(x, y, a, b) = \sqrt{(x - a)^2 + (y - b)^2}
\]

(3.14)

defined over \(U \times V\), where \(V = \{(a, b) \in \mathbb{R}^2 : -\infty < a < \varepsilon, \ -\infty < b < \varepsilon\}\), \(U = \mathbb{R}^2 \setminus \tilde{V}\), and \(\varepsilon > 0\) is a fixed number. It easy to see that (3.14) defines a complete integral of (2.10) over \(U \times V\) different from those introduced in Example 2.2 (the variables \((x, y, a, b)\) run here over a different subdomain of \(\mathbb{R}^4\)). It is readily verified that for a \(C^1\) solution of (2.10) defined over \(U\) as

\[
u(x, y) = \begin{cases} 
y & \text{if } y > 0 \text{ and } x \leq 0, \\
\sqrt{x^2 + y^2} & \text{if } y > 0 \text{ and } x > 0, \\
x & \text{if } x > 0 \text{ and } y \leq 0,
\end{cases}
\]

the corresponding function \(f\) can be defined over \(U\) as

\[
f(x, y) = \begin{cases} 
(a(x, y), b(x, y)) = (x, 0) & \text{if } y > 0 \text{ and } x \leq 0, \\
(a(x, y), b(x, y)) \equiv (0, 0) & \text{if } y > 0 \text{ and } x > 0, \\
(a(x, y), b(x, y)) = (0, y) & \text{if } x > 0 \text{ and } y \leq 0,
\end{cases}
\]

(note that \((a(x, y), b(x, y)) \in V\)). It is clear that in any neighbourhood \(U_{m_0}\) of the point \(m_0 = (x_0, 0)\) or \(m_0 = (0, y_0)\) (where \(x_0 > \varepsilon\) and \(y_0 > \varepsilon\)) rank(df) varies from 0 to 1. Therefore \(f(U_{m_0})\) is not a submanifold of \(\mathbb{R}^2\). Note, moreover, that for any subregion \(\tilde{U} \subset U\) containing \(x\)-axis and \(y\)-axis the rank(df) changes twice and that \(f(\tilde{U})\) has a cusp at \((0, 0)\). Note also that \(u\) has a non-uniform parametrization over \(\tilde{U}\), namely

\[
u(x, y) = \begin{cases} 
G(x, y, t_1(x, y)) = x, \Phi(t_1(x, y)) \equiv 0 & \text{if } y > 0 \text{ and } x \leq 0, \\
G(x, y, 0, 0) & \text{if } y > 0 \text{ and } x > 0, \\
G(x, y, \Psi(t_2(x, y)) \equiv 0, t_2(x, y) = y) & \text{if } x > 0 \text{ and } y \leq 0.
\end{cases}
\]
For a similar example (within the set of $C^2$ functions), when $\text{rank}(df(x,y))$ varies while $(x,y)$ runs over $U$, the interested reader is referred to [8]. It should also be noted that a “flip” in parametrisation is also possible for $\text{rank}(df) \equiv 1$ over $\Omega$ (see Example 3.3).

(3ii) We can now easily find a geometric interpretation of the contradiction arising in Brooks’ uniqueness proof for an image of the Lambertian hemisphere. Note that if $u_-$ happens to be a one-parameter (two-parameter) envelope of (2.19), then (see Remark 1) each graph of any element of (2.19) should “touch” the graph of $u_-$, at (along) a limiting characteristic point (curve). In particular, this should be true for the graph of $u_+$. The latter is possible only for $(x_0, y_0) = (0, 0)$ which was excluded from the analysis in Subsection 2.2 (the case when both hemispheres touch each other on the boundary is also by default excluded). Note also that Theorem 3.1 cannot be used in a neighbourhood of point $(0, 0)$ as condition (3.1) is not satisfied here. A similar geometric argument applies in Example 2.3(i).

(3iv) In Example 2.2 it was shown that the function $v_1$ cannot be represented as an envelope of (2.11). A simple geometric observation shows that none of the graphs of $G(x,y,a,b)$, with $(x,y)$ running over $U_1$ and with $(a,b) \in \mathbb{R}^2 \setminus \bar{U}_1$ fixed, “touches” the graph of $v_1$. In order to see that it is not possible to generate an appropriate “initial germ” assume that conditions (3.2) and (3.3) are satisfied. It is then clear that

$$\left( x_0, y_0, \sqrt{(x_0-a)^2 + (y_0-b)^2}, \frac{x_0-a}{\sqrt{(x_0-a)^2 + (y_0-b)^2}}, \frac{y_0-b}{\sqrt{(x_0-a)^2 + (y_0-b)^2}} \right)$$

$$= (x_0, y_0, x_0, 1, 0).$$

Hence, $y_0 = b$ and thus, as $x_0$ has to be different from parameter $a$, we finally obtain $x_0 - a = x_0$ resulting in $a = 0$. Consequently, $(a,b) \in U$, which is a contradiction.

