A Note on Uniqueness in Shape from Shading

Ryszard Kozera

Department of Computer Science
University of Western Australia
Nedlands, WA 6907
Australia
ryszard@cs.uwa.edu.au

Technische Universität Berlin
Fachbereich Informatik 13
Franklinstraße 28/29, D-10587 Berlin
Germany
kozera@cs.tu-berlin.de

Abstract

In this paper we analyse the problem of representing solutions of first-order partial differential equations in terms of complete integral and envelopes. In this context, we revisit the uniqueness results already existing in the shape-from-shading literature that concern eikonal equations corresponding to the images of a Lambertian hemisphere and a Lambertian plane. We show that the approach adopted by Brooks in [1, 2] is erroneous and subsequently re-establish its uniqueness claims.

1 Introduction

The shape-from-shading problem has been shown by Horn [12, Sections 10, 11] to correspond to that of solving a first-order partial differential equation. Specifically, one seeks a function $u$ (up to a constant), representing surface depth in the direction of the $z$-axis, satisfying the image irradiance equation

$$R(u_x, u_y) = E(x, y)$$

over some domain $\Omega \subset \mathbb{R}^2$. Here $R$ is a known function (the so-called reflectance map) capturing the illumination and surface reflecting conditions, $E$ is an image function formed by orthographic projection of light onto a plane parallel to the $xy$-plane, and $\Omega$ is the image domain.

An interesting case occurs, when the reflectance map is specified so as to correspond to the situation in which a distant point source illuminates a Lambertian surface. A small portion of such a surface acts as a perfect diffruser appearing equally bright from all directions. According to Lambert’s law, if a small portion of a Lambertian surface with normal direction $n = (u_x, u_y, -1)$ is illuminated by a distant point source of unit power in direction $p = (p_1, p_2, p_3)$, then, assuming that image irradiance is equal to scene radiance, the reflectance map is given by the cosine of the angle between these two directions. Thus, if $E(x, y)$ denotes the corresponding image, the image irradiance equation for the above situation takes the form

$$\langle p|n \rangle ||p||^{-1} ||n||^{-1} = E(x, y),$$

(1)

where $\langle \cdot | \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^3$, and $|| \cdot ||$ denotes the corresponding norm.

Given $0 < E(x, y) \leq 1$, the questions of the existence and uniqueness of solutions to (1) arise naturally. Existence corresponds to the problem of whether a given shading pattern with intensity between 0 and 1 is generated by a genuine Lambertian surface. Uniqueness corresponds to that of whether a shading pattern is due to one of several Lambertian shapes. Some progress in the elucidation of these issues has been made under the assumption that a light source is situated overhead. In this case $p = (0, 0, -1)$, and with $\mathcal{E}(x, y) = [E(x, y)]^{-2} - 1$, one can rewrite the image irradiance equation as the eikonal equation $u_x^2 + u_y^2 = \mathcal{E}(x, y)$. Brooks, Chojnacki, and Kozera [5, 6], Deift and Sylvester [9], and Oliensis [17] contributed important uniqueness results for this equation, whereas Brooks, Chojnacki, and Kozera [4, 6], Brooks and Chojnacki [3], Kimmel and Bruckstein [13], Rouy and Tourin [19], and Dupuis and Oliensis [11] established existence results. While all these results are far from being complete, they indicate, however, that uniqueness is rather exceptional and that existence is subject to many constraints.

Recently, rigorous uniqueness results concerning two-source and three-source photometric stereo have been obtained. As it turns out, any Lambertian surface, illuminated consecutively from two (three) linearly independent directions, can be generically uniquely (uniquely) recovered from its two (three images) (see e.g. Kozera...
In this paper we inspect critically the uniqueness results claimed by Brooks [1, 2] concerning the images of a Lambertian hemi-sphere and plane which, as we will show, are given invalid proofs. Both results resort to the incomplete Sneddon’s claim [20] about representability of any solution to a given first-order partial differential equation in terms of a complete integral and envelopes. In order to understand the flaws of Sneddon erroneous assertion we shall resort (in Section 3) to a recent Chojnacki’s result [7] formulating sufficient conditions for the representability of the solutions of the first-order partial differential equation in terms of complete integral and envelopes. Finally, in Section 4, we refer to a different approach so that the results claimed by Brooks, subject to minor reformulations, become valid statements. For a more extended version of this work the interested reader is referred to [7] or to Kozen [16].

