ANALYTIC CONTINUATION OF THE RESOLVENT OF THE LAPLACIAN AND THE DYNAMICAL ZETA FUNCTION

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Abstract. Let $s_0 < 0$ be the abscissa of absolute convergence of the dynamical zeta function $Z(s)$ for several disjoint strictly convex compact obstacles $K_i \subset \mathbb{R}^N$, $i = 1, \ldots, k_0$, $k_0 \geq 3$, and let $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1} \chi$, $\chi \in C_0^\infty(\mathbb{R}^N)$, be the cut-off resolvent of the Dirichlet Laplacian $-\Delta_D$ in $\Omega \setminus \cup_{i=1}^{k_0} K_i$. We prove that there exists $\sigma_1 < s_0$ such that $Z(s)$ is analytic for $\Re(s) \geq \sigma_1$ and the cut-off resolvent $R_\chi(z)$ has an analytic continuation for $\Im(z) < -i\sigma_1$, $|\Re(z)| \geq C > 0$.

1. Introduction

Let $K$ be a subset of $\mathbb{R}^N$ ($N \geq 2$) of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where $K_i$ are compact strictly convex disjoint domains in $\mathbb{R}^N$ with $C^\infty$ boundaries $\Gamma_i = \partial K_i$ and $k_0 \geq 3$. Set $\Omega = \mathbb{R}^N \setminus \bigcup_{i=1}^{k_0} K_i$ and $\Gamma = \partial K$. We assume that $K$ satisfies the following (no-eclipse) condition:

(H) \[ \text{for every pair } K_i, K_j \text{ of different connected components of } K \text{ the convex hull of } K_i \cup K_j \text{ has no common points with any other connected component of } K. \]

With this condition, the billiard flow $\phi_t$ defined on the cosphere bundle $S^*(\Omega)$ in the standard way is called an open billiard flow. It has singularities, however its restriction to the non-wandering set $\Lambda$ has only simple discontinuities at reflection points. Moreover, $\Lambda$ is compact, $\phi_t$ is hyperbolic and transitive on $\Lambda$, and it follows from [St1] that $\phi_t$ is non-lattice and therefore by a result of Bowen [Bo], it is topologically weak-mixing on $\Lambda$.

Given a periodic reflecting ray $\gamma \subset \Omega$ with $m_\gamma$ reflections, denote by $d_\gamma$ the period (return time) of $\gamma$, by $T_\gamma$ the primitive period (length) of $\gamma$ and by $P_\gamma$ the linear Poincaré map associated to $\gamma$. Denote by $P$ the set of all periodic rays in $\Omega$ and let $\lambda_i, \gamma, i = 1, \ldots, N - 1$, be the eigenvalues of $P_\gamma$ with $|\lambda_i, \gamma| > 1$ (see [PS1]).

Let $P$ be the set of primitive periodic rays. Set

\[ \delta_\gamma = -\frac{1}{2} \log(\lambda_1, \gamma \ldots \lambda_{N-1}, \gamma), \quad \gamma \in P, \]

\[ r_\gamma = \begin{cases} 0 & \text{if } m_\gamma \text{ is even,} \\ 1 & \text{if } m_\gamma \text{ is odd} \end{cases} \]

and consider the dynamical zeta function

\[ Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in P} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}. \]

It is easy to show that there exists $s_0 \in \mathbb{R}$ such that for $\Re(s) > s_0$ the series $Z(s)$ is absolutely convergent and $s_0$ is minimal with this property. The number $s_0$ is called abscissa of absolute...
convergence. On the other hand, using symbolic dynamics and the results of [PP], we deduce that $Z(s)$ is meromorphic for $\Re(s) > s_0 - a$, $a > 0$ (see [14]). According to some recent results ([St2] for $N = 2$, [St3] for $N \geq 3$) there exists $0 < \epsilon < a$ so that the dynamical zeta function $Z(s)$ admits an analytic continuation for $\Re(s) \geq s_0 - \epsilon$.

For $\Im(z) < 0$ consider the cut-off resolvent

$$R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi : L^2(\Omega) \rightarrow L^2(\Omega),$$

where $\chi \in C^\infty_0(\mathbb{R}^N)$, $\chi = 1$ on $K$ and $\Delta_D$ is the Dirichlet Laplacian in $\Omega$. The cut-off resolvent $R_\chi(z)$ has a meromorphic continuation in $\mathbb{C}$ for $N$ odd with poles $z_j$ such that $\Im(z_j) > 0$ and in $\mathbb{C} \setminus \{i\mathbb{R}^+\}$ for $n$ even. The analytic properties and the estimates of $R_\chi(z)$ play a crucial role in many problems related to the local energy decay, distribution of the resonances etc. In the physical literature and in many works concerning the numerical calculation of the resonances (see [CE], [W], [L], [LZ], [LSZ]) the following conjecture is often made.

Conjecture: The poles $\mu_j$ (with $\Re(\mu_j) < 0$) of $Z(s)$ and the poles $(-i\mu_j)$ of $R_\chi(z)$ are related by $z_j = -i\mu_j$.

At least one would expect that the poles $z_j$ of $R_\chi(z)$ lie in sufficiently small neighborhoods of $-i\mu_j$. Presumably for this reason the numbers $-i\mu_j$ are called pseudo-poles of $R_\chi(z)$.

The case of several disjoint disks has been treated in many works (see [W] for a comprehensive list of references), and a certain method for numerical computation of the resonances has been used. Although it is not rigorously known whether the numerically found resonances approximate the (true) resonances in the exterior of the discs, and whether the dynamical zeta function has an analytic continuation to the left of the line of absolute convergence, this way of computation is widely accepted in the physical literature.

In the case of two strictly convex disjoint domains it was proved ([I1], [G]) that the poles of $R_\chi(\lambda)$ are in a small neighborhoods of the pseudo-poles

$$m\pi d + i\alpha_k, \ m \in \mathbb{Z}, k \in \mathbb{N}.$$  

Here $d > 0$ is the distance between the obstacles and $\alpha_k > 0$ are determined by the eigenvalues $\lambda_j$ of the Poincaré map related to the unique primitive periodic ray.

It is known that the above conjecture is true for convex co-compact hyperbolic manifolds $X = \Gamma \setminus \mathbb{H}^{n+1}$, where $\Gamma$ is a discrete group of isometries with only hyperbolic elements admitting finite fundamental domain (then $X$ is a manifold of constant negative curvature). More precisely, the zeros of the corresponding Selberg’s zeta function coincide with the poles (resonances) of the Laplacian $\Delta_g$ on $X$ [PPe]. In particular, for modular surfaces the scattering operator $S(s)$ and the Riemann zeta function $\zeta(s)$ are related by the formula (see [CZ])

$$S(s) = \sqrt{-\pi} \frac{\Gamma(s - 1/2) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)}.$$

The analysis of the case of several convex obstacles is generally speaking much more complicated. However the case $s_0 > 0$ is much easier, since we know that for $-is_0 \leq \Im(z) \leq 0$ the cut-off resolvent $R_\chi(z)$ is analytic (see [16]). In the following we assume that $s_0 < 0$. The first problem
is to examine the link between the analyticity of $Z(s)$ for $\Re(s) > s_0$ and the behavior of $R_\chi(z)$ for $0 \leq \Im(z) < -is_0$. In this direction Ikawa established the following

**Theorem 1.** ([I3]) Assume $s_0 < 0$. Then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ so that the cut-off resolvent $R_\chi(z)$ is analytic for

$$\Im(z) < -i(s_0 + \epsilon), \quad |\Re(s)| \geq C_\epsilon.$$  

A similar result for a control problem has been established by Burq [B]. The proofs in [I3] and [B] are based on the construction of an asymptotic solution $U_M(x, s; k)$ with boundary data

$m(x, k) = e^{ik\varphi(x)}h(x), k \in \mathbb{R}, k \geq 1$, where $\varphi$ is a phase function ($\|
abla \varphi\| = 1$) and $h \in C^\infty(\Gamma)$ has a small support. More precisely, $U_M(., s; k)$ is $C^\infty(\Omega)$-valued holomorphic function in

$$D_0 = \{ s \in \mathbb{C} : \Re(s) > s_0 \},$$  

and we have

$$\begin{align*}
(\Delta - s^2)U_M(., s; k) &= 0 \text{ for } x \in \Omega^0, \Re(s) > s_0, \quad (1.1) \\
U_M(., s; k) &\in L^2(\Omega^0) \text{ if } \Re(s) > 0, \quad (1.2) \\
U_M(x, s; k) &= m(x, k) + r_M(x, s; k) \text{ on } \Gamma, \Re(s) > s_0, \quad (1.3)
\end{align*}$$

where, for $r_M(x, s; k)$ and $s \in D_0$, $|s + i\kappa| \leq 1$, we have the estimates

$$\|r_M(., s; k)\|_{C^p(\Gamma)} \leq C_p k^{-M+p} \left( \|\nabla \varphi\|_{C^{M+2+p}(\Gamma)} + 1 \right) \|h\|_{C^{M+2+p}(\Gamma)}, \forall p \in \mathbb{N}. \quad (1.4)$$

The function $U_M(x, s; k)$ is given by a finite sum of series having the form

$$\sum_{n=0}^{\infty} \sum_{|j|=n+3} \sum_{i=1}^{M} \sum_{q=0}^{2q} a_{j,*,*,*}(x, s; k)(s + i\kappa)^r (i\kappa)^q, \quad (1.5)$$

where $j = (j_0, \ldots, j_{n+2})$ are configurations of length $|j| = n + 3$, $\varphi_j(x)$ are phase functions and the amplitudes $a_{j,*,*,*}(x, s; k)$ depend on $s \in \mathbb{C}$ and $k \in \mathbb{R}$ (see Sections 4 and 6 for the notation and more details). The main difficulty is to establish the summability of these series and to obtain suitable $C^p$ estimates of their traces on $\Gamma$ for $\Re(s) > s_0$. The absolute convergence of $Z(s)$ makes it possible to study the absolute convergence of these series and to get crude estimates which lead to the properties (1.1)-(1-4) above. The dynamical zeta function $Z(s)$ is related to the periods of periodic rays and to the corresponding Poincaré maps and formally from $Z(s)$ we get no information about the dynamics of the rays in a neighborhood of the non-wandering set. On the other hand, the absolute convergence of $Z(s)$ is a strong condition which enables us to justify the existence of $U_M(x, s; k)$ establishing the absolute convergence of (1.5).

Passing to the case $\Re(s) < s_0$, it is an interesting problem to examine the link between the analytic continuation of $R_\chi(z)$ for $\Im(z) \geq -is_0$ and that of the dynamical zeta function $Z(s)$. Several years ago, Ikawa [15] announced a result concerning an analytic continuation of $R_\chi(z)$ in a domain $-i\mathcal{D}_{\epsilon,\alpha}$, where

$$\mathcal{D}_{\epsilon,\alpha} = \{ s \in \mathbb{C} : \Re(s) > s_0 - |\Im(s)|^{-\alpha}, \Im(z) \geq C_\epsilon, 0 < \alpha < 1 \}$$

assuming the following conditions:

(i) $Z(s)$ is analytic in $\mathcal{D}_{\epsilon,\alpha}$ and

$$|Z(s)| \leq C|s|^{1-\epsilon}, 0 < \epsilon < 1, \ s \in \mathcal{D}_{\epsilon,\alpha}, \quad (1.6)$$
(ii) if $w(\eta) > 0$ is an eigenfunction of the Ruelle operator $L_{s_0}$ (see Section 4 for the notations $\Sigma^+_A, L_{s_0}, \tilde{r}(\xi,s)$) with eigenvalue 1, then the constants

$$M = \max_{\xi \in \Sigma^+_A} \frac{w(\xi)}{w(\eta)} , \quad m = \min_{\xi \in \Sigma^+_A} |e^{\tilde{r}(\xi,s_0)}|$$

satisfy the inequality $\frac{M}{m} \sqrt{\theta} < 1$ with a global constant $0 < \theta < 1$ depending on the dispersing properties of the billiard flow (see [I3], [I4], [St1])

Ikawa announced in [I5] that (ii) holds in the case of three balls centered at the vertices of an equilateral triangle, provided the radii of the balls are sufficiently small. In general the condition (ii) is rather restrictive. On the other hand, it is difficult to check the condition (i) if we have no precise information about the spectral properties of $L_s$ for $\Re(s)$ close to $s_0$. To our best knowledge the above result of Ikawa has not been published anywhere.

Using the spectral properties of the Ruelle operator $L_s = L_{-sf+\tilde{g}}$ introduced in Section 4, our main result in this paper is the following

**Theorem 2.** Let $s_0 < 0$. Suppose that the operator $L_s$ satisfies the estimates (4.3). Then there exist $\sigma_1 < s_0$ and $C > 0$ such that $Z(s)$ is analytic for $\Re(s) \geq \sigma_1$ and the cut-off resolvent $R_\chi(z)$ is analytic for

$$\Im(z) < -i\sigma_1, \quad |\Re(s)| \geq C .$$

The estimate (4.3) was established in [St2] for $N = 2$ and under some pinching conditions in [St3] for $N \geq 3$. We expect that these Dolgopyat type estimates are true without any additional conditions for $N \geq 3$. We should mention that our result implies an analytic continuation of $R_\chi(z)$ in a strip $\sigma_1 \leq \Re(s) \leq s_0$, $|\Re(s)| > C$, and we have no restrictions on the eigenfunction $w(\eta)$ nor on the behavior of $Z(s)$ for $\sigma_1 \leq \Re(s) \leq s_0$. The proof of Theorem 2 is long and technical. Below we discuss the main points.

As in [I3], [I5] the idea is to construct an approximative solution $U_M(x,s;k)$ which is analytic for $\sigma_1 \leq \Re(s) \leq s_0$, $|s + ik| \leq 1$, $k \geq 1$ so that $U_M(x,s;k)$ satisfies the conditions (1.1) - (1.3) with estimates for $r_M(x,s;k)$ having higher order. The first approximation of $U_M(x,s;k)$ becomes a finite sum over $l = 1, \ldots, k_0$ of series

$$w_{0,l}(x,-is) = \sum_{n=0}^{\infty} \sum_{|j| = n+3, j_{n+2} = l} (-1)^{n+2} e^{-s\varphi_j}(x) a_j(x) = \sum_{n=0}^{\infty} U_{n+2,l}(x,-is),$$

where $j = (j_0, \ldots, j_n, j_{n+2})$ are configurations of length $|j| = n + 3$, $\varphi_j(x)$ are phase functions and $a_j(x)$ are amplitudes determined by a recurrent procedure. The crucial point is to obtain an analytic continuation of $w_{0,l}(x,s;k)$ from $\Re(s) > s_0$ to a strip $\sigma_1 \leq \Re(s) \leq s_0$ with $\sigma_1 < s_0$. To do this, the strategy is to establish suitable estimates for $U_{n+2,l}(x,-is)$ and to apply a summation by packages. The structure of $U_{n+2,l}$ is complicated since the phases $\varphi_j(x)$ and the amplitudes $a_j(x)$ are related to the dynamics of the reflecting rays having $|j|$ reflexions and issued from the unstable manifold $\{(x, \nabla \varphi(x)), x \in \supp(h)\}$. It seems unlikely that an explicit relationship exists between $U_{n+2,l}(x,-is)$ and the iterations $L_{-sf+\tilde{g}}^{n}$ of the Ruelle operator $L_{-sf+\tilde{g}}$ (see Section 4). Consequently, one would not expect such relationship between $\sum_{n=0}^{\infty} U_{n+2,l}(x,-is)$ and the zeta function $Z(s)$. Thus, it appears the situation considered here is rather different to the case of compact co-convex surfaces where one knows that the singularities of the Selberg zeta function coincide
with the singularities of the corresponding Poincaré series which in turn is related to the resolvent of the Laplacian [PPe].

It was observed by Ikawa [I5] that \( U_{n+2,1}(x, -is) \) can be compared with
\[
L^n_{-sf+\tilde{g}} \mathcal{M}_{n,s}(x) \mathcal{G}_s v_0(\xi),
\]
where \( \mathcal{M}_{n,s}(x) \) and \( \mathcal{G}_s \) are suitable operators defined by means of rays issued from appropriate unstable or stable manifolds, while \( v_0(\xi) \) is a function related to the boundary data \( m(x, s) = e^{-s\varphi(x)}h(x) \). The precise definitions (slightly different from these in [I5]) are given in Section 4.

The first main step in the proof of the main result is Theorem 3 in Sections 4-5 below which provides an estimate of the form
\[
|L^n_{-sf+\tilde{g}} \mathcal{M}_{n,s}(x) \mathcal{G}_s v_0(\xi) - U_{n+2,1}(x, -is)|_{C^p(\Gamma)} \leq C_p(s, \varphi, h)(\theta + ca)^n, \forall p \in \mathbb{N}, \forall n \in \mathbb{N},
\]
where \( a = s_0 - \Re(s) \) and \( 0 < \theta < 1 \), \( C_p > 0 \) are global constants. The part of Theorem 3 corresponding to \( p = 0 \) has been announced by Ikawa [I5] under the restrictions (i) and (ii) mentioned above. The proof of Theorem 3 is rather long and technical, however we consider it in detail since it is of fundamental importance for the considerations later on. Section 4 contains the proof of the case \( p = 0 \), while Section 5 deals with \( p \geq 1 \).