Note that the same type of difficulty in creating an “initial germ” appears also in Subsection 2.2. Indeed, if for some $(r^0, \theta^0, u^0_-, u^0_{-r}, u^0_{-\theta})$, we have

$$(r^0, \theta^0, -\sqrt{1-r^0 \theta^0}, r^0/\sqrt{1-r^0 \theta^0}, 0) = (r^0, \theta^0, z(r^0, \theta^0; k^0, M^0), z_r(r^0, \theta^0; k^0, M^0), k^0),$$

then $k^0 = 0$ and so $z(r^0, \theta^0; 0, M^0) = \sqrt{1-r^0 \theta^0} = -\sqrt{1-r^0 \theta^0}$. Thus $r^0 = \pm 1$, a contradiction. □
We close this subsection with an example illustrating the final comments made during the analysis of Case 2:

**EXAMPLE 3.3.** Consider a smooth function (see also Example 2.2) \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), defined over \( \Omega = \mathbb{R}^2 \setminus \{(0, 0)\} \) as follows:

\[
    f(x, y) = (f_1(x, y), f_2(x, y)) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).
\]

It is easily verified that

\[
    df(x, y) = \begin{pmatrix} \frac{y^2}{(x^2 + y^2)^{3/2}} & -\frac{xy}{(x^2 + y^2)^{3/2}} \\ -\frac{xy}{(x^2 + y^2)^{3/2}} & \frac{x^2}{(x^2 + y^2)^{3/2}} \end{pmatrix}
\]

and therefore, over \( \Omega \), \( \text{rank}(df) \equiv 1 \). Let

\[
    \phi_+(t) = \sqrt{1 - t^2} \quad \text{and} \quad \phi_-(t) = -\sqrt{1 - t^2}.
\]

Define the following subregions of \( \Omega \):

\[
    D_1 = \{(x, y) \in \mathbb{R}^2 : y > x \quad \text{and} \quad y > -x\}, \quad D_2 = \{(x, y) \in \mathbb{R}^2 : y < -x \quad \text{and} \quad y > x\},
\]

\[
    D_3 = \{(x, y) \in \mathbb{R}^2 : y < x \quad \text{and} \quad y < -x\}, \quad D_4 = \{(x, y) \in \mathbb{R}^2 : y > -x \quad \text{and} \quad y < x\},
\]

and \( D = \{(x, y) \in \mathbb{R}^2 : y > 0\} \).

It is easy to check that

\[
    \phi_+(f_1(x, y)) = \frac{|y|}{\sqrt{x^2 + y^2}}, \quad \phi_+(f_2(x, y)) = \frac{|x|}{\sqrt{x^2 + y^2}}, \quad (3.15)
\]

\[
    \phi_-(f_1(x, y)) = -\frac{|y|}{\sqrt{x^2 + y^2}}, \quad \text{and} \quad \phi_-(f_2(x, y)) = -\frac{|x|}{\sqrt{x^2 + y^2}}.
\]

An easy inspection shows that

\[
    f_2(x, y) = \phi_+(f_1(x, y)) \quad \text{for} \quad (x, y) \in D_1,
\]

\[
    f_1(x, y) = \phi_-(f_2(x, y)) \quad \text{for} \quad (x, y) \in D_2,
\]

\[
    f_2(x, y) = \phi_-(f_1(x, y)) \quad \text{for} \quad (x, y) \in D_3,
\]

\[
    f_1(x, y) = \phi_+(f_2(x, y)) \quad \text{for} \quad (x, y) \in D_4.
\]

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The latter equations show that, over $D_1$ and $D_3$, we have a parametrisation $(t, \Phi(t))$, and that, over $D_2$ and $D_4$, we have a symmetric parametrisation $(\Psi(t), t)$, respectively. Note, moreover, that $f_1(x, y)$ cannot be expressed as $\Theta(f_2(x, y))$ over $D_1$ (either globally or locally in a neighbourhood of the $y$-axis, for any function $\Theta$). This can be inferred from (3.15) combined with $f_1^2(x, y) + f_2^2(x, y) = 1$. Similarly, for a subregion $D_2$, $f_2(x, y)$ cannot be expressed as $\Lambda(f_1(x, y))$ (either globally or locally in a neighbourhood of the $x$-axis, for any function $\Lambda$). Thus we have shown that for any disc neighbourhood of $(0, 0)$ there has to be a non-uniform parametrisation. Note, moreover, that

$$f_2(x, y) = \phi_+(f_1(x, y)) \quad \text{for} \quad (x, y) \in D,$$

and thus, in particular, for any $U \subset D_2$ such that $y > 0$ (having, in view of the previous analysis, the parametrisation $(\Psi(t), t)$), we have a symmetric parametrisation $(t, \Phi(t))$. Thus, as we have shown, in some cases it is also possible to have two types of parametrisations over the same region. □