2 Revisiting Uniqueness Results for Images of a Lambertian Hemi-sphere and a Plane

In this section, we revisit the uniqueness results (see [1, 2]) concerning the images of a Lambertian hemi-sphere and a Lambertian plane, illuminated by an overhead, distant point-light source. First, we show that both proofs are essentially incorrect as they resort to an invalid assertion of Sneddon [20] about representation of solutions to a first-order partial differential equation in terms of a complete integral and envelopes. The literature on complete integrals is a bewildering collection of incomplete (see Dou [10, Section 6]) and erroneous statements (see [20, Section 7]). A recent result by Chojnacki [7] sheds a new light on this topic and fills up an existing gap in the literature. In order to understand the invalidity of Sneddon’s claim (S) (and both uniqueness results from [1, 2]) we shall resort later to Chojnacki’s result by highlighting its main points and, in particular, by presenting its consequences for an arbitrary eikonal equation appearing in the analysis of the shape-from-shading problem.

2.1 Preliminaries

We first recall the notion of a complete integral (see e.g. [7]). For a given first-order partial differential equation

\[ F(x, y, u, u_x, u_y) = 0, \]  

defined over an open region \( \Omega \subset \mathbb{R}^2 \), a function \( G(x, y, P_1, P_2) \) of class \( C^2 \) over \( \Omega \times V \) (where \( V \) is an open region of \( \mathbb{R}^2 \)) is called a complete integral of (2) if:

(i) for each \( (P_1, P_2) \in V \), the function \( G \) is a \( C^2 \) solution to (2) on \( \Omega \),

(ii) for each \( (x, y) \in \Omega \) and for each \( (P_1, P_2) \in V \), the rank of the matrix

\[ \begin{pmatrix} G_{P_1} & G_{x_1 P_1} & G_{x_2 P_1} \\ G_{P_2} & G_{x_1 P_2} & G_{x_2 P_2} \end{pmatrix} (x, y, P_1, P_2) \]  

is equal to two.

Any graph of \( G(x, y, P_{1}^{0}, P_{2}^{0}) \), with both \( P_{1}^{0} \) and \( P_{2}^{0} \) temporarily fixed, is called a zero-parameter envelope of \( G \) (as an envelope of the zero-parameter family of surfaces \( (x, y, G(x, y, P_{1}^{0}, P_{2}^{0})) \)). Moreover, for a given \( C^1 (C^2) \) function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \), we can form a one-parameter subfamily of functions \( G(x, y, P_1, \phi(P_1)) \), and next can generate, either locally or globally (if possible), its general integral, i.e. a function

\[ u(x, y) = G(x, y, P_1(x, y), \phi(P_1(x, y))), \]  

by eliminating parameter \( P_1 \) from the following system

\[ u(x, y) = G(x, y, P_1, \phi(P_1)) \]  

and \( (G(x, y, P_1, \phi(P_1)))_1 = 0. \)  

The graph of the general integral is called a one-parameter envelope of \( G \) (as an envelope of the one-parameter family of surfaces \( (x, y, G(x, y, P_1, \phi(P_1))) \)). One can show then that under certain conditions (see e.g. [7] or [10]), formula (5) defines a new \( C^1 (C^2) \) solution to (2). By choosing different functions \( \phi \) we can obtain many distinct solutions to (2) and hence many distinct one-parameter envelopes. Furthermore, given a complete integral \( G \), we can form, either locally or globally (if possible), its singular integral, i.e. a function

\[ u(x, y) = G(x, y, P_1(x, y), P_2(x, y)), \]  

by eliminating parameters \( P_1 \) and \( P_2 \) from the following system

\[ u(x, y) = G(x, y, P_1, P_2), \]  

\( G_{P_1}(x, y, P_1, P_2) = 0, \)  

and \( G_{P_2}(x, y, P_1, P_2) = 0. \)

The graph of the singular integral is called a two-parameter envelope of \( G \) (as an envelope of the two-parameter family of surfaces \( (x, y, G(x, y, P_1, P_2)) \)). One can also show that under certain conditions (see e.g. [7] or [10]), formula (7) defines a \( C^1 (C^2) \) solution to (2). For more detailed information about sufficient conditions that assure the local existence of a \( C^1 (C^2) \) solution to (2) which graph is expressed as a one-parameter or a two-parameter envelope the interested reader is referred to [7] and [10]. We shall also shortly treat this problem in Section 3.