In Section 6 we obtain estimates for the traces \( w_{0,1}(x, -is) \) applying Theorem 3 and estimating the term \( |L^n_{-sf+\tilde{g}} \mathcal{M}_{n,s}(x) \mathcal{G}_s v_0(\xi)| \). Here the Dolgopyat type estimates (4.3) for the iterations \( L^n_{-sf+\tilde{g}} \) play a crucial role and we can justify the analyticity of \( w_{0,1}(x, -is) \) for \( \Re(s) \geq \sigma_0 \) with \( \sigma_0 < s_0 \). The estimates for \( w_{0,1}(x, -is) \) for \( \sigma_0 \leq \Re(s) \leq s_0 \) are different from those in the domain of absolute convergence \( \Re(s) > s_0 \) and we must deal with two problems related to the analyticity of our construction and to the control of the corresponding \( C^p(\Gamma) \) norms. The consequence of this procedure reflects on the first approximation \( W^{(0)}(x, -is; k) \) satisfying the conditions:
\[
\begin{align*}
(\Delta_x - s^2)W^{(0)}(x, -is; k) &= 0, \quad x \in \tilde{\Omega}, \sigma_1 \leq \Re(s) \leq 1, \\
W^{(0)}(x, -is; k) &\in L^2(\tilde{\Omega}) \text{ for } \Re(s) > 0, \\
W^{(0)}(x, -is; k) - m(x, s) &= (ik)^{-1}R_0(x, s; k) \text{ on } \Gamma, \sigma_1 \leq \Re(s) \leq 1
\end{align*}
\]
with \( \sigma_0 < \sigma_1 < s_0 \). Choosing \( \sigma_1 \) close to \( s_0 \), we arrange the estimates
\[
||R_0(x, s; k)||_{C^p(\Gamma)} \leq C_p k^{p+\beta}, \quad 0 < \beta < 1, \forall p \in \mathbb{N}.
\]
 Finite lower order approximations are examined in Section 8 where similar estimates are obtained. The final step of our argument is in Section 9 where we solve an integral equation on the boundary. To do this, we invert in \( L^2(\Gamma) \) an operator \( I + L(s; k) \), where \( L(s; k) \) has a small \( L^2(\Gamma) \) norm for \( k \geq k_0 \). It is important to note that we need to construct only a finite number \( M > (N - 1)/2 \) lower order approximations.

2. Preliminaries

This section contains some basic facts about the dynamics of the billiard flow in the exterior \( \Omega \) of \( K \). Our main reference is [I3]; see also [B] and [PS1]. The notation follows mainly [I3].
Here and in the rest of the paper we assume that $K$ is as in Sect. 1. Denote by $A$ the $k_0 \times k_0$ matrix with entries $A(i, j) = 1$ if $i \neq j$ and $A(i, i) = 0$ for all $i$, and set

$$
\Sigma_A = \{ (\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0, \eta_1, \ldots, \eta_m, \ldots) : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z} \},
$$

$$
\Sigma_A^+ = \{ (\eta_0, \eta_1, \ldots, \eta_m, \ldots) : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \geq 0 \},
$$

$$
\Sigma_A^- = \{ (\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0) : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_{j-1} \neq \eta_j \text{ for all } j \leq 0 \}.
$$

Let $\text{pr}_1 : S^*(\Omega) = \Omega \times \mathbb{S}^{N-1} \rightarrow \Omega$ and $\text{pr}_2 : S^*(\Omega) \rightarrow \mathbb{S}^{N-1}$ be the natural projections. Introduce the shift operator $\sigma : \Sigma_A \rightarrow \Sigma_A$ and $\sigma : \Sigma_A^+ \rightarrow \Sigma_A^+$ by $(\sigma(\xi))_i = \xi_{i+1}$, $i \in \mathbb{Z}$, $\xi \in \Sigma_A$ and $(\sigma(\xi))_i = \xi_{i+1}$, $i \in \mathbb{N}$, $\xi \in \Sigma_A^+$.

Fix a large ball $B_0$ containing $K$ in its interior. For any $x \in \Gamma = \partial K$ we will denote by $\nu(x)$ the outward unit normal to $\Gamma$ at $x$. For any $\delta > 0$ and $V \subset \Omega$ denote by $S_\delta^*(V)$ the set of those $(x, u) \in S^*(\Omega)$ such that $x \in V$ and there exist $y \in \Gamma$ and $t \geq 0$ with $y + tu = x$, $y + su \in \mathbb{R}^N \setminus K$ for all $s \in (0, t)$ and $\langle u, \nu_T(y) \rangle \geq \delta$.

**Remark 1.** Notice that the condition (H) implies the existence of a constant $\delta_0 > 0$ depending only on the obstacle $K$ such that any $(x, u) \in S^*(\Omega)$ whose backward and forward billiard trajectories both have common points with $\Gamma$ belongs to $S_{\delta_0}^*(\Omega)$.

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. Denote by

$$
X_1(z_0), X_2(z_0), \ldots, X_m(z_0),
$$

the successive reflection points (if any) of the forward trajectory

$$
\gamma_+(z_0) = \{ \text{pr}_1(\phi_t(z_0)) : 0 \leq t \}.
$$

If $\gamma_+(z_0)$ is bounded (i.e. it has infinitely many reflection points), we will say that it has a forward itinerary $\eta = (\eta_1, \eta_2, \ldots)$ (or that it follows the configuration $\eta$) if $X_j(z_0) \in \partial K_{\eta_j}$ for all $j \geq 1$.

Similarly, we will denote by $\gamma_-(z_0)$ the backward trajectory determined by $z_0$ and by

$$
\ldots, X_{-m}(z_0), \ldots, X_{-1}(z_0), X_0(z_0)
$$

its backward reflection points (if any). For any $j \in \mathbb{Z}$ for which $X_j(z_0)$ exists denote by $\Xi_j(z_0)$ the direction of $\gamma(z_0) = \gamma_-(z_0) \cup \gamma_+(z_0)$ at $X_j(z_0)$, i.e. $\Xi_j(z_0) = \lim_{t \downarrow t_j} \text{pr}_2(\phi_t(z_0))$, where $X_j(z_0) = \text{pr}_1(\phi_{t_j}(z_0))$. Thus, $\phi_{t_j}(z_0) = (X_j(z_0), \Xi_j(z_0))$. A finite string $j = (j_0, j_1, j_2, \ldots, j_m)$ of numbers $j_i = 1, 2, \ldots, k_0$ will be called an admissible configuration (of length $|j| = m + 1$) if $j_i \neq j_{i+1}$ for all $i = 0, 1, \ldots, m - 1$. We will say that a billiard trajectory $\gamma$ with successive reflection points $x_0, x_1, \ldots, x_m$ follows the configuration $j$ if $x_i \in \Gamma_{j_i}$ for all $i = 0, 1, \ldots, m$.

A phase function on an open set $U$ in $\mathbb{R}^N$ is a smooth ($C^\infty$) function $\varphi : U \rightarrow \mathbb{R}$ such that $\|\nabla \varphi\| = 1$ everywhere in $U$. For $x \in U$ the level surface

$$
C_\varphi(x) = \{ y \in U : \varphi(y) = \varphi(x) \}
$$

has a unit normal field $\pm \nabla \varphi(y)$.

**Remark 2.** It should be mentioned that in Sects. 2-5 the $C^\infty$ smoothness assumption can be replaced by $C^k$ for any $k \geq 1$.

The phase function $\varphi$ defined on $U$ is said to satisfy the condition (P) on $\Gamma_j$ if:

(i) the normal curvatures of $C_\varphi$ with respect to the normal field $-\nabla \varphi$ are non-negative at every point of $C_\varphi$;

(ii) $U^+(\varphi) = \{ y + t\nabla \varphi(y) : t \geq 0, y \in U \cap \Gamma_j \} \supset \bigcup_{i \neq j} K_i$. 
A natural extension of $\varphi$ on $\mathcal{U}^+(\varphi)$ is obtained by setting $\varphi(y + t\nabla\varphi(y)) = \varphi(y) + t$ for $t \geq 0$ and $y \in \mathcal{U} \cap \Gamma_j$.

Given such a phase function $\varphi$ and $i \neq j$, denote by $\mathcal{U}_i(\varphi)$ the set of all points $x$ of the form $x = X_1(y, \nabla\varphi(y)) + t \Xi_1(y, \nabla\varphi(y))$, where $y \in \mathcal{U} \cap \Gamma_j$ and $t \geq 0$ are such that $X_1(y, \nabla\varphi(y)) \in \Gamma_{i(j)}$, where

$$\Gamma_{i(j)} = \{ x \in \Gamma_i : (\nu(x), (y-x)/\|y-x\|) \geq \delta_0 \text{ for all } y \in \Gamma_j \} ,$$

Then setting $\varphi_i(x) = \varphi(X_1(y, \nabla\varphi(y)) + t$, one gets a phase function $\varphi_i$ satisfying the Condition (P) on $\Gamma_i$ ([I3]). The operator sending $\varphi$ to $\varphi_i$ is denoted by $\Phi^i_j$, i.e. $\Phi^i_j(\varphi) = \varphi_i$.

Given an admissible configuration $j = (j_0, j_1, \ldots, j_m)$ and a phase function $\varphi$ satisfying the Condition (P) on $\Gamma_{j_0}$, define

$$\varphi_j = \Phi^{j_m}_{j_{m-1}} \circ \Phi^{j_{m-1}}_{j_{m-2}} \circ \ldots \circ \Phi^{j_1}_{j_0}(\varphi) .$$

Notice that for any $z$ in the domain $\mathcal{U}_j(\varphi_j)$ of $\varphi_j$ there exists $(x, u) \in S^+_z(\Gamma_{j_0})$ such that $x \in \mathcal{U}$ and $\gamma(x, u)$ follows the configuration $j$, i.e. it has at least $m$ reflection points and $X_i(x, u) \in \Gamma_{j_i}$ for all $i = 1, \ldots, m$, and $z = X_m(x, u) + t \Xi_m(x, u)$ for some $t \geq 0$. Denote $X^{-\ell}(z, \varphi_j) = X_{m-\ell}(x, u)$, $0 \leq \ell \leq m$.

Several well-known facts about the dynamics of the billiard in $\Omega$, phase functions and related objects will be frequently used throughout the paper and for convenience of the reader we state them here.

The following is a consequence of the hyperbolicity of the billiard flow in the exterior of $K$ and can be derived from the works of Sinai on general dispersing billiards ([Si1], [Si2]) and from Ikawa’s papers on open billiards ([I3]; see also [B]). In this particular form it can be found in [S] (see also Ch. 10 in [PS1]).

**Proposition 1.** There exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that for any admissible configuration $j = (j_0, j_1, \ldots, j_m)$ and any two billiard trajectories in $\Omega$ with successive reflection points $x_0, x_1, \ldots, x_m$ and $y_0, y_1, \ldots, y_m$, both following the configuration $j$, we have

$$\|x_i - y_i\| \leq C (\alpha^i + \alpha^{m-i}) , \quad 0 \leq i \leq m .$$

Moreover, $C$ and $\alpha$ can be chosen so that if $(x_0, (x_1 - x_0)/\|x_1 - x_0\|)$ and $(y_0, (y_1 - y_0)/\|y_1 - y_0\|)$ belong to the same unstable manifold of the billiard flow, i.e. there exists a phase function $\varphi$ satisfying the condition (P) on some open set $\mathcal{U}$ containing $x_0$ and $y_0$ and such that $\nabla \varphi(x_0) = (x_1 - x_0)/\|x_1 - x_0\|$ and $\nabla \varphi(y_0) = (y_1 - y_0)/\|y_1 - y_0\|$, then

$$\|x_i - y_i\| \leq C \alpha^{m-i} , \quad 0 \leq i \leq m .$$

Next, given a vector $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, denote

$$D_a = a_1 \frac{\partial}{\partial x_1} + \ldots + a_N \frac{\partial}{\partial x_N} ,$$

and for any $C^1$ vector field $f : U \longrightarrow \mathbb{R}^N$ ($U \subset \mathbb{R}^N$) and any $V \subset U$ set $\|f\|_0(V) = \sup_{x \in V} \|f(x)\|$ and $\|f\|_0 = \|f\|_0(U)$. Next, assume $f$ has continuous derivatives of all orders $\leq p$ ($p \geq 1$), set

$$\|f\|_p(x) = \max_{a^{(1)}, \ldots, a^{(p)} \in \mathbb{R}^{n-1}} \|D_{a^{(1)}} \ldots D_{a^{(p)}} f(x)\| , \quad \|f\|_p(V) = \sup_{x \in V} \|f\|_p(x) , \quad \|f\|_p = \|f\|_p(U) ,$$

$$\|f\|_{(p)}(x) = \max_{0 \leq j \leq p} \|f\|_{(p)}(x) , \quad \|f\|_{(p)}(V) = \sup_{x \in V} \|f\|_{(p)}(x) , \quad \|f\|_{(p)} = \|f\|_{(p)}(U) .$$
Similarly, for \( x \in \Gamma \) and \( V \subset \Gamma \) set
\[
\|f\|_{\Gamma, p}(x) = \max_{a^{(1)}, \ldots, a^{(p)} \in S_p \Gamma} \|((D_{a^{(1)}} \cdots D_{a^{(p)}} f)(x))\|, \quad \|f\|_{\Gamma, p}(V) = \sup_{x \in V} \|f\|_{\Gamma, p}(x), \quad \|f\|_{\Gamma, p} = \|f\|_{\Gamma, p}(U),
\]
where \( S_p \Gamma \) is the unit sphere in the tangent plane \( T_x \Gamma \) to \( \Gamma \) at \( x \). Finally, set
\[
\|f\|_{\Gamma, (p)}(x) = \max_{0 \leq j \leq p} \|f\|_{\Gamma, p}(x), \quad \|f\|_{\Gamma, (p)}(V) = \sup_{x \in V} \|f\|_{\Gamma, (p)}(x), \quad \|f\|_{\Gamma, (p)} = \|f\|_{\Gamma, (p)}(U).
\]

**Remark 3.** It follows easily from the definitions that for any \( \delta > 0 \) and any integer \( p \geq 1 \) there exists a constant \( A_p = A_p(\delta, K) > 0 \) such that if \( \psi \) is a phase function which is at least \( C^{p+1}\)-smooth on some subset \( V \) of \( \Omega \) and \( x \in V \cap \Gamma \) with \((x, \nabla \psi(x)) \in S_p^*(V)\), then \( \|\nabla \psi\|_{p}(x) \leq A_p \|\nabla \psi\|_{\Gamma, p}(x)\).

The following comprises Proposition 5.4 in [I1], Propositions 3.11 and 3.12 in [I3] and Lemma 4.1 in [I2] (see also the proof of the estimate (3.64) in [B]).

**Proposition 2.** For every integer \( p \geq 1 \) there exist global constants \( C_p > 0 \) and \( \alpha \in (0, 1) \) such that for any admissible configuration \( j = (j_0, j_1, \ldots, j_m) \) and any phase functions \( \varphi \) and \( \psi \) satisfying the Condition \( (P) \) on \( \Gamma_{j_0} \) on some open set \( U \), we have
\[
\|\nabla \varphi_j\|_{p}(x) \leq C_p \|\nabla \varphi\|_{(p)}(U \cap B_0) \tag{2.1}
\]
for any \( x \in U_{j}(\varphi) \cap B_0 \), and
\[
\|\nabla \varphi_j - \nabla \psi_j\|_{p}(x) \leq C_p \alpha^m \|\nabla \varphi - \nabla \psi\|_{p}(U \cap B_0), \tag{2.2}
\]
\[
\|X^{-\ell}(\cdot, \nabla \varphi_j) - X^{-\ell}(\cdot, \nabla \psi_j)\|_{p}(x) \leq C_p \alpha^{m-\ell} \|\nabla \varphi - \nabla \psi\|_{(p)}(U \cap B_0) \tag{2.3}
\]
for any \( x \in U_{j}(\varphi) \cap U_{j}(\psi) \cap B_0 \) and \( 0 \leq \ell < m \). Finally, we can choose \( C_p > 0 \) so that
\[
\|X^{-\ell}(\cdot, \nabla \varphi_j)\|_{\Gamma, p}(x) \leq C_p \alpha^\ell \tag{2.4}
\]
for all \( x \in U_{j}(\varphi) \cap B_0 \) and \( 0 \leq \ell < m \).

Given \( x \) in the domain \( U \) of a phase function \( \varphi \), denote
\[
\Lambda_{\varphi}(x) = \left( \frac{G_{\varphi}(x)}{G_{\varphi}(X^{-1}(x, \nabla \varphi))} \right)^{1/(N-1)},
\]
where \( G_{\varphi}(y) \) is the Gauss curvature of \( C_{\varphi}(y) \) at \( y \). It follows from [I3] (or [B]) that there exist global constants \( 0 < \alpha_1 < \alpha < 1 \) such that
\[
0 < \alpha_1 \leq \Lambda_{\varphi}(x) \leq \alpha < 1 \tag{2.5}
\]
for any phase function \( \varphi \) and any \( y \in U(\varphi) \).

It follows from Lemma 3.1 in [I3] that there exist \( \delta_1 > 0 \) and \( 0 < d_0 < \frac{1}{2} \min_{i \neq j} \text{dist}(K_i, K_j) \) such that for any \( i = 1, \ldots, k_0 \), \( x \in \Gamma_i \) and \( \xi \in S^{N-1} \) with \( 0 \leq \langle \xi, \nu(x) \rangle \leq \delta_1 \), the ray \( \{x + t\xi : t \geq 0\} \) has no common points with \( \cup_{j \neq i} B(K_j, d_0) \), where
\[
B(A, d_0) = \{y + w : y \in A, w \in \mathbb{R}^N, ||w|| \leq d_0\}.
\]
Fix an arbitrary \( \chi_0 \in C_0^\infty(\mathbb{R}) \) such that \( \chi_0(t) = 0 \) for \( |t| < \delta_1/2 \) and \( \chi_0(t) = 1 \) for \( |t| \geq \delta_1 \). Now for any \( j = (j_0 = 1, j_1, \ldots, j_m) \) and any \( x \in U_{j}(\varphi) \), following [I3], define
\[
(A_{\varphi, j}(x)) = \chi_0((\nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j)))) \Lambda_{\varphi, j}(x) h(X^{-m}(x, \nabla \varphi_j)),
\]
where
\[ \Lambda_{\varphi, j}(x) = \Lambda_{\varphi(j_1, \ldots, j_m)}(x) \Lambda_{\varphi(j_1, \ldots, j_{m-1})}(X^{-1}(x, \nabla \varphi)) \cdots \Lambda_{\varphi}(X^{-m}(x, \nabla \varphi)) \in (0, 1). \]

The following facts can be derived from \([I1], [I3]\) (see also Proposition 5.1 in \([B]\)).

**Proposition 3.** For every integer \( p \geq 1 \) there exists a global constant \( C_p > 0 \) such that for any admissible configuration \( j = (j_0, j_1, \ldots, j_m) \) and any phase function \( \varphi \) satisfying the Condition (\( P \)) on \( \Gamma_j \) on some open set \( U \), we have
\[ \| \Lambda_{\varphi, j} \|_p(x) \leq C_p \| \nabla \varphi \|_p(U \cap B_0), \quad x \in U_j(\varphi) \cap B_0. \]

### 3. Stable and Unstable Manifolds for Open Billiards

Let \( z_0 = (x_0, u_0) \in S^*(\Omega) \). For convenience we will assume that \( x_0 \notin K \). Assume that the backward trajectory \( \gamma_- (z_0) \) determined by \( z_0 \) is bounded, and let \( \eta \in \Sigma_A \) be its itinerary.