REMARK 7. Observe that for a given $C^2$ ($C^1$) solution $u$ of the equation (2.1), the corresponding function $f$ is, as a rule (due to Theorem 3.1), merely of class $C^1$ ($C^0$), whereas for a given $C^2$ ($C^1$) function $f$, the solution $u$ defined by

$$u(x, y) = G(x, y, f_1(x, y), f_2(x, y)) \quad \text{(3.16)}$$

is of class $C^2$ ($C^1$), respectively. In particular if $f$ is of class $C^1$ then for any envelope $u$, given by (3.16), the question of its $C^2$ differentiability requires separate treatment. In fact, observe that a function $f$, appearing in the Example 3.2, is globally (locally in the vicinity of either the $x$- or $y$-axis) of a mere $C^0$ class (as $u$ is only a $C^1$ solution in any neighbourhood of either the $x$ or $y$ axis). On the other hand, it is easily verified that for all $C^2$ envelopes, appearing in Example 2.2, the corresponding function $f$ is not only a $C^1$ (as assured by Theorem 3.1) but also a $C^2$ class mapping. In conclusion, still one more aspect needs to be emphasized here. Recall that, as the analysis of Case 2 and 3 reveals, the notion of general and singular integrals is a particular case of a more general definition.
of an envelope introduced in Remark 2. Note, moreover, that it is assumed therein that \( u \) is of class \( C^2 \) and that the corresponding function \( f \), due to Theorem 3.1, is merely of class \( C^1 \). Inspection of Cases 2 or 3 reveals that the whole analysis appearing therein cannot be directly extended to the case when \( u \in C^1(\Omega) \), unless \( f \), belonging to \( C^0(\Omega) \), is at least one-time continuously differentiable.

4. UNIQUENESS FOR IMAGES OF LAMBERTIAN HEMISPHERE AND PLANE

A natural question arises as to whether incomplete uniqueness assertions (see [2, 3]) are true. Uniqueness for equation (2.16) has been demonstrated by Deift and Sylvester [9], who showed that \( \pm(1 - x^2 - y^2)^{1/2} + k \) are the only \( C^2 \) solutions to this equation over the unit disc \( D(1) = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \). Interestingly, Deift and Sylvester also showed that this result fails in class of \( C^1 \) solutions (there exist infinitely many \( C^1 \) functions over \( D(1) \) satisfying (2.16)). The interested reader is also referred to [5, 6], where a \( C^1(D(1)) \cap C^2(D(1) \setminus \{(0, 0)\}) \) saddle-like solution to (2.16) is constructed. Note also that formula (2.19) defines infinitely many \( C^2 \) class solutions to (2.16) over \( \Omega_{(r, \theta)}\) \( \setminus ([0, 1] \times \{0\}) \). As for the image of the Lambertian plane, one can easily prove the following:

**Proposition 4.1.** The graph of any \( C^2 \) solution to the eikonal equation (2.30) defined over some region \( \Omega \subset \mathbb{R}^2 \), with \( c \geq 0 \), is a developable surface (and so its Gaussian curvature vanishes).

**Proof.** The validity of the proposition is obviously true for \( c \) vanishing (the only solution to (2.30) is a constant function).

Assume now that \( c > 0 \). Suppose that \( u \) is a solution of class \( C^2 \) to (2.30) over some region \( U \). For each \( s \in (-s_0, s_0) \), let

\[
t \mapsto (\tilde{x}(t, s), \tilde{y}(t, s), \tilde{u}(t, s), \tilde{p}(t, s), \tilde{q}(t, s))
\]
be the solution of the characteristic system of equations associated with (2.30)

\[(i) \quad \frac{d\tilde{x}}{dt}(t, s) = 2\tilde{p}(t, s),
\]
\[(ii) \quad \frac{d\tilde{y}}{dt}(t, s) = 2\tilde{q}(t, s),
\]
\[(iii) \quad \frac{d\tilde{u}}{dt}(t, s) = 2c,
\]
\[(iv) \quad \frac{d\tilde{p}}{dt}(t, s) = 0,
\]
\[(v) \quad \frac{d\tilde{q}}{dt}(t, s) = 0,
\]