So far, we have recalled a method of generating new solutions to (2) based on complete integrals. In other words, given an equation (2) with complete integral \( G \) we may generate locally (or globally) \( C^1 (C^2) \) solutions to (2) (see e.g. [7] or [10]) which graphs turn out to be either one-parameter or two-parameter envelopes of \( G \). We shall now describe a converse problem. Given a complete integral \( G \) and a solution \( u \) of class \( C^1 (C^2) \) to (2), the problem is to represent the graph of \( u \) in terms of some envelope of a complete integral \( G \). There are many results in the literature addressing this issue, some of them being in fact false. One such result reads as follows (see [20, Section 7; pp. 59-61]):
When, however, one complete integral has been obtained, every other solution, including every other complete integral, appears among the solution of type (5) or (7) corresponding to the complete integral we have found."

In this paper (apart from showing the incompleteness of both uniqueness claims contained in [1, 2]) we shall also indicate that the above assertion (see Example 1) is invalid.

**Example 1** Consider the following image irradiance equation (corresponding to the image of a Lambertian plane \( u(x,y) = ax + by + c \), with \( a^2 + b^2 = 1 \), illuminated from an overhead distant point light-source direction)

\[
\frac{1}{\sqrt{1 + u_x^2(x,y) + u_y^2(x,y)}} = \frac{1}{\sqrt{2}}
\]
defined over some region \( \Omega \subset U_1 \), where \( U_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \). This equation can be rewritten in the equivalent eikonal form

\[
u_x^2(x,y) + u_y^2(x,y) = 1. \tag{8}
\]

Consider now a two-parameter \( C^2 \) class family of cones

\[
G(x,y,a,b) = \sqrt{(x-a)^2 + (y-b)^2} \tag{9}
\]
defined over \( \Omega \times V \), where \( V \subset (\mathbb{R}^2 \setminus U_1) \). It is easy to observe that, for any fixed \((a^*, b^*) \in V \), the function \( G(x,y,a^*,b^*) \) is a \( C^2 \) solution to (8) over \( \Omega \). An easy verification shows also that the rank of the matrix (3) is here equal to two. Thus the formula (9) defines a complete integral for the eikonal equation (8).

We shall show now that the function \( v(x,y) = x + 2 \), being a \( C^2 \) solution to (8), cannot be represented as a general integral of (9) expressed in the form \( v(x,y) = G(x,y,a(x,y),b(a(x,y))) \). Suppose the contrary. Then, for some \( a \to b(a) \) and for some \((x,y) \to a(x,y)\), we have

\[
\begin{align*}
(x - a)^2 + (y - b(a))^2 &= (x + 2)^2, \\
(x - a) + (y - b(a))b'(a) &= 0. \tag{10}
\end{align*}
\]

Differentiating (10) with respect to \( y \) we get

\[
[(x - a) + (y - b(a))b'(a)] \frac{\partial b}{\partial y} + b(a) - y = 0.
\]

Hence, in view of (11), \( y = b(a) \) and further, still by (11), \( x = a \). Finally, \( y = b(x) \), which is absurd. On the other hand, it should be noted that the graph of the function \( v \) is a one-parameter envelope of \( G(x,y,a(b),b) \) with \( a(b) \equiv -2 \). Upon analyzing the above case we may come to the following critical conclusion:

*the definition of a one-parameter envelope, specified by (5) needs to be treated in a symmetric manner if Sneddon’s assertion (S) is to be correct.*