Given \( x \in \mathbb{R}^N \) and \( \epsilon > 0 \), by \( B(x, \epsilon) \) we denote the open ball with center \( x \) and radius \( \epsilon \) in \( \mathbb{R}^N \).

In this section we use some tools from \([I3]\) to construct the local unstable manifold \( W^u_{\text{loc}}(z_0) \) of \( z_0 \) in \( S^*(\Omega) \) and show that it is Lipschitz in \( z_0 \) (and \( \eta \)). In a similar way one deals with local stable manifolds.

**Proposition 4.** There exists a constant \( \epsilon_0 > 0 \) such that for any \( z_0 = (x_0, u_0) \in S^*_{b_0}(\Omega \cap B_0) \) whose backward trajectory \( \gamma_-(z_0) \) has an infinite number of reflection points \( X_j = X_j(z_0) \) \( (j \leq 0) \) and \( \eta \in \Sigma_A^L \) is its itinerary, the following hold:

(a) There exists a smooth \( (C^\infty) \) phase function \( \psi = \psi_\eta \) satisfying the condition (\( P \)) on \( U = B(x_0, \epsilon_0) \cap \Omega \) such that \( \psi(x_0) = 0, u_0 = \nabla \psi(x_0) \), and such that for any \( x \in C_\psi(x_0) \cap U^+(\psi) \) the billiard trajectory \( \gamma_-(x, \nabla \psi(x)) \) has an itinerary \( \eta \) and therefore \( d(\phi_t(x, \nabla \psi(x)), \phi_t(z_0)) \to 0 \) as \( t \to -\infty \). That is,
\[ W^u_{\text{loc}}(z_0) = \{ (x, \nabla \psi(x)) : x \in C_\psi(x_0) \cap U^+(\psi) \} \]

is the local unstable manifold of \( z_0 \). Moreover, for any \( p = 1, \ldots, k \) there exists a global constant \( C_p > 0 \) (independent of \( z_0 \) and \( \eta \)) such that
\[ \| \nabla \psi_\eta \|_p(U) \leq C_p. \quad \text{(3.1)} \]

(b) If \( (y, v) \in S^*(\Omega \cap B_0) \) is such that \( y \in C_\psi(x_0) \) and \( \gamma_-(y, v) \) has the same itinerary \( \eta \), then \( v = \nabla \psi(y) \), i.e. \( (y, v) \in W^u_{\text{loc}}(z_0) \).

(c) There exist a constant \( \alpha \in (0, 1) \) depending only on the obstacle \( K \) and for every \( p \geq 1 \) a constant \( C'_p > 0 \) such that for any integer \( r \geq 1 \) and any \( \zeta, \eta \in \Sigma_A^L \) with \( \zeta_j = \eta_j \) for \( -r \leq j \leq 0 \), we have \( \| \nabla \psi_\eta - \nabla \psi_{\zeta} \|_p(V) \leq C'_p \alpha^r \), where \( V = \mathcal{U}(\psi_\eta) \cap \mathcal{U}(\psi_\zeta) \).

**Proof.** (a) Take \( \epsilon_0 > 0 \) so small that whenever \( (x, u) \in S^*_{b_0/2}(\Omega \cap B_0) \) and \( (y, v) \in S^*(\Omega) \) is such that \( \| x - y \| < \epsilon_0 \) and \( \| u - v \| < \epsilon_0 \) we have \( (y, v) \in S^*_{b_0}(\Omega) \). Then define \( \mathcal{U} \) as in part (a) above. Set
\[ d_m = \| X_{m+1} - X_m \|, \quad u_m = \frac{X_{m+1} - X_m}{\| X_{m+1} - X_m \|} \in S^{n-1} \quad (m \geq 1). \]
Given any integer \( m \geq 1 \), consider the linear phase function \( \psi^{(m)} = \psi^{(m, \eta)} \) in \( \Omega \) such that \( \nabla \psi^{(m)} \equiv u_{-m} \) and \( \psi^{(m)}(X_{-m}) = - (d_{-m} + d_{-m+1} + \ldots + d_{-1}) \). Then define
\[
\psi^{(m)} = \psi^{(m, \eta)} = \Phi^{\eta}_{\eta_0} \circ \Phi^{\eta-1}_{\eta_1} \circ \ldots \circ \Phi^{\eta_{-m+2}}_{\eta_{-m+1}} \circ \Phi^{\eta_{-m}}_{\eta_{-m+1}} (\psi^{(m)}) \, .
\]

Clearly \( \psi^{(m)} \) is a smooth phase function defined everywhere on \( U \) (in fact, on a much larger subset of \( \Omega \)) with \( \psi^{(m)}(X_0) = 0 \). Moreover, it follows from Proposition 2.1 above that
\[
\| \nabla \psi^{(m)} - \nabla \psi^{(m+1)} \|_{p(U)} \leq \text{Const}_p \alpha^m \, , \quad m \geq 1
\]
for some global constant \( \text{Const}_p > 0 \) depending only on \( K \) and \( p \). Here we use the fact that \( \| \nabla \psi^{(m)} - \nabla \psi^{(m+1)} \|_{(p)} \leq \text{Const} \), due to the special choice of the phase functions \( \psi^{(m)} \) and \( \psi^{(m+1)} \).

Since \( \psi^{(m)}(X_0) = \psi^{(m+1)}(X_0) = 0 \), it now follows that there exists a constant \( C_p \alpha^m > 0 \) such that
\[
\| \psi^{(m)}(x) - \psi^{(m+1)}(x) \| \leq C_p \alpha^m \, , \quad x \in U \cap B_0 \, .
\]

This implies that for every \( x \in U \) there exists \( \psi(x) = \lim_{m \to \infty} \psi^{(m)}(x) \). Now (3.2) shows that \( \psi \) is \( C^\infty \)-smooth in \( U \) and
\[
\| \nabla \psi^{(m)} - \nabla \psi \|_{p(U)} \leq \text{Const}_p \alpha^m \, , \quad m \geq 1 \, .
\]
In particular, \( \| \nabla \psi \| \equiv 1 \) in \( U \). Extending \( \psi \) in a trivial way along straight line rays, we get a phase function \( \psi \) satisfying the condition (P) in \( U \).

Let us now show that \( W = \{(x, \nabla \psi(x)) : x \in C_{\psi}(x_0) \cap U^+(\psi)\} \) is the local unstable manifold of \( z_0 \). Given \( x \in C_{\psi}(x_0) \cap U^+(\psi) \) sufficiently close to \( x_0 \) and an arbitrary integer \( r \geq 0 \), consider the points \( X^{-r}(x, \psi^{(m)}) \in \partial K_{\eta-r} \) for \( m \geq r \). By Proposition 1, there exist global constants \( \text{Const} > 0 \) and \( \alpha \in (0,1) \) such that
\[
\| X^{-r}(x, \psi^{(m)}) - X^{-r}(x, \psi^{(m')}) \| \leq \text{Const} \alpha^{m-r} \, , \quad m' \geq m > r \, .
\]
Thus, there exists \( X^{-r} = \lim_{m \to \infty} X^{-r}(x, \psi^{(m)}) \in \partial K_{\eta-r} \) and
\[
\| X^{-r}(x, \psi^{(m)}) - X^{-r} \| \leq \text{Const} \alpha^{m-r} \, , \quad m > r \, .
\]

It is now easy to see that \( \{X^{-J}\}_{J=0}^{\infty} \) are the successive reflection points of a billiard trajectory in \( \Omega \) and this is the trajectory \( \gamma_{-}(x, \nabla \psi) \). The backward itinerary of the latter is obviously \( \eta \). Moreover, (3.3) implies \( d(\phi_t(x, \nabla \psi(x)), \phi_t(z_0)) \to 0 \) as \( t \to - \infty \), so \( (x, \nabla \psi(x)) \in W^u_{\text{loc}}(z_0) \).

Finally, by (2.1),
\[
\| \psi^{(m)} \|_{p(U)} \leq \text{Const}_p \| \psi^{(m)} \|_{p} \leq \text{Const}_p \, ,
\]
and combining this with (3.3) gives (3.1).

(b) Let \( (y, v) \in S^*(\Omega) \) be such that \( y \in C_{\psi}(x_0) \) and \( \gamma_{-}(y, v) \) has the same itinerary \( \eta \). Define the phase functions \( \varphi^{(m)} \) and \( \varphi \) as in part (a) replacing the point \( z_0 = (x_0, u_0) \) by \( z = (y, v) \), and let \( \varphi(x) = \lim_{m \to \infty} \varphi^{(m)}(x) \). Then by part (a), we have \( W^u_{\text{loc}}(z) = \{(x, \nabla \psi(x)) : x \in C_{\varphi}(y) \cap U^+(\varphi)\} \).

On the other hand, it follows from Proposition 2 that there exist constants \( \text{Const} > 0 \) and \( \alpha \in (0,1) \) such that \( \| \nabla \varphi^{(m)} - \nabla \varphi \| \leq \text{Const} \alpha^m \) for all \( m \geq 0 \), which implies \( \varphi = \psi \). Thus, \( v = \nabla \varphi(y) = \nabla \psi(y) \in W^u_{\text{loc}}(z_0) \).

(c) Choose the constants \( \alpha \in (0,1) \) and \( \text{Const}_p > 0 \) \( (p = 1, \ldots, k) \) as in part (a). Let \( \zeta, \eta \in \Sigma^-_A \) be such that \( \zeta_j = \eta_j \) for all \( -r \leq j \leq 0 \) for some \( r \geq 1 \). Construct the phase functions \( \psi^{(m, \eta)} \) and \( \psi^{(m, \zeta)} \) \( (m \geq 1) \) as in part (a); then \( \eta = \lim_{m \to \infty} \psi^{(m, \eta)} \), \( \zeta = \lim_{m \to \infty} \psi^{(m, \zeta)} \). It follows from
Proposition 2 that $\|\nabla \psi^{(r,\eta)} - \nabla \psi^{(r,\xi)}\| \leq \text{Const}_p \alpha^r$. Combining this with (3.3) with $m = r$ for $\eta$ and then with $\eta$ replaced by $\xi$, one gets

$$\|\nabla \psi_\eta - \nabla \psi_\xi\| \leq \|\nabla \psi_\eta - \nabla \psi^{(r,\eta)}\| + \|\nabla \psi^{(r,\eta)} - \nabla \psi^{(r,\xi)}\| + \|\nabla \psi^{(r,\xi)} - \nabla \psi_\xi\| \leq \text{Const}_p \alpha^r .$$

This proves the assertion. ■

4. RUELLE OPERATOR AND ASYMPTOTIC SOLUTIONS

Given $\xi \in \Sigma_A$, let

$$\ldots, P_{-2}(\xi), P_{-1}(\xi), P_{0}(\xi), P_{1}(\xi), P_{2}(\xi), \ldots$$

be the successive reflection points of the unique billiard trajectory in the exterior of $K$ such that $P_{j}(\xi) \in K_{\xi_j}$ for all $j \in \mathbb{Z}$. Set

$$f(\xi) = \|P_{0}(\xi) - P_{1}(\xi)\| .$$

Following [I3] (see also Section 3), one constructs a sequence $\{\varphi_{\xi,j}\}_{j=-\infty}^{\infty}$ of phase functions such that for each $j$, $\varphi_{\xi,j}$ is defined and smooth in a neighborhood $U_{\xi,j}$ of the segment $[P_{j}(\xi), P_{j+1}(\xi)]$ in $\Omega$ and:

(i) $\|\nabla \varphi_{\xi,j}\| = 1$ on $U_{\xi,j}$ and satisfies the condition (P) on $U_{\xi,j}$;

(ii) $\nabla \varphi_{\xi,j}(P_{j}(\xi)) = \frac{P_{j+1}(\xi) - P_{j}(\xi)}{\|P_{j+1}(\xi) - P_{j}(\xi)\|}$ ;

(iii) $\varphi_{\xi,j} = \varphi_{\xi,j+1}$ on $\Gamma_{\xi,j+1} \cap U_{\xi,j} \cap U_{\xi,j+1}$ ;

(iv) for each $x \in U_{\xi,j}$ the surface

$$C_{\xi,j}(x) = \{y \in U_{\xi,j} : \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x)\}$$

is strictly convex with respect to its normal field $\nabla \varphi_{\xi,j}$.

More precisely, one can proceed as follows. Given $\xi \in \Sigma_A$, let $\xi^- = (\ldots, \xi_{-2}, \xi_{-1}, \xi_0)$ and let $\psi_{\xi^-}$ be the phase function with $\psi_{\xi^-}(P_0) = 0$ and $\nabla \psi_{\xi^-}(P_0) = (P_1 - P_0)/\|P_1 - P_0\|$ constructed in Proposition 4(a). Set $\varphi_{\xi,0} = \psi_{\xi^-}$ and

$$\varphi_{\xi,j} = (\psi_{\xi^-})_{(\xi_{0},\xi_{1},\ldots,\xi_{j})}$$

for any $j > 0$. For $j < 0$, setting $\xi^{(j)} = (\ldots, \xi_{-j-2}, \xi_{-j-1}, \xi_j)$ and using again Proposition 4, we get a phase function $\psi_{\xi^{(j)}}$ with $\psi_{\xi^{(j)}}(P_j) = 0$ and $\nabla \psi_{\xi^{(j)}}(P_j) = (P_{j+1} - P_j)/\|P_{j+1} - P_j\|$. By the uniqueness of the phase functions $\psi_{\eta}$ (see Proposition 4(c)), it follows that there exists a constant $c_j$ such that $\psi_{\xi^-} = (\psi_{\xi^{(j)}} + c_j)_{(\xi_j, \xi_{j+1}, \ldots, \xi_0)}$ (locally near the segment $[P_0, P_j]$). Setting $\varphi_{\xi,j} = \psi_{\xi^{(j)}} + c_j$, one obtains a phase function defined on some naturally determined (see the proof of Proposition 3 (a)) open set $U_{\xi^-,j}$ such that

$$(\varphi_{\xi,j})_{(\xi_{0},\xi_{1},\ldots,\xi_{-j-1},\xi_{-j})} = \psi_{\xi^-}, \ j < 0 .$$

(4.1)

This completes the construction of the phase functions $\varphi_{\xi,j}$.

Moreover, it follows from Proposition 2 that for any $p = 1, \ldots, k$, there exists a global constant $C_p > 0$ such that

$$\|\nabla \varphi_{\xi,j}\|_{(p)} \leq C_p$$

(4.2)

for all $\xi \in \Sigma_A$ and $j \in \mathbb{Z}$. 


Remark 4. Notice that the above construction can be carried out for \( j < 0 \) for any \( \xi \in \Sigma_- \) and any billiard trajectory \( \gamma \) in \( \Omega \) with reflection points \( \ldots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi) \) such that \( P_j(\xi) \in K_{\xi} \) for all \( j \leq 0 \). Then one defines a phase function \( \psi_\xi \) with \( \psi_\xi(P_0) = 0 \) as above, and using (4.1) one gets a sequence \( \{\varphi_{\xi,j}\}_{j \leq 0} \) of phase functions such that for each \( j < 0 \), \( \varphi_{\xi,j} \) is defined and smooth in a neighborhood \( U_{\xi,j} \) of the segment \( [P_j(\xi), P_{j+1}(\xi)] \) in \( \Omega \) and satisfies the conditions (i)-(iv). Moreover (4.2) holds for any \( p = 1, \ldots, k \) and any \( j \leq 0 \).

For any \( y \in U_{\xi,j} \) denote by \( G_{\xi,j}(y) \) the Gauss curvature of \( C_{\xi,j}(x) \) at \( y \). Now define \( g : \Sigma_A \longrightarrow \mathbb{R} \) by

\[
g(\xi) = \frac{1}{N - 1} \ln \frac{G_{\xi,0}(P_1(\xi))}{G_{\xi,0}(P_0(\xi))}
\]

Given a function \( F(\xi) \) defined on \( \Sigma_A \), we set

\[
\var_n F = \sup \{|F(\xi) - F(\eta)| : \xi_i = \eta_i \text{ for } |i| \leq n\},
\]

and for \( 0 < \theta < 1 \) we define \( \|F\|_\theta = \sup \var_n F \), \( \|F\|_\infty = \|F\|_1 + \|F\|_\theta \) and introduce the space \( \mathcal{F}_\theta(\Sigma_A) = \{F(\xi) : \|F\|_\theta < \infty\} \). The functions \( f(\xi), g(\xi) \in \mathcal{F}_\theta(\Sigma_A) \) with a suitable \( 0 < \theta < 1 \) (see [14]). By Sinai’s Lemma (see e.g. [PP]), there exist \( \hat{f}, \hat{g} \in \mathcal{F}_\theta(\Sigma_A) \) depending on future coordinates only and \( \chi_1, \chi_2 \in \mathcal{F}_\theta(\Sigma_A) \) such that

\[
f(\xi) = \tilde{f}(\xi) + \chi_1(\xi) - \chi_1(\sigma \xi), \xi \in \Sigma_A,
\]

and

\[
g(\xi) = \tilde{g}(\xi) + \chi_2(\xi) - \chi_2(\sigma \xi), \xi \in \Sigma_A.
\]

As in the proof of Sinai’s Lemma for any \( k = 1, \ldots, k_0 \) choose and fix an arbitrary sequence \( \eta^{(k)} = (\ldots, \eta_{-m}^{(k)}, \ldots, \eta_{-1}^{(k)}, \eta_0^{(k)}) \in \Sigma_A \) with \( \eta_0^{(k)} = k \). Then for any \( \xi \in \Sigma_A \) (or \( \xi \in \Sigma_A^+ \)) set

\[
e(\xi) = (\ldots, \eta_{-m}^{(\xi_0)}, \ldots, \eta_{-1}^{(\xi_0)}, \eta_0^{(\xi_0)} = \xi_0, \xi_1, \ldots, \xi_m, \ldots) \in \Sigma_A.
\]

Then we have

\[
\chi_1(\xi) = \sum_{n=0}^{\infty} \left[f(\sigma^n(\xi)) - f(\sigma^n e(\xi))\right],
\]

and the function \( \chi_2 \) is defined similarly, replacing \( f \) by \( g \).