that satisfies the initial conditions

\[(i) \quad \tilde{x}(0, s) = x_0(s),
\]
\[(ii) \quad \tilde{y}(0, s) = y_0(s),
\]
\[(iii) \quad \tilde{u}(0, s) = u_0(s),
\]
\[(iv) \quad \tilde{p}(0, s) = p_0(s),
\]
\[(v) \quad \tilde{q}(0, s) = q_0(s),
\]

where

\[u_0(s) = u(x_0(s), y_0(s)),
\]
\[p_0(s) = u_x(x_0(s), y_0(s)),
\]
\[q_0(s) = u_y(x_0(s), y_0(s)),
\]

and is defined on a maximal interval. It is readily verified that (4.1i) and (4.1ii) combined first with (4.1iv), (4.1v), (4.2iv), and (4.2v) and then with (4.2i), and (4.2ii) yield

\[\tilde{x}(t, s) = 2tq_0(s) + x_0(s) \quad \tilde{y}(t, s) = 2tp_0(s) + y_0(s).\]

Clearly, (4.1iii) with (4.2iii) imply

\[\tilde{u}(t, s) = 2ct + u_0(s).\]

By the fundamental property of solutions to characteristic system [24, Chapter 2, Paragraph 3] we have

\[\tilde{u}(t, s) = u(x(t, s), y(t, s)).\]
Thus, by (4.3) and (4.4), a $C^2$ surface $S_u$, being a graph of $u$, can be represented in the following parametric form

$$
\begin{pmatrix}
\tilde{x}(t, s) \\
\tilde{y}(t, s) \\
\tilde{u}(t, s)
\end{pmatrix} =
\begin{pmatrix}
x_0(s) \\
y_0(s) \\
u_0(s)
\end{pmatrix} +
t
\begin{pmatrix}
2p_0(s) \\
2q_0(s) \\
2c
\end{pmatrix}.
$$

Thus $S_u$ is a ruled surface.

Observe now that by differentiating both sides of (2.30) with respect to $x$ and then with respect to $y$, we obtain the following system of equations

$$
-uy u_{xy} = u_x u_{xx}, \quad u_x u_{yx} = -u_y u_{yy}. \tag{4.5}
$$

Multiplying the first equation by the second one and taking into account that $u$ is a $C^2$ function we deduce that $u$ satisfies the following equation

$$
u_x u_y (u_{xx} u_{yy} - u_{xy}^2) = 0. \tag{4.6}
$$

Let $(x', y')$ be an arbitrary point in $\Omega$. We now show that equation (4.6) implies that the Gaussian curvature of the graph $S_u$ of $u$ at point $(x', y', u(x', y'))$ expressed as

$$
K_u(x', y') = \frac{u_{xx}(x', y') u_{yy}(x', y') - u_{xy}^2(x', y')}{(1 + u_x^2(x', y') + u_y^2(x', y'))^2}
$$

vanishes.

To this, assume that $u_x(x', y') u_y(x', y') \neq 0$. Then clearly (4.6) yields that $K_u(x', y') = 0$. If, on the other hand, $u_x(x', y') u_y(x', y') = 0$, then as $c > 0$ only one derivative can vanish (say $u_x(x', y') = 0$). Hence by simple inspection of (4.5) we deduce that $u_{xy}(x', y') = u_{yy}(x', y') = 0$, and thus we also have $K_u(x', y') = 0$. As surface $S_u$ is ruled and its Gaussian curvature everywhere vanishes [14, Proposition 3.7.5] assures that $S_u$ is developable. □

**REMARK 8.** Note that a function $u(x, y) = xy$, whose graph constitutes a ruled surface, does not satisfy (2.30). In light of this observation and Proposition 4.1 it is clear that the uniqueness assertion (appearing in [2, 3]) about the image of the Lambertian plane

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captures too vast a class of possible solutions to (2.30). Note also that Proposition 4.1 combined with (2.31), (2.32), and (2.26) assures a wide uniqueness class for the equation (2.30) (either locally or globally).

REMARK 9. An example due to Heintze (see [14, Example 3.9.4]) shows that there are surfaces with vanishing Gaussian curvature everywhere which are not ruled and hence not developable. Thus we cannot immediately substantiate Proposition 4.1 by a mere conclusion resulting from the fact that \( S_u \) is a flat surface. In order to prove that \( S_u \) is developable (by following the above observation) one can resort to [14, Theorem 3.7.9] which establishes a sufficient condition for a flat surface \( S_u \) to be a developable surface. The author encountered some difficulties while trying to prove the fulfillment of this sufficient condition requiring a non-existence of planar points on the surface \( S_u \) (where both principal curvatures are vanishing—see [14, Definition 3.5.9]). Finally, in an attempt to prove Proposition 4.1, this initial idea has been dropped and an alternative method resorting to the “characteristic strip” approach was applied.

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