Analogously, the graph of the function \( v_1(x,y) = x \) cannot be represented as a one-parameter envelope of any subfamily of (9) expressed in the form \( G(x,y,a,b(a)) \). On the other hand, if the graph of \( v_1 \) happens to be a one-parameter envelope of \( G(x,y,a(b),b) \), we obtain

\[
\begin{align*}
(x - a(b))^2 + (y - b)^2 &= x^2, \tag{12}
\end{align*}
\]

\[
(x - a(b))a'(b) + (y - b) = 0. \tag{13}
\]

By differentiating (12) with respect to \( y \), we get

\[
[(x - a(b))a'(b) + (y - b)] \frac{\partial b}{\partial y} + b - y = 0.
\]

Hence, in view of (13), \( b(x,y) = y \), and further, by (12), we have \( (x - a(b))^2 = x^2 \). The latter one is only possible for \( a(b) = 0 \) or \( a(b) = 2x \). The first case is impossible as \((a(b), b) \notin (\mathbb{R}^2 \setminus U_1) \). So is the second, as then \( 2x = a(y) \), which is absurd. A straightforward verification of conditions (7) shows, that \( v_1 \) cannot be represented as a singular integral of (8) either.

Thus we have arrived at the conclusion that Sneddon’s assertion (S) cannot be universally true.

Summing up observations made so far, it is clear that we meet serious difficulties in treating claim (S) as a valid assertion. The Example 1 explicitly shows that some of the solutions may also be represented in terms of a symmetric uniform parametrisation \( G(x,y,a(b(x,y)), b(x,y)) \), whereas the other cannot be expressed in terms of a uniform parametrisation either of the form \( G(x,y,a(b(x,y)), b(x,y)) \) or \( G(x,y, a(x,y), b(a(x,y))) \).

### 2.2 Uniqueness Results for a Lambertian Hemi-sphere and a Plane Revisited

We shall now shortly outline Brooks’s uniqueness proofs for images of a Lambertian hemi-sphere and a Lambertian plane (see [1, 2]). As both rely on Sneddon’s erroneous assertion (S) they remain invalid. We begin by quoting the precise statements of the uniqueness results (contained in [1, 2]) which are given flawed proofs.

**Theorem 2.1** Consider an image of the Lambertian hemi-sphere \( u_S(x,y) = \sqrt{1 - x^2 - y^2} \) illuminated from an overhead distant point light-source direction. Then the functions \( u(x,y) = \pm u_S(x,y) + C \), are the only solutions to the corresponding image irradiance equation

\[
\frac{1}{\sqrt{1 + u_x^2 + u_y^2}} = \sqrt{1 - x^2 - y^2} \tag{14}
\]

defined over \( \Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \).

**Theorem 2.2** Consider an image of the Lambertian plane \( u(x,y) = ax + by + C \) illuminated from an overhead
distant point light-source direction. Then any solution to the corresponding image irradiance equation

$$
\frac{1}{\sqrt{1+u_x^2+u_y^2}} = \frac{1}{\sqrt{1+a^2+b^2}}
$$

(15)
is a ruled surface.

In order to substantiate the first claim, Brooks adopts (in [1, 2]) the following proving pattern (for a more detailed discussion see also [16]):

(i) first, to generate a complete integral to (14),

(ii) next, to generate all of its one-parameter and two-parameter envelopes (introduced in (5) and (7)) and to show that they are not smooth over \( \Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\} \),

(iii) finally, by using (ii) combined with the assertion (S), to claim a uniqueness result for (14) over \( \Omega \).

A similar approach is also adopted in [1, 2] for proving a corresponding uniqueness result for the image of a Lambertian plane. Clearly, due to the intrinsic invalidity of Sneddon’s claim (S), we cannot assume the correctness of the proofs for both Theorems 2.1 and 2.2.

3 Applications of Complete Integrals in Representing Solutions

In this section, we tackle the problem of finding sufficient conditions for representing any solution to (2) as an appropriate envelope of a complete integral. Some of the existing results point out difficulties in generating such statements (see e.g. Courant and Hilbert [8, Chapter 1, Paragraph 4; pp. 25]) whereas others contain erroneous assertions (see e.g. [20, Chapter 7: pp. 59-61]). Recently, a paper by Chojnacki [7] rectified the flawed statements and filled an existing gap in the literature. We shall use here Chojnacki’s result and rediscuss its meaning in the context of the erroneous Sneddon’s assertion (S) (for more detailed analysis see also [16]).