Setting \( \chi(\xi, s) = -s \chi_1(\xi) + \chi_2(\xi), \) for the function

\[
r(\xi, s) = -s f(\xi) + g(\xi) + i \pi
\]

we have

\[
r(\xi, s) = \tilde{r}(\xi, s) + \chi(\xi, s) - \chi(\sigma \xi, s), \xi \in \Sigma_A, s \in \mathbb{C}
\]

where

\[
\tilde{r}(\xi, s) = -s \tilde{f}(\xi) + \tilde{g}(\xi) + i \pi
\]

depends on future coordinates of \( \xi \) only (so it can be regarded as a function on \( \Sigma_A^+ \times \mathbb{C} \)). Below we need the Ruelle transfer operator \( L_s : C(\Sigma_A^+) \longrightarrow C(\Sigma_A^+) \) defined by

\[
L_s u(\xi) = \sum_{\sigma \eta = \xi} e^{\tilde{r}(\eta, s)} u(\eta)
\]
for any continuous (complex-valued) function \( u \) on \( \Sigma_A^+ \) and any \( \xi \in \Sigma_A^+ \). Notice that 
\[
L^n_{s} u(\xi) = (-1)^n L^n_{-sf+\tilde{g}}(\xi) = (-1)^n \sum_{\sigma\eta = \xi} e^{-sf(\eta)+\tilde{g}(\eta)}u(\eta), \quad n \geq 0
\]
hence 
\[
\|L^n_{s}u\|_{\infty} = \| \sum_{\sigma\eta = \xi} e^{-sf_\sigma(\eta)+\tilde{g}_\sigma(\eta)}u(\eta)\|_{\infty}.
\]
To estimate the norm of \( \tilde{L}_s^n u \) (for some special functions \( u \)), where \( \tilde{L}_s = L_{-sf+\tilde{g}} \), we will apply Dolgopyat type estimates \([D]\) established in \([St2]\) for \( N = 2 \) and under some conditions in \([St3]\) for \( N \geq 3 \) (see also \([PS2]\) for the such type of estimates related to a different coding of the billiard trajectories). These results imply that there exist constants \( \sigma_0 < s_0 \) and \( 0 < \rho < 1 \) so that for a special class of functions \( u \) and \( s = \tau + it \) with \( \tau \geq \sigma_0 \) and \( n = p[\log |t|]+l, \quad p \in \mathbb{N}, \quad 0 \leq l \leq \lfloor \log |t| \rfloor - 1, \) we have 
\[
\|\tilde{L}_s^n u\|_{\infty} \leq C\rho^{[\log |t|]}e^{lPr(-\tau f+\tilde{g})}\|u\|_{\infty}.
\]
(4.3)

We will deal with oscillatory data on \( \Gamma_1 \) (which can be replaced by any \( \Gamma_j \)) of the form 
\[
u_0(x;s) = e^{-is\varphi(x)}h(x) \quad , \quad x \in \Gamma_1 , s \in \mathbb{C}.
\]
(4.4)
Here \( \varphi \) is a \( C^\infty \) phase function satisfying the condition \((P)\) on \( \Gamma_1 \) (see Sect. 2 above) and \( h \) is a \( C^\infty(\Gamma) \) function with small support on \( \Gamma_1 \). For every configuration \( j = (j_0,j_1,\ldots,j_n), \quad j_0 = 1, \) we can construct a function \( u_j(x;s) \) following a recurrent procedure (see \([I5]\)). We construct a sequence of phase functions \( \varphi_j(x) \) and amplitudes \( a_j(x) \) so that setting 
\[
u_j(x;s) = (-1)^{|j|}e^{-s\varphi_j(x)}a_j(x),
\]
for the configurations \( j \) and \( j' = (j_0,j_1,\ldots,j_m,j_{m+1}) \) we have 
\[
u_{j_0}(x;s) = \nu_0(x;s) \quad \text{on} \quad \Gamma_1
\]
\[
u_j(x;s) + \nu_{j'}(x;s) = 0 \quad \text{on} \quad \Gamma_{j_{m+1}}.
\]
The phase functions \( \varphi_j \) are determined following the procedure in Section 2, while the amplitudes \( a_j(x) \) are determined as the solutions of the transport equations and, using the notations of Section 2, we set \( a_j(x) = (A_j(\varphi)h)(x) \).

Next, let \( \mu = (\mu_0 = 1,\mu_1,\ldots) \in \Sigma_A^+ \). It follows from \([I3]\) that there exists a unique point \( y(\mu) \in \Gamma_1 \) such that the ray \( \gamma(y,\varphi) \) issued from a point \( y(\mu) \) in direction \( \nabla \varphi(y(\mu)) \) follows the configuration \( \mu \). Let \( Q_0(\mu) = y(\mu), Q_1(\mu),\ldots, \) be the consecutive reflection points of this ray. Define 
\[
f_i^+(\mu) = \|Q_i(\mu) - Q_{i+1}(\mu)\|
\]
and 
\[
g_i^+(\mu) = \frac{1}{N-1} \ln \frac{G_{\mu,i}^\varphi(Q_{i+1}(\mu))}{G_{\mu,i}^\varphi(Q_i(\mu))} < 0,
\]
where \( G_{\mu,i}^\varphi(y) \) denotes the Gauss curvature of the surface 
\[
G_{\mu,i}^\varphi(x) = \{ z \in U(\mu_0,\mu_1,\ldots,\mu_i)(\varphi) : \varphi(\mu_0,\mu_1,\ldots,\mu_i)(z) = \varphi(\mu_0,\mu_1,\ldots,\mu_i)(x) \}
\]
at \( y \).
Next, for \( s \in \mathbb{C} \) and \( \xi \in \Sigma_A^+ \) with \( \xi_0 = 1 \), following [I5], set
\[
\phi^+(\xi, s) = \sum_{n=0}^{\infty} \left(-s \left[f(\sigma^n e(\xi)) - f_n^+(\xi)\right] + [g(\sigma^n e(\xi)) - g_n^+(\xi)]\right).
\]
Formally, define \( \phi^+(\xi, s) = 0 \) when \( \xi_0 \neq 1 \), thus obtaining a function \( \phi^+: \Sigma_A^+ \times \mathbb{C} \to \mathbb{C} \). Now for any \( s \in \mathbb{C} \) define the operator \( \mathcal{G}_s : C(\Sigma_A^+) \to C(\Sigma_A^+) \) by
\[
\mathcal{G}_s v(\xi) = \sum_{\sigma \eta = \xi} e^{-\phi^+(\eta, s)} + \chi(e(\eta), s) e^{-s f(\eta)} + \hat{g}(\eta) v(\eta)
\]
for any \( v \in C(\Sigma_A^+) \) and \( \xi \in \Sigma_A^+ \). (This is slightly different from the corresponding definition in [I5].)

We will replace the function \( v_0 \) from Sect. 4.2 in [I5] by \( \tilde{v} \in \mathcal{F}_\phi(\Sigma_A^+) \) defined by
\[
\tilde{v}_0(\xi) = e^{-s \phi(Q_0(\xi))} h(Q_0(\xi))
\]
if \( \xi_0 = 1 \) and \( \tilde{v}_0(\xi) = 0 \) otherwise. Recall that \( h \) comes from the boundary data (4.4).

**Fix an arbitrary** \( \ell = 1, \ldots, k_0 \) **and an arbitrary point** \( x_0 \in \Gamma_\ell \). Define the function \( \phi^-(x_0; \cdot, \cdot) : \Sigma_A \times \mathbb{C} \to \mathbb{C} \) (depending on \( \ell \) as well) as follows. First, set \( \phi^-(x_0; \eta, s) = 0 \) if \( \eta_0 \neq \ell \). Next, assume that \( \eta \in \Sigma_A \) satisfies \( \eta_0 = \ell \). There exists a unique billiard trajectory in \( \Omega \) with successive reflection points \( \tilde{P}_i(x_0; \eta) \in \partial K_0 \) \((-\infty < i \leq 0)\) such that \( x_0 = \tilde{P}_{-1}(x_0; \eta) + t \nabla \varphi_{\eta,-}(\tilde{P}_{-1}(x_0; \eta)) \) for some \( t > 0 \). In general the segment \( [\tilde{P}_{-1}(x_0; \eta), x_0] \) may intersect the interior of \( K_\ell \). If this is the case, set again \( \phi^-(x_0; \eta, s) = 0 \). Otherwise, denote \( \tilde{P}_0(x_0; \eta) = x_0 \), and for any \( i < 0 \) set
\[
f_i^-(x_0; \eta) = \| \tilde{P}_{i+1}(x_0; \eta) - \tilde{P}_i(x_0; \eta) \|
\]
\[
g_i^-(x_0; \eta) = \frac{1}{N - 1} \ln \frac{G_{\eta,i}(\tilde{P}_{i+1}(x_0; \eta))}{G_{\eta,i}(\tilde{P}_i(x_0; \eta))}
\]
Then define
\[
\phi^-(x_0; \eta, s) = -s \sum_{i=-1}^{-\infty} \left[f(\sigma^i(\eta)) - f_i^-(x_0; \eta)\right] + \sum_{i=-1}^{-\infty} \left[g(\sigma^i(\eta)) - g_i^-(x_0; \eta)\right].
\]
We will show later that this series is absolutely convergent.

Next, similarly to [I5], define the operator \( \mathcal{M}_{n,s}(x_0) : C(\Sigma_A^+) \to C(\Sigma_A^+) \) (depending also on \( \ell \)) by
\[
(\mathcal{M}_{n,s}(x_0)v)(\xi) = \sum_{\sigma \eta = \xi} e^{-\phi^-(x_0; \sigma^{n+1} e(\eta), s)} - \chi(\sigma^{n+1} e(\eta), s) - s f(\eta) + \hat{g}(\eta) v(\eta)
\]
for any \( v \in C(\Sigma_A^+) \) and \( \xi \in \Sigma_A^+ \).

Let \( s_0 \in \mathbb{R} \) be the abscissa of absolute convergence of the dynamical zeta function (see Sect. 1) determined by \( \text{Pr}(-s_0 f + \hat{g}) = 0 \). Here \( \text{Pr}(G) \) is the pressure of \( G \) defined as
\[
P(G) = \sup_{\mu \in \mathcal{M}} [h(\mu) + \int_{\Sigma_A^+} G \mu],
\]
where \( \mathcal{M} \) is the set of all probabilistic measures on \( \Sigma_A^+ \) invariant with respect to \( \sigma \) and \( h(\mu) \) is the metric entropy of \( \mu \).

The part (a) in the following theorem is similar to (4.10) in [I5]:
Theorem 3. There exist global constants $C > 0$, $c > 0$ and $\theta \in (0, 1)$ depending only on $K$ such that for any choice of $\ell = 1, \ldots, k_0$, $x_0 \in \Gamma_\ell$ and $a > 0$ the following hold:

(a) For all integers $n \geq 1$, all $\xi \in \Sigma_+^\ell$ with $\xi_0 = \ell$ and all $s \in \mathbb{C}$ with $\text{Re}(s) \geq s_0 - a$ we have

$$\left|\left( L_n^s \mathcal{M}_{n,s}(x_0) \mathcal{G}_s \tilde{v}_s \right) (\xi) - \sum_{|j|=n+3, j_{n+2}=\ell} u_j(x_0; -is) \right| \leq C (\theta + ca)^n e^{C|\text{Re}(s)| (1 + \|\varphi\|_{\Gamma_0} + \|\nabla \varphi\|_{\Gamma_0})} \left( |s| + \|\nabla \varphi\|_{\Gamma_0} \right) \|h\|_{\Gamma_0} + \|h\|_{\Gamma_0} \right) \cdot \ (4.6)$$

(b) For any integers $p \geq 1$ and $n \geq 1$, any $\xi \in \Sigma_+^\ell$ with $\xi_0 = \ell$ and any $s \in \mathbb{C}$ with $\text{Re}(s) \geq s_0 - a$ we have

$$\left|\left( L_n^s \mathcal{M}_{n,s}(\cdot) \mathcal{G}_s \tilde{v}_s \right) (\xi) - \sum_{|j|=n+3, j_{n+2}=\ell} u_j(\cdot; -is) \right|_{\Gamma_\ell} \leq C (\theta + ca)^n e^{C|\text{Re}(s)| (1 + \|\varphi\|_{\Gamma_0} + \|\nabla \varphi\|_{\Gamma_0})} \left( |s| + \|\nabla \varphi\|_{\Gamma_0,i} + \|\nabla \varphi\|_{\Gamma_0,i+1} \right) \|h\|_{\Gamma_\ell} \cdot \ (4.7)$$

In this section we deal with part (a). The proof of part (b) is given in Section 5 below.

Proof of Theorem 3(a). Fix $\ell$, $x_0 \in \Gamma_\ell$ and $\xi \in \Sigma_+^\ell$ with $\xi_0 = \ell$. Then for any $s \in \mathbb{C}$ and $n \geq 1$, using Sect. 4.1 in [15], we get

$$u_{(j_1, j_2, \ldots, j_{n+1}, \ell)}(x_0, -is) = (-1)^{n+2} e^{-s[\varphi(Q_0(j)) + f^+_0(x_0;j) + \ldots + f^+_{n+1}(x_0;j)]} \left( A_j(\varphi) \right) (x_0) \cdot \ (4.8)$$

where $f^+_i(x_0; j) = \|Q_i(x_0;j) - Q_{i+1}(x_0;j)\|$ ($i = 0, 1, \ldots, n + 1$), $Q_i(x_0;j)$ being the reflection points of the billiard trajectory issued from a point $y \in \Gamma_1$ in direction $\nabla \varphi(y)$ which follows the configuration $j$ for its first $n + 1$ reflections and is such that $Q_{n+2}(x_0;j) = x_0$. We assume that $x_0$ and $j$ are such that the segment $[Q_{n+1}(x_0;j), x_0]$ does not intersect the interior of $K_\ell$. Then there is exactly one such trajectory.

On the other hand,

$$\left( L_n^s \mathcal{M}_{n,s}(\cdot) \mathcal{G}_s \tilde{v}_s \right) (\xi)$$

$$= (-1)^n \sum_{\sigma^n \eta = \xi} e^{-s f_n(\eta) + \tilde{g}_n(\eta)} \left( \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s \right) (\eta) = (-1)^n \sum_{\sigma^n \eta = \xi} e^{-s f_n(\eta) + \tilde{g}_n(\eta)} \sum_{\sigma \zeta = \eta} e^{-\phi(x_0; \sigma^{n+1}e(\zeta), s) - h(\sigma^{n+1}e(\zeta), s) - \phi(x_0; \sigma^{n+1}e(\zeta), s) - h(\sigma^{n+1}e(\zeta), s)} \left( \mathcal{G}_s \tilde{v}_s \right) (\zeta)$$

$$= (-1)^n \sum_{\sigma^n \eta = \xi} e^{-s f_n(\eta) + \tilde{g}_n(\eta)} \sum_{\sigma \zeta = \eta} e^{-\phi(x_0; \sigma^{n+1}e(\zeta), s) - h(\sigma^{n+1}e(\zeta), s) - \phi(x_0; \sigma^{n+1}e(\zeta), s) - h(\sigma^{n+1}e(\zeta), s)}$$

$$\times \sum_{\sigma \mu = \zeta} e^{-\phi(\mu, s) + h(\mu, s) - \phi(\mu, s) - h(\mu, s)} \tilde{v}_s (\mu)$$

$$= (-1)^n \sum_{\sigma^{n+2} \mu = \xi, \mu_0 = 1} e^{-s f_{n+2}(\mu) + \tilde{g}_{n+2}(\mu)} W^{(n+2)}(x_0; \mu, s) \cdot \ (4.9)$$

where the function

$$W^{(n+2)}(x_0; \cdot, \cdot) = W_{1,\ell}^{(n+2)}(x_0; \cdot, \cdot) : \Sigma_A^+ \times \mathbb{C} \rightarrow \mathbb{C}$$
is defined by $W^{(n+2)}(x_0; \mu, s) = 0$ when $\mu_0 \neq 1$ or $\mu_{n+2} \neq \ell$ and

$$W^{(n+2)}(x_0; \mu, s) = e^{-\phi^-(x_0; \sigma^{n+1} e(\sigma \mu), s)} - \chi(\sigma^{n+1} e(\sigma \mu), s) - \phi^+(\mu, s) + \chi(e(\mu), s) - s \varphi(Q_0(\mu)) h(Q_0(\mu)) \quad (4.10)$$

whenever $\mu_0 = 1$ and $\mu_{n+2} = \ell$. It follows from (4.9) that

$$(L_{n+2}^{\mu} \mathcal{M}_{n,s} g_s \mathcal{G}_s) (\xi) = (-1)^n \left( L_{n+2}^{\mu} W^{(n+2)}(x_0; \cdot, s) \right) (\xi). \quad (4.11)$$

Clearly, in (4.9) the summation is over sequences

$$\mu = (1, j_1, j_2, \ldots, j_{n+1}, \ell, \xi, \xi_2, \ldots) = (j, \xi), \quad (4.12)$$

with $\mu_{n+2} = \ell$, where $j = (1, j_1, j_2, \ldots, j_{n+1}, \ell)$.

Write for convenience

$$W^{(n+2)}(x_0; \mu, s) = e^{w(x_0; \mu, s)} e^{-s \varphi(Q_0(\mu)) h(Q_0(\mu))}, \quad (4.13)$$

where

$$w(x_0; \mu, s) = -\phi^-(x_0; \sigma^{n+1} e(\sigma \mu), s) - \chi(\sigma^{n+1} e(\sigma \mu), s) - \phi^+(\mu, s) + \chi(e(\mu), s). \quad (4.14)$$

It follows from Propositions 1 and 2 that there exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$|f(\sigma^n e(\xi)) - f_i^+(\xi)| \leq C \alpha^n, \quad |g(\sigma^n e(\xi)) - g_i^+(\xi)| \leq C \|\nabla \varphi\|_{\Gamma; (1)} \alpha^n$$

for all $\xi \in \Sigma_A$ and all integers $n \geq 1$, so

$$\phi^+(\mu, s) = (|s| + \|\nabla \varphi\|_{\Gamma; (1)}) O(\alpha^n) + \sum_{i=0}^{n+1} \left( -s \left[ f(\sigma^i e(\mu)) - f_i^+(\mu) \right] + [g(\sigma^i e(\mu)) - g_i^+(\mu)] \right).$$

Thus, using the definitions of $\tilde{f}, \tilde{g}$ and $\chi$ and the fact that $\chi(\sigma^{n+2} e(\mu), s) = \chi(\sigma^{n+1} e(\sigma \mu), s) + |s| O(\alpha^n)$, we get

$$-s[f_i^+(\mu) + f_1^+(\mu) + \ldots + f_{n+1}^+(\mu)] + [g_0^+(\mu) + g_1^+(\mu) + \ldots + g_{n+1}^+(\mu)]$$

$$= (s + \|\nabla \varphi\|_{\Gamma; (1)}) O(\alpha^n) - \phi^+(\mu, s) - s[f(\sigma \mu) + f(\sigma e(\mu)) + \ldots + f(\sigma^{n+1} e(\mu))]$$

$$+ [g(e(\mu)) + g(\sigma e(\mu)) + \ldots + g(\sigma^{n+1} e(\mu))]$$

$$= (s + \|\nabla \varphi\|_{\Gamma; (1)}) O(\alpha^n) - \phi^+(\mu, s) - s[\tilde{f}(\sigma \mu) + \tilde{f}(\sigma e(\mu)) + \ldots + \tilde{f}(\sigma^{n+1} e(\mu))]$$

$$+ [\tilde{g}(e(\mu)) + \tilde{g}(\sigma e(\mu)) + \ldots + \tilde{g}(\sigma^{n+1} e(\mu))]$$

$$- s[\chi(1(\mu)) + \chi(\sigma e(\mu)) + \ldots + \chi(\sigma^{n+1} e(\mu))]$$

$$+ s[\chi(1(\sigma \mu)) + \chi(\sigma^2 e(\mu)) + \ldots + \chi(\sigma^{n+2} e(\mu))]$$

$$+ [\chi(2(\sigma e(\mu)) + \chi(2(\sigma^2 e(\mu)) + \ldots + \chi(2(\sigma^{n+1} e(\mu)))]$$

$$- [\chi(2(\sigma \mu)) + \chi(2(\sigma^2 e(\mu)) + \ldots + \chi(2(\sigma^{n+2} e(\mu)))]$$

$$= (|s| + \|\nabla \varphi\|_{\Gamma; (1)}) O(\alpha^n) - \phi^+(\mu, s) - s \tilde{f}(\mu) - \tilde{g}(\mu) + \chi(e(\mu), s) - \chi(\sigma^{n+1} e(\sigma \mu), s).$$

Now, fix a moment $n \geq 1$ and $\mu$ as in (4.12), and set $\eta = \sigma^{n+1} e(\sigma \mu)$. Then we have

$$\eta = \sigma^{n+1} e(\sigma \mu) = (\ldots, *, \mu_1, \mu_2, \ldots, \mu_{n+1}; \mu_{n+2} = \ell, \mu_{n+3}, \ldots), \quad (4.15)$$

and as for $\phi^+$ one gets

$$\phi^-(x_0; \eta, s) = (|s| + \|\nabla \varphi\|_{\Gamma; (1)}) O(\alpha^n) - s \sum_{i=-1}^{n-1} [f(\sigma^i \eta) - f_i^-(x_0; \eta)] + \sum_{i=-1}^{n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)].$$
From these estimates and (4.14) one derives that
\[w(x_0; \mu, s) = s f_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - \phi^-(x_0; \eta, s) - s \sum_{i=0}^{n+1} f_i^+(\mu) + \sum_{i=0}^{n+1} g_i^+(\mu) + (|s| + \|\nabla \varphi\|_{\Gamma, (1)}) O(\alpha^n)\]

\[= s f_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s u(x_0; \mu) + v(x_0; \mu) + (|s| + \|\nabla \varphi\|_{\Gamma, (1)}) O(\alpha^n),\]

(4.16)

where
\[u(x_0; \mu) = - \sum_{i=0}^{n+1} [f(\sigma^i \eta) - f_i^-(x_0; \eta)] + \sum_{i=0}^{n+1} f_i^+(\mu)\]

and
\[v(x_0; \mu) = - \sum_{i=-1}^{-n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)] + \sum_{i=0}^{n+1} g_i^+(\mu)\]

We will show that
\[\left| u(x_0; \mu) - \sum_{i=0}^{n+1} f_i^+(x_0; j) \right| \leq \text{Const} \alpha^n, \quad (4.17)\]

and
\[\left| e^{\nu(x_0; \mu)} h(Q_0(\mu)) - (A_j(\varphi) h)(x_0) \right| \leq \text{Const} \left( \|\nabla \varphi\|_{\Gamma, (1)} \|h\|_{\Gamma, 0} + \|h\|_{\Gamma, (1)} \right) \theta^n \quad (4.18)\]

for some global constant \(\text{Const} > 0\), where
\[\theta = \sqrt{\alpha} \in (0, 1).\]

There exists a unique ray \(\gamma(y, \varphi)\) issued from a point \(y = y_n(x_0; \mu) \in \Gamma_1\) in direction \(\nabla \varphi(y)\), following the configuration \(\mu\), and such that its \((n + 2)\)nd reflection point is \(x_0\). That is, if \(\tilde{Q}_i(x_0; \mu) (0 \leq i)\) are the consecutive reflection points of \(\gamma(y, \varphi)\), then \(\tilde{Q}_{n+2}(x_0; \mu) = x_0\). Again it is essential to consider the case when the segment \([\tilde{Q}_{n+1}(x_0; \mu), x_0]\) does not intersect the interior of \(K_k\).

Before we continue, let us make a few simple (however essential) remarks concerning the sequences of reflection points
\[Q_0(\mu) \in \Gamma_1 = \Gamma_{\mu_0}, Q_1(\mu) \in \Gamma_{\mu_1}, \ldots, Q_{n+1}(\mu) \in \Gamma_{\mu_{n+1}}, Q_{n+2}(\mu) \in \Gamma_{\mu_{n+2}} = \Gamma_\ell, \ldots, \quad (4.19)\]
\[\tilde{Q}_0(x_0; \mu) \in \Gamma_1 = \Gamma_{\mu_0}, \tilde{Q}_1(x_0; \mu) \in \Gamma_{\mu_1}, \ldots, \tilde{Q}_{n+1}(x_0; \mu) \in \Gamma_{\mu_{n+1}}, \tilde{Q}_{n+2}(x_0; \mu) \in \Gamma_\ell, \ldots, \quad (4.20)\]
\[\ldots, P_{\eta-n-1}(\eta) \in \Gamma_{\eta-n-1} = \Gamma_{\mu_1}, \ldots, P_{-1}(\eta) \in \Gamma_{\eta-1} = \Gamma_{\mu_{n+1}}, P_0(\eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_\ell, \ldots, \quad (4.21)\]
\[\ldots, \tilde{P}_{\eta-n-1}(x_0; \eta) \in \Gamma_{\eta-n-1} = \Gamma_{\mu_1}, \ldots, \tilde{P}_{-1}(x_0; \mu) \in \Gamma_{\eta-1} = \Gamma_{\mu_{n+1}}, \tilde{P}_0(x_0; \eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_\ell. \quad (4.22)\]

It is clear that the sequences (4.19) and (4.20) ‘start’ from the same unstable manifold (determined by \(\varphi = \text{const}\)), therefore by Proposition 1 there exist constants \(C > 0\) and \(\alpha \in (0, 1)\) such that
\[\|Q_i(\mu) - \tilde{Q}_i(x_0; \mu)\| \leq C \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2. \quad (4.23)\]

Similarly, the right ends of sequences (4.21) and (4.22) determine points in the same unstable manifold, so these sequences ‘converge backwards’, i.e.
\[\|P_i(\eta) - \tilde{P}_i(x_0; \eta)\| \leq C \alpha^{|i|}, \quad i \leq 0. \quad (4.24)\]
On the other hand, notice that the sequences (4.19) and (4.21) continue indefinitely to the right following the same patterns. Thus, these sequences ‘converge forwards’. More precisely, using Proposition 1 again, we have

$$\|Q_i(\mu) - P_{i-n-2}(\eta)\| \leq C \alpha^i, \quad 1 \leq i,$$

(4.25)

Similarly, the sequences (4.20) and (4.22) ‘converge forwards’ to \(\tilde{Q}_{n+2}(x_0;\mu) = \tilde{P}_0(x_0;\eta) = x_0\), namely

$$\|\tilde{Q}_i(x_0;\mu) - \tilde{P}_{i-n-2}(x_0;\eta)\| \leq C \alpha^i, \quad 1 \leq i \leq n + 2,$$

(4.26)

It now follows from (4.2) and (4.24) that, taking a larger constant \(C > 0\) if necessary,

$$|g(\sigma^i(\eta)) - g_i^-(x_0;\eta)| = \left| \frac{1}{N-1} \ln \frac{G_{\eta,i}(P_{i+1}(\eta))}{G_{\eta,i}(Q_i(x_0;\mu))} \right| \leq C \alpha^{|i|}$$

(4.27)

for all \(i \leq 0\). In particular, the second series in (4.5) is absolutely convergent, and by (4.27) and Proposition 3,

$$|v(x_0;\mu)| \leq \text{Const}$$

for some global constant \(\text{Const} > 0\).

Next, setting

$$\tilde{a}_i(x_0;\mu) = \frac{1}{N-1} \ln \left( \frac{G_{\mu,i}(\tilde{Q}_{i+1}(x_0;\mu))}{G_{\mu,i}(\tilde{Q}_i(x_0;\mu))} \right),$$

(4.28)

and using (4.23) and Proposition 2, one gets

$$|\tilde{a}_i(x_0;\mu) - g^+_i(\mu)| = \left| \frac{1}{N-1} \ln \frac{G_{\mu,i}(\tilde{Q}_{i+1}(x_0;\mu))}{G_{\mu,i}(\tilde{Q}_i(x_0;\mu))} - \ln \frac{G_{\mu,i}(P_{i+1}(\mu))}{G_{\mu,i}(Q_i(x_0;\mu))} \right|$$

$$\leq C \|\nabla \varphi\|_{\Gamma,(1)} \left( \|\tilde{Q}_i(x_0;\mu) - Q_i(\mu)\| + \|\tilde{Q}_{i+1}(x_0;\mu) - Q_{i+1}(\mu)\| \right)$$

$$\leq C \|\nabla \varphi\|_{\Gamma,(1)} \alpha^{n+2-i}. \quad (4.29)$$

for all \(i = 0, 1, \ldots, n + 2\), where \(C > 0\) is some sufficiently large global constant.

Next, notice that by construction

$$\varphi_{n,i} = (\varphi_{n-n-2,\mu(1),\ldots,\mu_{n+2+i})} + \text{const}, \quad -n - 1 \leq i \leq -1.$$  

Thus, by (2.2), (4.2) and (4.25), for all \(-n - 1 \leq i \leq -1\) we have

$$|g^+_{n+2+i}(\mu) - g(\sigma^i\eta)| = \left| \frac{1}{N-1} \ln \frac{G_{\mu,n+2+i}(Q_{n+2+i+1}(\mu))}{G_{\mu,n+2+i}(Q_{n+2+i}(\mu))} - \ln \frac{G_{\eta,i}(P_{i+1}(\eta))}{G_{\eta,i}(P_i(\eta))} \right|$$

$$\leq \text{Const} \left( \|\nabla \varphi\|_{\Gamma,(1)} + \|\nabla (\varphi_{n-n-2,\mu(1),\ldots,\mu_{n+2+i})\|_{\Gamma,(1)} \right) \alpha^{n+2+i} + \text{Const} \alpha^{n+2+i}$$

$$\leq \text{Const} \|\nabla \varphi\|_{\Gamma,(1)} \alpha^{n+2+i}$$

(4.30)

In a similar way (4.26) implies

$$|\tilde{a}_{n+2+i}(x_0;\mu) - g_i^-(x_0;\eta)| \leq C \|\nabla \varphi\|_{\Gamma,(1)} \alpha^{n+2+i}, \quad -n - 1 \leq i \leq -1.$$  

(4.31)
To prove (4.18), notice that \( \chi_0((\nabla \varphi(x_0), \nu(\tilde{Q}_{n+1}(x_0; \mu))) = 1 \) (see Sect. 2 for the choice of the function \( \chi_0 \)), so

\[
(A_j(\varphi)h)(x_0) = \Lambda_{\varphi,j}(x_0) h(\tilde{Q}_0(x_0; \mu)) .
\]

The definition of \( \Lambda_{\varphi,j} \) gives

\[
\ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu)) = \ln \left( \Lambda_{\varphi,\mu_1,\ldots,\mu_{n+1}}(\tilde{Q}_{n+2}(x_0; \mu)) \Lambda_{\varphi,\mu_1,\ldots,\mu_n}(\tilde{Q}_{n+1}(x_0; \mu)) \cdots \Lambda_{\varphi,\mu_0}(\tilde{Q}_0(x_0; \mu)) \right)
= \frac{1}{N-1} \ln \left( \frac{G_{\mu,n+1}^2(\tilde{Q}_{n+2}(x_0; \mu))}{G_{\mu,n+1}(\tilde{Q}_{n+1}(x_0; \mu))} \right) + \frac{1}{N-1} \ln \left( \frac{G_{\mu,n}(\tilde{Q}_{n+1}(x_0; \mu))}{G_{\mu,n}(\tilde{Q}_n(x_0; \mu))} \right)
+ \ldots + \frac{1}{N-1} \ln \left( \frac{G_{\mu_0}^2(\tilde{Q}_1(x_0; \mu))}{G_{\mu_0}(\tilde{Q}_0(x_0; \mu))} \right) = \sum_{i=0}^{n+1} \bar{a}_i(x_0; \mu) .
\]

Next, assume for simplicity that \( n \) is odd (the other case is similar), and set \( m = (n + 1)/2 \). Using (4.27) – (4.31) we get

\[
\ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu)) - v(x_0; \mu) = \sum_{i=0}^{n+1} \bar{a}_i(x_0; \mu) + \sum_{i=-1}^{-n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)] - \sum_{i=0}^{n+1} g_i^+(\mu)
= \sum_{i=-m-1}^{-n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)] + \sum_{i=0}^{m} \bar{a}_i(x_0; \mu) - g_i^+(\mu)
+ \sum_{i=m+1}^{n+1} [\bar{a}_i(x_0; \mu) - g_{-n-2}^-(x_0; \eta)] + \sum_{i=-1}^{-m} [g(\sigma^i \eta) - g_{n+2+i}^+(\mu)]
= O(\alpha^m) \|\nabla \varphi\|_{\Gamma,(1)} = O(\theta^m) \|\nabla \varphi\|_{\Gamma,(1)} .
\]

Since by (4.23),

\[
|h(\tilde{Q}_0(x_0; \mu)) - h(Q_0(\mu))| = \|h\|_{\Gamma,1} O(\alpha^n) ,
\]

the above gives

\[
\left| e^{v(x_0; \mu)} h(Q_0(\mu)) - (A_j(\varphi)h)(x_0) \right| \leq \left| e^{v(x_0; \mu)} - e^{\ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu))} \right| \|h(Q_0(\mu))\|
+ \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu)) \|h(Q_0(\mu)) - h(\tilde{Q}_0(x_0; \mu))\|
\leq e^{\max\{v(x_0; \mu), \ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu))\}} \|v(x_0; \mu) - \ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0; \mu))\| \|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)} O(\alpha^n)
\leq \text{Const} \left( \|\nabla \varphi\|_{\Gamma,(1)} \|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)} \right) \theta^n ,
\]

which proves (4.18).

Similarly to (4.27) one gets

\[
|f(\sigma^i(\eta)) - f_i^-(x_0; \eta)| \leq \text{Const} \alpha^{|i|} ,
\]

and also

\[
|f_i^+(\mu) - f_i^+(x_0; j)| = \|Q_i(\mu) - Q_{i+1}(\mu)\| - \|Q_i(x_0; j) - Q_{i+1}(x_0; j)\| \leq \text{Const} \alpha^{n+2-|j|} .
\]

Combining these two estimates yields (4.17).
Next, using the notation from the beginning of this proof, notice that for any \( \mu \) as in (4.11) we have \( Q_i(x_0;j) = \tilde{Q}_i(x_0;\mu) \) for all \( i = 0,1,\ldots,n+2 \) and therefore \( f_i^+(x_0;j) = \|\tilde{Q}_i(x_0;\mu) - \tilde{Q}_{i+1}(x_0;\mu)\| \) for all \( i = 0,1,\ldots,n+1 \). (This has been used already in the proof of (4.17).)

Define the function
\[
\tilde{W}^{(n+2)}(x_0;\cdot,\cdot) : \Sigma_A^+ \times \mathbb{C} \longrightarrow \mathbb{C}
\]
by \( \tilde{W}^{(n+2)}(x_0;\mu,s) = 0 \) when \( \mu_0 \neq 1 \) or \( \mu_{n+2} \neq \ell \) and
\[
\tilde{W}^{(n+2)}(x_0;\mu,s) = e^{sf_{n+2}(\mu) - \tilde{g}_{n+2}(\mu)} - e^{s\tilde{Q}_i(x_0;\mu)} - s\sum_{i=0}^{n+1} \|\tilde{Q}_i(x_0;\mu) - \tilde{Q}_{i+1}(x_0;\mu)\|
\times \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x_0;\mu)) h(\tilde{Q}_0(x_0;\mu)) ,
\]
whenever \( \mu_0 = 1 \) and \( \mu_{n+2} = \ell \), where \( j = j^{(n+2)}(\mu) \) is defined by (4.12).