3.1 Representation of the Solutions in Terms of Complete Integral and Envelopes

We begin by quoting the main result of Chojnacki [7] (when \( n = 2 \)):

**Theorem 3.1** Let \( G(x,y,P_1,P_2) \) be a complete integral of the equation (2) defined over some region \( N \subset M \times P \), where \( M, P \subset \mathbb{R}^2 \), and let \( n^0 = (x^0,y^0,P_1^0,P_2^0) \in \mathbb{R}^4 \) and \((y^0,q^0) \in \mathbb{R}^2\) be such that

$$
(dF)(x^0,y^0,u_0^0,p_0^0,q_0^0) \neq 0,
$$

(16)

$$
(x^0,y^0,G(n^0),G_x(n^0),G_y(n^0)) = (x^0,y^0,u^0,p^0,q^0),
$$

(17)


If \( u \) is a \( C^k (k = 1, 2) \) solution to (2) on an open neighbourhood of \( \Omega \subset M \) of \((x^0,y^0)\), with

$$
(x^0,y^0,u(x^0,y^0),u_x(x^0,y^0),u_y(x^0,y^0)) = (x^0,y^0,u_0^0,p_0^0,q_0^0)
$$

(18)

then there exists an open neighbourhood \( U \subset \Omega \) of \((x^0,y^0)\) and a \( C^{k-1} \) function \( f = (f_1,f_2) \) from \( U \) into \( P \) such that

$$
f(x^0,y^0) = (P_1^0,P_2^0)
$$

(19)

and, for each \((x,y) \in U\),

$$
(x,y,u(x,y),u_x(x,y),u_y(x,y)) = (x,y,G(x,y,f_1(x,y),f_2(x,y))),
$$

(20)

$$
G_x(x,y,f_1(x,y),f_2(x,y)), G_y(x,y,f_1(x,y),f_2(x,y)).
$$

Moreover, if, for some open neighbourhood \( V \subset M \) of \((x^0,y^0)\), a continuous function \( g \) from \( V \) into \( P \) satisfies (19) and (20), then \( g \) and \( f \) coincide on an open neighbourhood \( V' \subset V \cup U \) of \((x^0,y^0)\).

**Remark 1.** Note that in light of the above theorem the notion of an envelope (not necessarily a one-parameter or a two-parameter one) can be introduced in a different and broader way (see also [7] or [16]). Namely, assume that \( u \) is a \( C^1 (C^2) \) function. Given a family of \( C^2 \) functions \( G(x,y,P_1,P_2) \) defined over \( \Omega \subset M \times P \), then if there exists a \( C^1 (C^2) \) function \( f: \Omega \to P \) such that for any \((x,y) \in \Omega \) equation (20) is satisfied then we say that the graph of \( u \) is a \( C^1 (C^2) \) envelope of \( G \). Furthermore, if \( f(\Omega) \) is a \( k \)-dimensional \((k = 0, 1, 2)\), \( C^1 (C^2) \) submanifold of \( P \), then we say that the graph of \( u \) is a \( k \)-parameter envelope of \( G \) with selector \( f \).

It is easy to observe that the last theorem (see [7] or [16]) reduces the problem of finding \( C^2 \) solutions to (2) to that of solving the following system of quasi-linear equations with respect to the unknown \( C^1 \) function \( f = (f_1,f_2) \):

$$
\frac{\partial G}{\partial P_1}(x,y,f_1(x,y),f_2(x,y)) \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial G}{\partial P_2}(x,y,f_1(x,y),f_2(x,y)) \frac{\partial f_2}{\partial x}(x,y) = 0,
$$

(21)

$$
\frac{\partial G}{\partial P_1}(x,y,f_1(x,y),f_2(x,y)) \frac{\partial f_1}{\partial y}(x,y) + \frac{\partial G}{\partial P_2}(x,y,f_1(x,y),f_2(x,y)) \frac{\partial f_2}{\partial y}(x,y) = 0.
$$