Using (4.8), we can now write
\[
\sum_{|j|=n+3,j_0=1,j_{n+2}=\ell} u_j(x_0, -1s) = (-1)^{n+2} \sum_{|j|=n+3,j_0=1,j_{n+2}=\ell} e^{-s [\varphi(Q_0(i)) + f^+_n(jx_0) + \cdots + f^+_1(x_0j)]} (A_j(\varphi)h)(x_0)
\]
\[
= (-1)^n \sum_{\sigma^{n+2}\mu = \xi, \mu_0 = 1} e^{-sf_{n+2}(\mu) - \tilde{g}_{n+2}(\mu)} \tilde{W}^{(n+2)}(x_0;\mu,s)
\]
\[
= (-1)^n \left( L_{-s\tilde{f} + \tilde{g}}^{n+2} \tilde{W}^{(n+2)}(x_0;\cdot,\cdot) \right)(\xi) .
\]

This and (4.11) imply
\[
\left| \left( L_n^p \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s \right) (\xi) - \sum_{|j|=n+3,j_0=1,j_{n+2}=\ell} u_j(x_0, -1s) \right|
\]
\[
= \left| L_{-s\tilde{f} + \tilde{g}}^{n+2} \left( W^{(n+2)}(x_0;\cdot,\cdot) - \tilde{W}^{(n+2)}(x_0;\cdot,\cdot) \right)(\xi) \right| .
\]

On the other hand, the so called Basic Inequality for Ruelle transfer operators (see e.g. [PP]) gives that there exists a global constant \( C > 0 \) (depending only on \( K \)) such that
\[
\left| L_{-s\tilde{f} + \tilde{g}}^p u \right|_{\infty} \leq C e^{C|R_e(s)|} e^{p \Pr(-R_e(s) \tilde{f} + \tilde{g})} \|u\|_{\infty} , \quad p \geq 0 , \quad s \in \mathbb{C} ,
\]
for any continuous function \( u : \Sigma_A^+ \longrightarrow \mathbb{C} \).

Fix for a moment \( s \in \mathbb{C} \). We will now estimate \( \| W^{(n+2)}(x_0;\cdot,\cdot) - \tilde{W}^{(n+2)}(x_0;\cdot,\cdot) \|_{\infty} \) as a function of \( \mu \in \Sigma_A^+ \). According to the definitions of \( W^{(n+2)} \) and \( \tilde{W}^{(n+2)} \), it is enough to consider \( \mu \in \Sigma_A^+ \) with \( \mu_0 = 1 \) and \( \mu_{n+2} = \ell \). For such \( \mu \), using (4.13), (4.16), (4.32), (4.33) and (4.35), we
have

\[
W^{(n+2)}(x_0; \mu, s) - \widetilde{W}^{(n+2)}(x_0; \mu, s) = \\
= \left| e^{s \varphi(Q_0(\mu))} e^{-s \varphi(Q_0(\mu))} h(Q_0(\mu)) \\
- e^{s \varphi(Q_0(\mu))} e^{-s \varphi(Q_0(\mu))} h(Q_0(\mu)) \right| \\
- s \sum_{i=0}^{n+1} \| \tilde{Q}_i(x_0; \mu) - \tilde{Q}_{i+1}(x_0; \mu) \| + s \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) h(Q_0(\mu)) |h(Q_0(\mu))| \\
= \left| e^{s \varphi(Q_0(\mu))} e^{-s \varphi(Q_0(\mu))} h(Q_0(\mu)) \\
- e^{s \varphi(Q_0(\mu))} e^{-s \varphi(Q_0(\mu))} h(Q_0(\mu)) \right| \\
\times \left| e^{g_{n+2}(\mu) - g_{n+2}(\mu)} - s \varphi(Q_0(\mu)) - s \varphi(Q_0(\mu)) \right| \\
\times \left| \sum_{i=0}^{n+1} \tilde{f}_i(x_0; \mu) + \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) \right| \\
- h(Q_0(\mu)) \right| . \tag{4.38}
\]

To estimate (4.38), first notice that by (4.15) and Proposition 1,

\[|f(\sigma^i \mu) - f(\sigma^{i-(n+2)} \eta)| \leq \text{Const} \alpha^i, \quad 0 \leq i \leq n + 2.\]

Using this, (4.24), (4.26) and Proposition 1 again, one gets

\[
\left| \tilde{f}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{f}_i(x_0; \mu) \right| \leq \text{Const} \left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) \right| \\
\leq \text{Const} \left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) \right| \\
\leq \text{Const} \left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) \right| \tag{4.39}
\]

for some global constant \(\text{Const} > 0\). Similarly, it follows from (4.15), (4.29) and (4.30) that

\[
\left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \mu) \right| \leq \text{Const} \left| \tilde{f}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{f}_i(x_0; \mu) \right| \tag{4.40}
\]

Next, notice that

\[
\left| e^{s + \| \nabla \varphi \|_{\Gamma,(1)} O(\theta^n)} - 1 \right| \leq \text{Const} e^{\text{Const} (\| \text{Re}(s) \|_{\Gamma,(1)} + \| \nabla \varphi \|_{\Gamma,(1)}) \left( |s| + \| \nabla \varphi \|_{\Gamma,(1)} \right) \theta^n}. \tag{4.41}
\]

Using the latter, (4.17), (4.18), (4.39) and (4.40) in (4.38) yields

\[
W^{(n+2)}(x_0; \mu, s) - \widetilde{W}^{(n+2)}(x_0; \mu, s) \leq \text{Const} e^{\text{Const} (\| \text{Re}(s) \|_{\Gamma,(1)} + \| \nabla \varphi \|_{\Gamma,(1)} \left( |s| + \| \nabla \varphi \|_{\Gamma,(1)} \right) \theta^n}. \tag{4.42}
\]
Thus, choosing the global constant $C > 0$ sufficiently large, combining the above with (4.37) gives
\[
\left| L^{n+2}_{-s\tilde{f}+\tilde{g}} \left( W^{(n+2)}(x_0;\cdot,s) - \tilde{W}^{(n+2)}(x_0;\cdot,s) \right) (\xi) \right| 
\leq \left( e^{C(\|\Re(s)\| + \|\varphi\|) + \|\nabla \varphi\| + \|\gamma\|)} + \|h\| \right) \|h\| \cdot \|\gamma\| \cdot e^{C(\|\Re(s)\| + \|\varphi\|) + \|\nabla \varphi\| + \|\gamma\|)} \times \left( e^{\Pr(-\Re(s))(\tilde{f}+\tilde{g})} \right)^{n+2}. \tag{4.41}
\]

Next notice that
\[
\frac{d}{ds} \Pr(-s\tilde{f}+\tilde{g}) = -\int_{\Sigma^+} \tilde{f} d\nu = -\int_{\Sigma^+} f d\nu = -c_0 < 0,
\]
where $\nu$ is the equilibrium state of $(-s_0\tilde{f}+\tilde{g})$. Recall that $\Pr(-s_0\tilde{f}+\tilde{g}) = 0$, so $\Pr(-\Re(s)\tilde{f}+\tilde{g}) < 1$ for $\Re(s) > s_0$. Now assume $s_0 - a \leq \Re(s)$ for some constant $a > 0$. Then
\[
e^{\Pr(-\Re(s)(\tilde{f}+\tilde{g}))} = 1 + c_0(s_0 - \Re(s)) + O((\Re(s) - s_0)^2) \leq 1 + c_1 a
\]
for some constant $c_1 > 0$. Thus,
\[
e^{\Pr(-\Re(s)(\tilde{f}+\tilde{g}))} \theta \leq \theta + ca \tag{4.42}
\]
for some global constant $c = c_1 \theta > 0$. Combining this with (4.41), completes the proof of (4.6).

5. Estimates of the derivatives

In this section we prove Theorem 3(b). Throughout we assume that $p \geq 1$.

For any $x \in \Gamma_\ell$ close to $x_0$ and any $\eta \in \Sigma_0$ with $\eta_0 = \ell$ define the points $\tilde{P}_j(x;\eta)$ and the functions $f^-_j(x;\eta)$, $g^-_j(x;\eta)$, $\phi^-_j(x;\eta,s)$, etc., as in the beginning of Section 4 replacing the point $x_0$ by $x$. We will assume that the segment $[\tilde{P}_-1(x_0;\eta),x_0]$ has no common points with the interior of $K_\ell$ and $x$ is close enough to $x_0$ so that the same holds with $x_0$ replaced by $x$.

By Proposition 4 there exists a unique phase function $\psi_\eta$ (also depending on $x_0$) in a neighborhood $U$ of $x_0$ in $\Gamma_\ell$, such that $\psi_\eta(x_0) = 0$ and the backward trajectory $\gamma_-(x,\nabla \psi_\eta(x))$ of any point $x \in U$ with $\psi_\eta(x) = 0$ has an itinerary $(\ldots,\eta_{-j},\ldots,\eta_{-1},\eta_0)$, that is
\[
\nabla \psi_\eta(x) = \frac{\tilde{P}_0(x;\eta) - \tilde{P}_1(x;\eta)}{\|\tilde{P}_0(x;\eta) - \tilde{P}_1(x;\eta)\|}
\]
for any $x \in C_{\psi_\eta} \cap U$. (Notice that in general $\psi_\eta$ is different from the functions $\varphi_{\eta,j}$ defined in the beginning of Sect. 4.) For any $i < 0$, denoting $J = (\eta_i,\eta_{i+1},\ldots,\eta_{-1},\eta_0)$, we can write $\psi_\eta = (\psi_{\eta,i},J)$ for some phase function $\psi_{\eta,i}$ (defined on some naturally defined open subset $V_{\eta,i}$ of $\mathbb{R}^N$) satisfying Ikawa’s condition (P) on $\Gamma_{\eta,i}$. We then have
\[
\tilde{P}_i(x;\eta) = X^{-1}(x,\nabla(\psi_{\eta,i},J)).
\]
As in the beginning of Sect. 4 (see (4.2) there) one derives that there exists a global constant $C_p > 0$ such that
\[
\|\psi_{\eta,i}\|_{(p)}(V_{\eta,i} \cap B_0) \leq C_p
\]
for all $\eta$ and $i < 0$. Using (2.4) in Proposition 2 with $\varphi = \psi_{\eta,m}$ for some $m \geq i$ and replacing $C_p$ with a larger global constant if necessary, we get
\[
\|\tilde{P}_i(\cdot;\eta)\|_{\Gamma,p}(x) \leq C_p \alpha^{|i|}, \quad j < 0. \tag{5.1}
\]
Similarly, for any $\mu \in \Sigma_0^+$ with $\mu_0 = 0$ and $\mu_{n+2} = k$ we have
\[
\|\tilde{Q}_i(\cdot;\eta)\|_{\Gamma,p}(x) \leq C_p \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2, \tag{5.2}
\]
and
\[ \|\tilde{Q}_i(\cdot; \mu) - \tilde{P}_{i-n-2}(\cdot; \eta)\|_{\Gamma_p(x)} \leq C_p \alpha^i, \quad 0 \leq i \leq n + 2. \]  
(5.3)

Next, recall the function \( \Lambda_{\sigma} \) from the beginning of this section. By Proposition 2,
\[ \|\nabla \varphi\|_{\Gamma_p} \leq C_p \|\nabla \varphi\|_{\Gamma_{(p)}} \]  
(5.4)
for any finite admissible configuration \( J \).

Since for any \( i < 0 \) we have
\[ g_i^{-}(x; \eta) = \ln \Lambda_{\psi_i}(\tilde{P}_{i+1}(x; \eta)) \],

it follows from (5.1), (5.2), (5.3) and Proposition 3 that for any \( p \geq 1 \) there exists a constant \( C_p' > 0 \) (independent of \( x, n, \eta \) and \( \mu \)) such that
\[ \|g_i^{-}(\cdot; \eta)\|_{\Gamma_p(x)} \leq C_p' \alpha^{|i|}, \quad i < 0. \]  
(5.5)
Similarly, according to (4.28) and Proposition 2, we can choose \( C_p'' \) so that
\[ \|\tilde{a}_i(\cdot; \mu)\|_{p(x)} \leq C_p'' \|\nabla \varphi\|_{\Gamma_{(p)}} \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2, \]  
(5.6)
and as in the proof of (4.30),
\[ \|\tilde{a}_i(\cdot; \mu) - g_{i-n-2}(\cdot; \eta)\|_{p(x)} \leq C_p'' \|\nabla \varphi\|_{\Gamma_{(p+1)}} \alpha^i, \quad 0 \leq i \leq n + 2. \]  
(5.7)

Next, given \( x \) as above, \( \mu \) and \( n \) with \( \mu_{n+2} = \ell \), define \( W^{(n+2)}(x; \mu, s) \) by (4.10), \( \eta \) by (4.15) and \( \check{W}^{(n+2)}(x; \mu, s) \) by (4.35) replacing \( x_0 \) by \( x \). We will estimate the derivatives of
\[ W^{(n+2)}(x; \mu, s) - \check{W}^{(n+2)}(x; \mu, s) \]  
(5.8)
with respect to \( x \).

First look at the first derivatives \( D_b[W^{(n+2)}(\cdot; \mu, \eta,s) - \check{W}^{(n+2)}(\cdot; \mu, \eta,s)](x) \), where \( b \in S_x \Gamma \). Writing
\[ \phi^{-}(x; \eta, s) = -s \phi_1^{-}(x; \eta) + \phi_2^{-}(x; \eta), \]

notice that for any \( x, x' \in \Gamma_\ell \) (close to \( x_0 \) we have
\[ \phi_1^{-}(x; \eta) - \phi_1^{-}(x'; \eta) = -\psi_\eta(x) + \psi_\eta(x'), \]
so \( D_b(\phi^{-}(\cdot; \eta))(x) = D_b(\psi_\eta(x)) \), and therefore by (4.35) and (4.15)
\[ D_b w(\cdot; \mu, s)(x) = -s D_b \psi_\eta(x) + \sum_{i=-1}^{-\infty} D_b(\tilde{g}_i^{-}(\cdot; \eta))(x). \]  
(5.9)
We will see later that the latter series is uniformly convergent.

Next, using the notation
\[ \mathbf{j} = (\mu_0, \mu_1, \mu_2, \ldots, \mu_{n+2}) \]
and
\[ z(x; \mu, s) = s \tilde{f}_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s \varphi(\tilde{Q}_0(x; \mu)) - s \sum_{i=0}^{n+1} \|\tilde{Q}_i(x; \mu) - \tilde{Q}_{i+1}(x; \mu)\| \]
\[ = s \tilde{f}_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s (\varphi_{\mu_0})_{\mathbf{j}}(x), \]
it follows from (4.38) that
\[
W^{(n+2)}(\cdot; \mu, s) - \tilde{W}^{(n+2)}(\cdot; \mu, s)(x) \\
= e^{w(x; \mu, s)-s \varphi(Q_0(\mu))} h(Q_0(\mu)) - e^{z(x; \mu, s)} \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu)) h(\tilde{Q}_0(x; \mu)) \\
= [e^{w(x; \mu, s)-s \varphi(Q_0(\mu))} - e^{z(x; \mu, s)+\ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu))}] h(Q_0(\mu)) \\
+ e^{z(x; \mu, s)} \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu)) [h(Q_0(\mu)) - h(\tilde{Q}_0(x; \mu))] \\
= (I)(x) + (II)(x) ,
\]
where
\[
(I)(x) = [e^{w(x; \mu, s)-s \varphi(Q_0(\mu))} - e^{z(x; \mu, s)+\ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu))}] h(Q_0(\mu)) ,
\]
and
\[
(II)(x) = e^{z(x; \mu, s)} \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu)) [h(Q_0(\mu)) - h(\tilde{Q}_0(x; \mu))] .
\]

Let \( \mathcal{O} \) be a small compact connected neighbourhood of \( x \) in \( \Gamma \). Fix \( \mu, s, n \) and \( \eta \) with (4.15) temporarily, and set
\[
\gamma(y) = w(y; \mu, s) - s \varphi(Q_0(\mu)) , \quad \delta(y) = z(x; \mu, s) + \ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x; \mu)) , \quad y \in \mathcal{O} .
\]

To estimate \( I \) first notice that by (4.16), (4.17), (4.27), (4.29), (4.32), (4.33), (4.39) and (4.40)
\[
\| \gamma \|_0(\mathcal{O}) = O(|s| + |s| \| \varphi \|_{\Gamma,0} + \| \nabla \varphi \|_{\Gamma,(1)}) ,
\]
and
\[
| e^\gamma |_{\Gamma,0}(\mathcal{O}) \leq \text{Const} e^{\text{Const} |\Re(s)| (1+|\| \varphi \|_{\Gamma,0} + \| \nabla \varphi \|_{\Gamma,(1)})} .
\]

It follows from (5.6) and (4.40) that \( |\tilde{g}_{n+2}(\mu)| \leq \text{Const} \| \nabla \varphi \|_{\Gamma,(1)}. \) Combining this with the definition of \( z(x; \mu, s) \) and (4.39) implies
\[
\| z(\cdot; \mu, s) \|_{0}(\mathcal{O}) = O(|s| + |s| \| \varphi \|_{\Gamma,0} + \| \nabla \varphi \|_{\Gamma,(1)}) , \quad \| \delta \|_{0}(\mathcal{O}) = O(|s| + |s| \| \varphi \|_{\Gamma,0} + \| \nabla \varphi \|_{\Gamma,(1)}) .
\]

Next, we will estimate the derivatives of \( \gamma \) and \( \delta \). For any \( q \geq 1 \) and any \( y \in \mathcal{O} \), using (5.9), (3.1) and (5.5) we get
\[
\| \gamma \|_{\Gamma,q}(y) = \| s \phi_1^+ (\cdot; \eta) - \phi_2^- (\cdot; \eta) \|_{\Gamma,q}(y) \leq |s| \| \nabla \psi_0 \|_{\Gamma,q}(y) + \sum_{i=-1}^{\infty} \| g_i^- (\cdot; \eta) \|_{\Gamma,q}(y)
\]\n\[
\leq |s| \text{Const}_q + \text{Const}_q \sum_{i=-1}^{\infty} \alpha_i \leq \text{Const}_q (|s| + 1) ,
\]
where \( \text{Const}_q \) denotes a positive global constant depending on \( q \). Thus, for any \( q \geq 0 \),
\[
| e^\gamma |_{\Gamma,q}(\mathcal{O}) \leq \text{Const}_q e^{|\Re(s)| (1+|\| \varphi \|_{\Gamma,0} + \| \nabla \varphi \|_{\Gamma,(1)})} .
\]

Similarly, (5.4) gives
\[
\| z(\cdot; \mu, s) \|_{\Gamma,q}(y) = \| s (\varphi_{\mu_0})_j \|_{\Gamma,q}(y) \leq \text{Const}_q |s| \| \nabla \varphi \|_{\Gamma,(q)} ,
\]
while (4.31) and (5.6) imply
\[
\| \ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(\cdot; \mu)) \|_{\Gamma,q}(y) \leq \sum_{i=0}^{n+1} \| \tilde{a}_i (\cdot; \mu) \|_{\Gamma,q}(y) \leq \text{Const}_q \| \nabla \varphi \|_{\Gamma,(q)}.
\]
for any $q \geq 0$, so
\[ \| \delta \|_{\Gamma, q}(y) \leq \text{Const}_q (|s| + 1) \| \nabla \varphi \|_{\Gamma, (q)} , \quad y \in \mathcal{O}. \] (5.17)

The next step is to estimate the derivatives of $\gamma - \delta$. First notice that by Proposition 2 and (3.1) we have
\[ \| \nabla \psi - \nabla (\varphi_{\mu_0})_J \|_{\Gamma, q}(\mathcal{O}) \leq \text{Const}_q \alpha^n \| \nabla \psi - \nabla \varphi_{\mu_0} \|_{\Gamma, (q)} \leq \text{Const}_q \alpha^n \| \nabla \varphi \|_{\Gamma, (q)}. \]

Set again $m = \frac{n+1}{2}$, assuming for simplicity that $n$ is odd, and $\theta = \sqrt{\alpha} \in (0, 1)$. As in the proof of (4.18) above, for any $y \in \mathcal{O}$ and any $q \geq 1$, using (5.5), (5.6) and (5.7), we have
\[ \| \gamma - \delta \|_{\Gamma, q}(y) \leq \left\| -s \psi + \sum_{i=-1}^{\infty} g_i^- (\cdot; \eta) + s (\varphi_{\mu_0})_J - \sum_{i=0}^{n+1} \tilde{a}_i (\cdot; \mu) \right\|_{\Gamma, q}(y) \leq |s| \| \psi - (\varphi_{\mu_0})_J \|_{\Gamma, q}(y) + \sum_{i=-m-1}^{\infty} \| g_i^- (\cdot; \eta) \|_{\Gamma, q}(y) + \sum_{i=0}^{n+1} \| \tilde{a}_i (\cdot; \mu) - g_i^- (\cdot; \eta) \|_{\Gamma, q}(y) \leq \text{Const}_q \| |s| \| \nabla \varphi \|_{\Gamma, (q)} \| \nabla \psi - (\varphi_{\mu_0})_J \|_{\Gamma, q}(y) + \| \nabla \varphi \|_{\Gamma, (q+1)} \theta^n. \]

From Sect. 4, a similar estimate holds for $q = 0$. Consequently,
\[ \| e^{\delta - \gamma} \|_{\Gamma, q}(\mathcal{O}) \leq \text{Const}_q \| e^{\delta - \gamma} \|_0(\mathcal{O}) \left( \max_{1 \leq i \leq q} \| \delta - \gamma \|_{\Gamma, i}(\mathcal{O}) \right)^q \leq \text{Const}_q e^\text{Const} \| \Re(s) + \| \nabla \varphi \|_{\Gamma, (1)} \| |s| \| \nabla \varphi \|_{\Gamma, (q)} + \| \nabla \varphi \|_{\Gamma, (q+1)} \theta^n \|_{\Gamma, (1)}. \]

Finally, as in the estimate just after (4.40), it follows that
\[ \| e^{\delta - \gamma} - 1 \|_0(\mathcal{O}) \leq \text{Const} e^\text{Const} \| \Re(s) + \| \nabla \varphi \|_{\Gamma, (1)} \| |s| \| \nabla \varphi \|_{\Gamma, (q)} + \| \nabla \varphi \|_{\Gamma, (q+1)} \theta^n. \]

It now follows from the above, (5.12) and (5.14) that for any $q \geq 1$,
\[ \| (I) \|_{\Gamma, q}(\mathcal{O}) \leq \| h \|_0(\Gamma) \| e^{\gamma} - e^{\delta} \|_{\Gamma, q}(\mathcal{O}) = \| h \|_0(\Gamma) \| e^{\gamma} (1 - e^{\delta - \gamma}) \|_{\Gamma, q}(\mathcal{O}) \leq \text{Const}_q \| h \|_0(\Gamma) \sum_{i=0}^{q} \| e^{\gamma} \|_{\Gamma, i}(\mathcal{O}) \| 1 - e^{\delta - \gamma} \|_{\Gamma, q-i}(\mathcal{O}) \leq \text{Const}_q \| h \|_0(\Gamma) e^\text{Const} \| \Re(s) \|_{\Gamma, 0} \sum_{i=0}^{q-1} (|s| + 1)^i \| e^{\delta - \gamma} \|_{\Gamma, q-i}(\mathcal{O}) + (|s| + 1)^q \| 1 - e^{\delta - \gamma} \|_0(\mathcal{O}) \leq \text{Const}_q \| h \|_0(\Gamma) e^\text{Const} \| \Re(s) \|_{\Gamma, 0} + \| \nabla \varphi \|_{\Gamma, (1)} \| |s| \| \nabla \varphi \|_{\Gamma, (q)} + \| \nabla \varphi \|_{\Gamma, (q+1)} \theta^n. \]
To estimate (II), first notice that by (4.33),
\[ \| h(Q_0(\mu)) - h(\widetilde{Q}_0(:\mu)) \|_0(O) \leq \| h \|_{\Gamma,1}(O) \text{Const } \alpha^n, \]
while for \( q \geq 1 \) and any \( y \in O \), (5.2) implies
\[ \| h(Q_0(\mu)) - h(\widetilde{Q}_0(:\mu)) \|_{\Gamma,q}(y) = \| h(\widetilde{Q}_0(:\mu)) \|_{\Gamma,q}(y) \leq \text{Const } \| h \|_{\Gamma,q}(O) \alpha^n. \]
Next, for any \( q \geq 0 \), (5.13) and (5.15) give
\[ \| e^{z(\mu,s)} \|_{\Gamma,q}(O) \leq \text{Const}_q e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} (\| \| \nabla \varphi \|_{\Gamma,(q)})^q. \]
Similarly, (5.16) implies
\[ \| A_{\varphi,j}(\widetilde{Q}_{n+2}(:\mu)) \|_{\Gamma,q}(O) \leq \text{Const}_q \| \nabla \varphi \|_{\Gamma,(q)}. \]
Combining the above estimates, for any \( q \geq 1 \) one gets
\[ (II)_{\Gamma,q}(O) \leq \text{Const}_q \sum_{\substack{i+j+k=q \\ i,j,k \geq 0}} \| e^{z(\mu,s)} \|_{\Gamma,i}(O) \| A_{\varphi,j}(\widetilde{Q}_{n+2}(:\mu)) \|_{\Gamma,j}(O) \]
\[ \times \| h(Q_0(\mu)) - h(\widetilde{Q}_0(:\mu)) \|_{\Gamma,k}(O) \]
\[ \leq \text{Const}_q \alpha^n e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} + \| \nabla \varphi \|_{\Gamma,(i)}} \]
\[ \times \left[ \sum_{\substack{i+j+k=q \\ i,j,k \geq 0}} (\| \| \nabla \varphi \|_{\Gamma,(i)} \|^2 (\| \nabla \varphi \|_{\Gamma,(j)})^2 \| h \|_{\Gamma,\ell}(O) \]
\[ + \sum_{\substack{i+j+k=q \\ i,j \geq 0, k = 1}} (\| \| \nabla \varphi \|_{\Gamma,(i)} \|^2 (\| \nabla \varphi \|_{\Gamma,(j)})^2 \| h \|_{\Gamma,1}(O) \] \]
\[ \leq \text{Const}_q \alpha^n e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} + \| \nabla \varphi \|_{\Gamma,(i)}} \]
\[ \times \sum_{r=0}^{q-1} (\| \| \nabla \varphi \|_{\Gamma,(r)} \|^2 (\| \nabla \varphi \|_{\Gamma,(r+1)})^{r+1} \| h \|_{\Gamma,q-r}(O). \]

It now follows from (5.10) and the estimates for (I) and (II) found above that for any \( p \geq 1 \) we have
\[ \| W^{(n+2)}(::\mu,s) - \widetilde{W}^{(n+2)}(::\mu,s) \|_{\Gamma,(p)}(x) \]
\[ \leq \text{Const}_p \theta^n e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} + \| \nabla \varphi \|_{\Gamma,(i)}} \]
\[ \times \sum_{r=0}^{q} \left( \| \| \nabla \varphi \|_{\Gamma,(r)} \|^2 + \| \nabla \varphi \|_{\Gamma,(r+1)} \right)^{r+1} \| h \|_{\Gamma,q-r}(O). \]
Combining this estimate with the argument from the end of Sect. 4 completes the proof of Theorem 3. \( \blacksquare \)

6. Estimates for \( w_0(x, -is) \)

Throughout this and the following sections we will use the notation
\[ E_p(s, \varphi, h) = \begin{cases} 
C_p e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} + \| \nabla \varphi \|_{\Gamma,(i)}} 
\sum_{j=0}^{p} \left( \| \| \nabla \varphi \|_{\Gamma,j} + \| \nabla \varphi \|_{\Gamma,j+1} \right)^{j+1} \| h \|_{\Gamma,p-j} & \text{if } p \geq 1, \\
C_0 e^{\text{Const} \| \mathfrak{R}(s) \|_{(1+\| \varphi \|_{\Gamma,0})} + \| \nabla \varphi \|_{\Gamma,(i)}} \left( \| \| \nabla \varphi \|_{\Gamma,1} \| h \|_{\Gamma,0} + \| h \|_{\Gamma,(1)} \right) & \text{if } p = 0,
\end{cases} \]
where by \( C_p \) we denote positive global constants depending on \( p \) which may change from line to line.
where \( L_s = -L_{-f+\bar{\sigma}} \) and \( \sigma_0 < s_0 \). The precise choice of \( \sigma_0 \) depends on the estimates (4.3) and will be discussed below. For this purpose we write

\[
\left( L_s^n \mathcal{M}_{n,s} - L_s^{n-1} \mathcal{M}_{n-1,s} \right) w(\xi) = -L_s^{n+1} \left[ V^{(n)}(x; s, \mu) - \tilde{V}^{(n)}(x; s, \mu) \right](\xi),
\]

where

\[
\begin{align*}
V^{(n)}(x; s, \mu) &= \exp \left( -\phi^-(x; \sigma^{n+1} e(\mu), s) - \chi(\sigma^{n+1} e(\mu), s) \right) w(\mu), \\
\tilde{V}^{(n)}(x; s, \mu) &= \exp \left( -\phi^-(x; \sigma^n e(\mu), s) - \chi(\sigma^n e(\mu), s) \right) w(\mu).
\end{align*}
\]

The inequality (6.1) follows from the estimates

\[
\left\| \phi^-(x; \sigma^{n+1} e(\xi), s) - \phi^-(x; \sigma^n e(\sigma(\xi)), s) \right\|_{\Gamma_p} \leq C_p E_p(s, \varphi, h) \theta^n,
\]

and the form of the operators \( \mathcal{M}_{n,s}(x) \). The estimate (6.3) is a consequence of the choice of \( \chi_1, \chi_2 \) and the fact that \( f, g \in \mathcal{F}_0(\Sigma_A) \). To prove (6.2), notice that

\[
\sum_{i=-\infty}^{-1} |f(\sigma^{n+1+i} e(\xi)) - f(\sigma^{n+i} e(\sigma(\xi)))| \leq C \theta^n,
\]

and similar estimates hold for the function \( g \). The terms involving \( f \) and \( g \) are independent on \( x \) and they are not important for the estimates of the derivatives. To deal with the terms depending on \( x \), recall that

\[
\phi^- (x; \eta) = -s \phi_1^- (x; \eta) + \phi_2^- (x; \eta)
\]

with \( D_b(\phi^- (\cdot; \eta))(x) = D_b(\psi^- (x)) \). Here and below we use the notations of the previous section. On the other hand,

\[
\left\| \nabla \psi_{\sigma^{n+1} e(\mu)}(x) - \nabla \psi_{\sigma^n e(\sigma(\mu))}(x) \right\|_{\Gamma_p} \leq C_p \alpha^n.
\]

In fact, the backward trajectories \( \gamma_-(x, \nabla \psi_{\sigma^{n+1} e(\mu)}(x)) \) and \( \gamma_-(x, \nabla \psi_{\sigma^n e(\sigma(\mu))}(x)) \) follow an itinerary \( (\mu_{n+1}, \mu_n, \ldots, \mu_1) \) and we can apply Proposition 2. Now we repeat the argument used in the previous section for the estimate of \( \| \gamma - \delta \|_{\Gamma_p} \). Set \( m = \frac{n+1}{2} \) and assume for simplicity that \( n \) is odd. For fixed \( n \) we set \( \eta = \sigma^{n+1} e(\mu), \tilde{\eta} = \sigma^n e(\sigma(\mu)) \). The estimates of

\[
\left\| \phi_1^- (x; \eta) - \phi_1^- (x; \tilde{\eta}) \right\|_{\Gamma_p}
\]

follows from (6.4). Next we write

\[
\begin{align*}
\sum_{i=-\infty}^{-1} \left( g^-_i (x; \eta) - g^-_i (x; \tilde{\eta}) \right) &= \sum_{i=-m-1}^{-1} \left( g^-_i (x; \eta) - g^-_i (x; \tilde{\eta}) \right) \\
+ \sum_{i=m+1}^{n+1} \left( g^-_{i-n-2} (x; \eta) - \tilde{a}_i (x; \mu) \right) - \sum_{i=m+1}^{n+1} \left( g^-_{i-n-2} (x; \tilde{\eta}) - \tilde{a}_i (x; \mu) \right).
\end{align*}
\]

The \( \| \cdot \|_{\Gamma_p} \) norms of the sums from \( i = m + 1 \) to \( n + 1 \) can be estimated as in Section 5 by using (5.7) since

\[
\begin{align*}
\eta &= \sigma^{n+1} e(\mu) = (\ldots, *, \mu_0, \mu_1, \ldots, \mu_{n+1} = \ell, \mu_{n+2}, \ldots), \\
\tilde{\eta} &= \sigma^n e(\sigma(\mu)) = (\ldots, *, \mu_1, \ldots, \mu_{n+1} = \ell, \mu_{n+2}, \ldots),
\end{align*}
\]
According to (6.1), the series defining $R_s$ is close to $s$ where the constant $\rho$ is reduced to that of the series $\sum_{i=m+1}^{n+1} \|g_{i-n-2}(x; \eta) - \tilde{a}_i(x; \mu)\|_{\Gamma, p} \leq \|g_{i-n-2}(x; \tilde{\eta}) - \tilde{a}_i(x; \mu)\|_{\Gamma, p} \leq \sum_{i=m+1}^{n+1} \alpha^i$.

Consequently, the problem of the analytic continuation of the left hand side of (6.5) for $R_s$ is reduced to that of the series $\sum_{n=1}^{\infty} L_s^n w = \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{n} \left( L_s^k M_{k,s} - L_s^{k-1} M_{k-1,s} L_s \right) L_s^{n-k} w + M_{0,s} L_s^n w \right]$.

Since $s_0 \in \mathbb{R}$ is the abscissa of absolute convergence, for $\Re(s) > s_0$ we have $\Pr(-\Re(s)\tilde{f} + \tilde{g}) < 0$ and $\|L_s^n\| \leq 1, \forall n$. Consequently, the double sum at the right hand side is absolutely convergent for $\Re(s) > s_0$ and we may change the order of summation. Applying Fubini theorem, we are going to examine

$$\sum_{n=0}^{\infty} L_s^n M_{n,s} G_s \tilde{v}_s = \left( M_{0,s} + R_s \right) \sum_{n=0}^{\infty} L_s^n G_s \tilde{v}_s,$$

where

$$R_s = \sum_{k=1}^{\infty} \left( L_s^k M_{k,s} - L_s^{k-1} M_{k-1,s} L_s \right).$$

According to (6.1), the series defining $R_s$ is absolutely convergent for $\sigma_0 \leq \Re(s) \leq 1$ and $\|R_s\|_{\Gamma, p} \leq C_p E_p(s, \varphi, h)$.

Consequently, the problem of the analytic continuation of the left hand side of (6.5) for $\Re(s) < s_0$ is reduced to that of the series $\sum_{n=0}^{\infty} L_s^n w = G_s \tilde{v}_s$.

For the analysis of $\sum_{n=0}^{\infty} L_s^n w$ we use the Dolgojat type estimate (4.3) with $u = w = G_s \tilde{v}_s$. Thus for $s = \tau + it$, $\tau \geq \sigma_0$, $|t| \geq 2$, we get

$$\sum_{n=0}^{\infty} L_s^n w \leq \sum_{p=0}^{\infty} \sum_{i=0}^{\lfloor \log |t| \rfloor - 1} C \rho^p |\log |t|| e^{i \Pr(-\tau \tilde{f} + \tilde{g})} \leq \frac{C}{1 - \rho |\log |t||} \sum_{i=0}^{\lfloor \log |t| \rfloor - 1} e^{i \Pr(-\tau \tilde{f} + \tilde{g})} \leq C_1 \max\{|\log |t||, |t| \Pr(-\tau \tilde{f} + \tilde{g})\},$$

where the constant $C > 0$ depends on $\tilde{v}_s$, i.e. on $\varphi$ and $h$. On the other hand, for $\sigma_0$ sufficiently close to $s_0$ the equality (4.42) implies $\Pr(-\sigma_0 \tilde{f} + \tilde{g}) = \beta_0 < 1$. 

and

$$\sum_{i=m+1}^{n+1} \|g_{i-n-2}(x; \eta) - \tilde{a}_i(x; \mu)\|_{\Gamma, p} \leq \sum_{i=m+1}^{n+1} \alpha^i.$$
Combining this with the estimate for $R_s$ and the fact that the operator $G_s$ is bounded, we conclude that for $\sigma_0 \leq \Re(s)$ and $|t| \geq 2$ we have

$$\| \sum_{n=0}^{\infty} L_s^n M_{n,j} G_s v_{s,j} \|_{\Gamma_j,0} \leq C_0 |t|^{\beta_0}.$$ 

Introduce the function

$$w_{0,j}(x, -is) = \sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2}=j} u_j(x, -is).$$

The analysis in Section 5 of [1] implies that the series defining $w_{0,j}(x, -is)$ is absolutely convergent for $x \in \Gamma_j$, $\Re(s) > s_0$ and we have

$$\|w_{0,j}(x, -is)\|_{\Gamma_j,0} \leq C_j, \delta, \Re(s) \geq s_0 + \delta, \delta > 0. \quad (6.6)$$

On the other hand, the analytic continuation of the series $\sum_{n=0}^{\infty} L_s^n M_{n,j} G_s$ established above and Theorem 3 (a) with sufficiently small $\epsilon = s_0 - \Re(s) > 0$ guarantee an analytic continuation of $w_{0,j}(x, -is)$ for $x \in \Gamma_j$, $\Re(s) \geq \sigma_0$, $|\Im(s)| \geq 2$ with $\sigma_0 = s_0 - \epsilon$. Applying Theorem 3 (a) once more for $s = \sigma_0 + it$, we get the estimate

$$\|w_{0,j}(x, -i\sigma_0 + t)\|_{\Gamma_j,0} \leq D_j |t|^{1+\beta_0}.$$ 

The same argument works for all $\ell = 1, \ldots, \kappa_0$ and we get the same estimate for

$$w_{0,\ell}(x, -is) = \sum_{n=0}^{\kappa_0} \sum_{|j|=n+3, j_{n+2}=\ell} u_j(x, -is), x \in \Gamma_\ell.$$ 

In particular, we obtain a $L^\infty(\Gamma)$ estimate for

$$w_0(x, -is) = \sum_{\ell=1}^{\kappa_0} w_{0,\ell}(x, -is)$$

and $\Re(s) \geq \sigma_0$. Clearly, we can choose $0 < \beta_0 < 1$ independent on $\ell = 1, \ldots, \kappa_0$.