Conversely, if \( f \) is a \( C^k (k = 1, 2) \) function satisfying (21) then \( u \) given by \( u(x,y) = G(x,y,f_1(x,y),f_2(x,y)) \) is a \( C^k \) solution to (2). There are cases when the function \( f \) can be easily handled. For example, when \( f \) has constant rank at each point in \((x,y) \in U \) (i.e. \( \text{rank}(df(x,y)) = k_1 \) where \( k_1 = 0, 1, 2 \)). Note also that if \( \text{rank}(df) \equiv \text{const} \), then \( f(U) \) is a \( k_1 \)-dimensional \( C^k \) submanifold of \( P \) (the graph of \( u \) is
an envelope of the $k_1$-parameter family of surfaces $(x, y, G(x, y, f_1(x, y), f_2(x, y)))$. To validate the above observation, as well as to understand better the incompleteness of Sneddon’s assertion $(S)$, we shall now briefly look at these three particular cases (for more extended and detailed analysis an interested reader is referred to [7] or [16]). Retaining the notation from the Theorem 3.1, assume that $u$ is a $C^2$ solution to (2) on an open neighbourhood of $(x_0, y_0) \in U$ such that (19) and (20) hold for some $C^1$ function.

**Case 1.** Assume that, for all $(x, y) \in U$, $\text{rank}(df(x, y)) = 0$ ($k_1 = 0$). Then, clearly $f$ is a constant function and so $f(x, y) = (f_1(x, y), f_2(x, y)) = (P_1(x, y), P_2(x, y))$ implies $u(x, y) = G(x, y, P_1, P_2)$. We thus, obtain a solution (which graph constitutes a zero-parameter envelope of $G$), of class $C^2$, which is one of the elements of the initial family of functions constituting the complete integral.

**Case 2.** Assume that, for all $(x, y) \in U$, $\text{rank}(df(x, y)) = 1$ ($k_1 = 1$). By shrinking $U$ if necessary, one can show (see [7] or [16]) that, for some smooth function $\Phi$ (or $\Psi$), $u$ satisfies the system (5) defining a one-parameter envelope (or a symmetric system, with $u(x, y) = G(x, y, \Phi(P_2(x, y)), P_2(x, y)))$. Note that $f(U)$ is here a one dimensional $C^1$ submanifold of $\mathcal{P}$ which can be parametrised as $(t, \Phi(t)) \mapsto ((\Psi(t), t))$. Note also that, as $\text{rank}(df) = 1$ over $U$ (by shrinking it if necessary), the rank theorem implies that, over $U$, we have a “uniform” parametrisation of $(f_1(x, y), f_2(x, y))$ i.e. either as $(t, \Phi(t))$ or as $(\Psi(t), t)$. It should be emphasised now, that two important factors have to be born in mind. First, that $\text{rank}(df(x, y)) = 1$, for each $(x, y) \in \Omega$, does not assure a “global uniform parametrisation” over $\Omega$. It may happen that there is a “flip” in parametrisation. Secondly, that even locally it is possible to represent $(f_1(x, y), f_2(x, y))$ by using either both symmetric forms $(t, \Phi(t))$ and $(\Psi(t), t)$ or only one of them (see [16]).

**Case 3.** Assume that, for all $(x, y) \in U$, $\text{rank}(df(x, y)) = 2$ ($k_1 = 2$). Clearly, $f(U)$ is then a two dimensional $C^1$ submanifold (an open subset of $\mathcal{P}$) and hence can be parametrised by two independent parameters. One can also show (see [7] or [16]) that the system (7), defining a singular integral (which graph constitutes a two-parameter envelope of $G$), is in this case also satisfied.

### 3.2 Final Remarks and Conclusions

We shall now briefly analyse the main flaws resulting from Sneddon’s assertion $(S)$ (also reappearing in both Brooks’ uniqueness claims in [1, 2]).

(3i) It may happen that (even locally), for any open neighbourhood of $(x_0, y_0)$, the $\text{rank}(df)$ at a point $(x, y) \in U$ varies as $(x, y)$ runs over $U$ and so $f(U)$ is not a $C^k$ ($k = 1, 2$) submanifold of $\mathcal{P}$. This leads to a flaw of Sneddon’s statement $(S)$ covering only the case with constant $\text{rank}(df(x, y))$ over $U$ (we assume that the system (5) is treated here symmetrically). There can be solutions for which $\text{rank}(df)$ varies and none of Sneddon’s representations is valid (see [7] or [16]). Interestingly enough, the case of changing, locally, the $\text{rank}(df)$ from 2 to 1 or from 2 to 0 is impossible (see [7]). Thus the $\text{rank}(df)$ (if non-constant) can only vary here between 0 and 1 as $(x, y)$ runs over $U$. Clearly in such a case we may have a non-uniform parametrisation. As mentioned in the Case 2, a “flip” is also possible for $\text{rank}(df) \equiv 1$. Clearly, none of the above cases is captured by Sneddon’s claim $(S)$.

(3ii) As the Theorem 3.1 is of local character care must be taken with complete integrals applied globally. Note also that for an eikonal equation condition (16) is not satisfied at any singular point. In such a case, the Theorem 3.1 is of no applicability whatsoever. Thus, any analysis of representability of solutions to the eikonal equation in terms of complete integrals, in any neighbourhood of singular point, requires a separate treatment. These difficulties were overlooked in Brooks’ proof of the Theorem 2.1, where the global analysis over a region $\Omega$ containing a singular point was performed.

(3iii) It may also happen that, given a complete integral $G$ of (2), an “initial germ” (for generating locally a new solution $u$ to (2) which graph is an envelope of $G$) may not satisfy conditions (17), (18), and (19). This is closely related to the range of choices of parameters $(P_1, P_2)$ which may be too restrictive to define a family of functions for which each graph “touches a graph of $u$” so that conditions (17), (18), and (19) are fulfilled (for a pertinent example see [16]). This can be another source of “losing solutions” while trying to treat Sneddon’s assertion $(S)$ as a valid statement.

(3iv) Note finally, that as the analysis of Case 1, 2, and 3 reveals, a general definition of an envelope (introduced in Remark 1) subsumes the three categories of envelopes considered by Sneddon. Additionally, as shown in [7] or [16], this definition has essentially broader meaning as the definition adopted by Sneddon. Furthermore, note that, as the Theorem 3.1 assures, this more general definition of an envelope together with conditions (16), (17), and (18), form a constrained set of sufficient conditions enough to generate a positive result about representability of the graph of any solution to a given first-order partial differential equation in terms of complete integral and envelopes.

A more interested reader is referred here to [7] and [16], where a number of illustrative examples pointing out the above difficulties are presented.

### 4 Uniqueness for Images of Lambertian Hemi-sphere and Plane

A natural question arises as to whether both uniqueness assertions from the Theorems 2.1 and 2.2 (given
incorrect proofs in [1, 2]) are in fact true.

As already mentioned, uniqueness for the equation (14) (corresponding to the image of a Lambertian hemisphere) has been proved by Deift and Sylvester [9], who showed that \( \pm(1 - x^2 - y^2)^{1/2} + C \) are the only \( C^2 \) solutions to this equation over the unit disc \( D(1) = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \). Interestingly, Deift and Sylvester also showed that this result fails in class of \( C^1 \) solutions (there exist infinitely many \( C^1 \) functions over \( D(1) \) satisfying (14)). An interested reader is also referred to [5, 6], where a \( C^1(D(1)) \cap C^2(D(1)) \setminus \{(0, 0)\} \) saddle-like solution to (14) is constructed.

As for the image of a Lambertian plane the following uniqueness result can be established:

**Theorem 4.1** The graph of any \( C^2 \) solution to the image irradiance equation (15) defined over some region \( \Omega \subset \mathbb{R}^2 \), with \( c = a^2 + b^2 \geq 0 \), is a developable surface and so its Gaussian curvature vanishes.

For a full proof, which resorts to the standard method of characteristic strips, the interested reader is referred to [16].

**Acknowledgements**

I should like to thank Prof. W. Chojnacki for his help in preparation of this paper. Part of this research was conducted under The University of Western Australia Research Launching Grant and under Alexander von Humboldt Foundation Research Reward.

**References**