Now we will obtain $C^p(\Gamma)$ estimates for $w_0(x, -is)$. To examine the regularity of the functions $w_{0,j}(x, -is)|_{\Gamma_j}$ on $\Gamma_j$, set

$$U_{n+2,j}(x, -is) = \sum_{|j|=n+3, j_{n+2}=j} u_j(x, -is).$$

We start with an estimate of the $C^p(\Gamma_j)$ norms of $U_{n+2,j}(x, -is)$. For this purpose, applying Theorem 3 (b) with $p \geq 1$, we must estimate the norms $\|L_s^p M_{n,j} G_s v_{s,j}\|_{\Gamma_j,p}$. We write

$$L_s^n M_{n,s} = M_{0,s} L_s^n + \sum_{k=1}^{m} \left( L_s^k M_{k,s} - L_s^{k-1} M_{k-1,s} L_s \right) L_s^{n-k} + \sum_{k=m+1}^{n} \left( L_s^k M_{k,s} - L_s^{k-1} M_{k-1,s} L_s \right) L_s^{n-k} = B_0 + B_1 + B_2,$$

where $m = \lfloor n/2 \rfloor$. For the term $B_0$, we use the estimate (4.3) with $0 < \rho < 1$ and we get

$$\|B_0\|_{\Gamma_j,p} \leq C_p E_p(s, \varphi, h) \rho^n.$$
For the term $B_1$ we get
$$
\|B_1\|_{\Gamma_j,p} \leq C'_p E_p(s, \varphi, h) \sum_{k=1}^{m} \theta^k \rho^m \leq C''_p E_p(s, \varphi, h)(\sqrt{\rho})^m.
$$
Finally, for $B_2$ we obtain
$$
\|B_2\|_{\Gamma_j,p} \leq D_p E_p(s, \varphi, h) \sum_{k=m+1}^{n} \theta^k \leq D'_p E_p(s, \varphi, h) \theta^{m+1}.
$$
So changing $\theta$ by another global constant $0 < \tilde{\theta} < 1$, $\tilde{\theta} \geq \max \{\sqrt{\rho}, \sqrt{\theta}\}$, we arrange an estimate
$$
\|L^u_n M_{\alpha,s}\|_{\Gamma_j,p} \leq B_p E_p(s, \varphi, h) \tilde{\theta}^n.
$$
Thus with global constants $C_p, D_p$ we deduce
$$
\|U_{n+2,j}(x, -is)\|_{\Gamma_j,p} \leq C_p E_p(s, \varphi, h) (\theta^n + \tilde{\theta}^n) \leq D_p E_p(s, \varphi, h) \tilde{\theta}^n, \forall n \in \mathbb{N}.
$$
Consequently, the series $w_{0,j}(x, -is)$ is convergent in $C^p(\Gamma_j)$ norm and for $\sigma_0 \leq \tau \leq s_0 + 1$ we have the estimates
$$
\|w_{0,j}(x, -it + t)\|_{\Gamma_j,p} \leq B_p E_p(s, \varphi, h), \quad p \geq 1,
$$
where the constants $B_p$ are independent on $j$. Summing over $\ell = 1, \ldots, \kappa_0$, we obtain the same estimate for $\|w_0(x, -it + t)\|_{\Gamma,p}$ and for $\operatorname{Re}(s) \geq \sigma_0$ the trace $w_0(x, -is)|_{\Gamma}$ is an analytic function with values in $C^\infty(\Gamma)$.

It is interesting to observe that contracting the domain $\sigma_0 \leq \operatorname{Re}(s) \leq s_0 + 1$ we may obtain better bounds for the $C^p(\Gamma)$ norms. For example, we treat below the case $p = 0$ and the same argument works for $p \geq 1$. In the domain $\sigma_0 \leq \operatorname{Re}(s) \leq s_0 + \delta$, $\operatorname{Im}(s) \geq 2$, we apply the Phragmen-Lindelöf theorem (see 5.65 in [T]). Notice that when we decrease $\delta > 0$ the constant $C_{j,\delta}$ in (6.6) change but we have always the bound (6.6). Consequently, for $\sigma_0 \leq \tau \leq s_0 + \delta$ we deduce
$$
\|w_{0,j}(x, -it + t)\|_{\Gamma_j,0} \leq B|t|^\kappa(x), \quad t \geq 2,
$$
where $\kappa(x)$ is a linear function such that
$$
\kappa(\sigma_0) = 1 + \beta_0, \quad \kappa(s_0 + \delta) = 0.
$$
It is clear that choosing $\delta > 0$ small enough, there exist $\sigma'_0, \sigma_0 < \sigma'_0 < s_0$ and $0 < \beta < 1$ so that for $\tau \geq \sigma'_0$ we have
$$
\|w_{0,j}(x, -it + t)\|_{\Gamma_j,0} \leq A_j |t|^\beta, \quad t \geq 2
$$
and similarly we treat the case $t \leq -2$. Finally, for $\tau \geq \sigma'_0, |t| \geq 2$ we have
$$
\|w_{0,j}(x, -it + t)\|_{\Gamma_j,0} \leq A_j |t|^\beta. \quad (6.7)
$$
Here the constants $A_j$ depend on the norms of $\nabla \varphi$ and $h$ and summing over $\ell = 1, \ldots, \kappa_0$ for $\sigma_0 < \sigma'_0 \leq \tau$ we get
$$
\|w_0(x, -it + t)\|_{\Gamma,0} \leq A|t|^\beta. \quad (6.8)
$$

Remark 5. In the following we will not use the estimate (6.8) but a similar argument based on Phragmen-Lindelöf theorem will be crucial in Section 9, where we need to control the behavior of the remainder $R_M(x, s; k)$ and its bounds when $|\operatorname{Im}(s)| \to \infty$. On the other hand, (6.8) is related to the assumption of Ikawa (1.6) mentioned in the Introduction.
7. The leading term $W^{(0)}(x, -is; k)$

For our construction we need the following definition introduced by Ikawa in [I3].

**Definition 1.** Let $\omega \subset \mathbb{R}^N$ be an open set and let $\mathcal{D}$ be a domain in $\mathbb{C}$. We say that the function $U(x, s; k)$ satisfies the condition (S) in $(\omega, \mathcal{D})$ if the following hold:

(i) for each $k \in \mathbb{R}$, $U(., s; k)$ is a $C^\infty(\mathbb{C}_+)$-valued holomorphic function in $\mathcal{D}$,

(ii) $U(., s; k) \in L^2(\omega)$ for $\Re s > 0$,

(iii) $(\Delta - s^2)U(x, s; k) = 0$ in $\omega$ for every $s \in \mathcal{D}$.

Let $S_j(s)$ be the operator constructed in Section 4 of [I3]. To recall this construction, set $\Omega_j = \Omega \setminus K_j$ and consider boundary data $m(x; k) = e^{i\psi(x)}b(x, s; k)$, $k \in \mathbb{R}$, satisfying the condition (A) on $\Gamma_j$. The condition (A) means that there exists a phase function $\varphi$ satisfying the condition $(P)$ in $\Gamma_j$ introduced in Section 2 such that

$$\varphi = \psi \text{ on supp } \bigcup_{s, k} g(., s; k)$$

and

$$\|b(x, s; k)\|_{\Gamma_j, p} \leq C_p, \forall k \geq 1, \forall p \in \mathbb{N}.$$ 

Then $v(x, s; k) = S_j(s)m(x; k)$ is a $C^\infty(\Omega)$-valued entire function with the following properties:

$$\begin{align*}
(\Delta - s^2)v(x, s; k) &= 0, \quad x \in \Omega_j, \\
v(., s; k) &\in L^2(\Omega_j) \text{ if } \Re(s) > 0, \\
v(x, s; k) &= \sum_{q=0}^M \left( \sum_{\nu=0}^{2q} a_{q, \nu}(x; k)(s + ik)^\nu \right)(ik)^{-q} \\
&\times e^{-(s+ik)(\varphi(x)-\psi(x^{-1}(x, \nabla \varphi)))} + r_M(x, s; k), \\
v(x, s; k) &= m(x; k) + r_{1, M}(x, s; k) \text{ on } \Gamma_j.
\end{align*}$$

Moreover, in $\Omega_j(R) = \Omega_j \cap \{ |x| \leq R \}$ for $-a \leq \Re(s) \leq 1, |\Im(s + ik)| \leq 1$, the following estimates hold:

$$\|a_{q, \nu}(x; k)\|_{m(\Omega_j(R))} \leq A_{R, m}\|\nabla \varphi\|_{\Gamma_j, m+2q}\|h\|_{\Gamma_j, m+2q}, \quad m \in \mathbb{N}, \quad (7.1)$$

$$\|r_M(x, s; k)\|_{m(\Omega_j(R))} \leq C_{R, m}e^{-\Re(s)(R+a+1)}k^{-M+m+2}\|\nabla \varphi\|_{\Gamma_j, m+2M}\|g\|_{\Gamma_j, m+2M}, \quad m \in \mathbb{N}. \quad (7.2)$$

Similar estimates hold for $r_{1, M}(x, s; k)$.

In the following we assume that $\sigma_0 \leq \Re(s) \leq 1$. For boundary data $m(x; k) = e^{-s\psi(x)}b(x, s; k)$ with $\psi(x)$ satisfying the condition $(P)$ on $\Gamma_j$ such that

$$\|b(., s; k)\|_{\Gamma_j, p} \leq C_p, \quad p \in \mathbb{N},$$

we write

$$m(x; k) = e^{ik\psi(x)}\hat{b}(x, s; k)$$

with $\hat{b}(x, s; k) = e^{-(s+ik)\psi(x)}b(x, s; k)$. For $\sigma_0 \leq \Re(s) \leq 1, |\Im(s) + ik| \leq 1$ the amplitude $\hat{b}$ and its derivatives remain bounded in $C^p(\Gamma_j)$ norms. Consequently, we can apply the operator $S_j(s)$
to the oscillatory term

\[ \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s\varphi_j(x)} a_j(x) \bigg|_{\Gamma_j} \]

\[ = e^{-s\varphi_j(x)} \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s\varphi_j(x)+s\varphi_j(x)} a_j(x) \bigg|_{\Gamma_j} \]

\[ = e^{-s\varphi_j(x)} \bigg|_{\Gamma_j} \left( e^{s\varphi_j(x)} U_{n+2,j}(x - is) \right) \bigg|_{\Gamma_j}, \]

where \( a_j(x) = (A_j(\varphi)h)(x) \) and \( A_j(\varphi)h \) are introduced in Section 2, while \( j = (j_0, j_1, \ldots, j_{n+2}) \) is a configuration such that \( j_{n+2} = j \). The choice of \( j \) is not important for our argument. We may consider this term as the product of \( e^{-s\varphi_j(x)} \) with the amplitude

\[ m_{j,n}(x, s) = \left( e^{s\varphi_j(x)} U_{n+2,j}(x - is) \right) \bigg|_{\Gamma_j}. \]

Moreover, since the norms \( \|\nabla \varphi_j\|_{\Gamma_j,(p)} \) are uniformly bounded for \( p \in \mathbb{N} \), taking into account the estimates for \( \|U_{n+2,j}(x, -is)\|_{\Gamma_j,(p)} \) established in Section 6, we get

\[ \|m_{j,n}(\cdot, s)\|_{\Gamma_j,(p)} \leq C_p \left\| U_{n+2,j}(x, -is) \right\|_{\Gamma_j,(p)} \leq C_p |\text{Im}(s)|^{p+1}\beta_0 \delta_n, \forall n \in \mathbb{N}. \]

Let

\[ S_j(s)e^{-s\varphi_j(x)} a_i|_{\Gamma_j} = e^{-s\varphi_j(x)} \left( \sum_{q=0}^{M} \left( \sum_{\nu=0}^{2q} \frac{a_j^{(2q)}(x)(s+ik)^\nu}{\nu!} \right) (ik)^{-q} + r_j^{(2)}(x, s; k) \right) \]

with \( a_{j,0,0}(x) = a_j(x) \). On the other hand, we have the estimates

\[ \left\| S_j(s) \left( \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s\varphi_j(x)} a_j(x) \bigg|_{\Gamma_j} \right) \right\|_{C^p(\Omega(R))} \]

\[ \leq B_{p,R} \left\| U_{n+2,j}(x, -is) \right\|_{\Gamma_j,(p+2)} \]

with constants \( B_{p,R} \) independent of \( n \). Consequently, for \( \sigma_0 \leq \Re(s) \leq 1, |\text{Im}(s) + ik| \leq 1 \), the series

\[ S_j(s)\left( w_{0,j}(x, -is)\bigg|_{\Gamma_j} \right) = S_j(s) \left( \sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2}=j} u_j(x, -is) \bigg|_{\Gamma_j} \right) \]

is convergent in \( C^p(\Omega(R)) \). Taking the sum over \( j = 1, \ldots, \kappa_0 \), we conclude that the function

\[ w^{(0)}(x, -is; k) = \sum_{j=1}^{\kappa_0} S_j(s)\left( w_{0,j}(x, -is)\bigg|_{\Gamma_j} \right) \]

satisfies the condition (S) in \( \tilde{\Omega}, \{s \in \mathbb{C}; \Re(s) \geq \sigma_0 \} \). Indeed, for each \( k \in \mathbb{R} \) the properties (i) and (iii) of Definition 1 follow directly from the convergence of the series. The condition (ii) follows easily since for \( \Re(s) > 0 \) we have absolutely convergent series and this case has been treated by Ikawa [13].

Next consider the boundary data

\[ \tilde{m}(x; k) = e^{ik\varphi(x)} \tilde{g}(x; k) \]
satisfying condition (E) on $\Gamma_j$ and the operator $\tilde{S}_j(s)$ introduced in Definition 4.6 and Proposition 4.7 in [I3]. This construction is easy since the rays involved in the construction leave a compact neighborhood of the obstacle. For more details we refer to [I3]. In particular, we consider the boundary data

$$e^{-s\psi(x)}(\tilde{A}_j(\varphi)\hat{g})(x)$$

with

$$(\tilde{A}_j(\varphi)\hat{g})(x) = \left(1 - u_0(\langle \nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j)) \rangle)\right) A_{\varphi,j}(x), \hat{g}(X^{-m}(x, \nabla \varphi_j))$$

and apply the operator $\tilde{S}_j(s)$ to the sum

$$\tilde{U}_{n+2,j}(x, -is)\big|_{\Gamma_j} = \sum_{j=n+3, j_{n+2}=j} a_j(x, -is)\big|_{\Gamma_j} = \sum_{j=n+3, j_{n+2}=j} (-1)^{n+2}e^{-s\varphi_j(x)}a_j(x)\big|_{\Gamma_j},$$

where $a_j(x) = (\tilde{A}_j(\varphi)\hat{g})(x)$. Repeating the above argument, we can justify the existence of

$$\tilde{w}^{(0)}(x, -is; k) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \tilde{S}_j(s) \left( \tilde{U}_{n+2,j}(x, -is)\big|_{\Gamma_j} \right)$$

and show that $\tilde{w}^{(0)}(x, -is; k)$ satisfies the condition (S) in $(\Omega, \{s \in \mathbb{C}; \Re(s) \geq \sigma_0\})$. Now introduce

$$W^{(0)}(x, -is; k) = w^{(0)}(x, -is; k) + \tilde{w}^{(0)}(x, -is; k).$$

The forms of $a_j(x)$ and $\tilde{a}_j(x)$ and the construction of Section 4 imply that

$$W^{(0)}(x, -is; k)\big|_{\Gamma - m(x; k)} = (ik)^{-1}R_0(x, s; k).$$

Our construction for $\Re(s) > s_0$ is the same as that of Ikawa [I3], so for $\Re(s) > s_0 + \delta > s_0$ we get the estimates

$$\|R_0(x, s; k)\|_{\Gamma,p} \leq C_{p,\delta}k^p, \quad p \in \mathbb{N},$$

established in [I3]. Here the constants $C_{p,\delta}$ depend on the norms of $\nabla \varphi$ and $h$. However the estimates of Section 6 for $\sigma_0 \leq \Re(s) \leq 1$, $|s + \text{i}k| \leq 1$, do not imply (7.3). The reason is that we have sums of terms

$$\sum_{j=n+3, j_{n+2}=j} e^{-s\varphi_j(x)} \sum_{\nu=0}^{2} a_{j,1,\nu}(x, s; k)(s + \text{i}k)^{-1}$$

and it is rather complicated to find an analogue of Theorem 3 in this situation, since the amplitudes $a_{j,1,\nu}$ are determined by the transport equations and their forms are not as simple as those of $\tilde{A}_j(\varphi)\hat{g}(x)$. Nevertheless, the analysis of Section 6 shows that $R_0(x, s; k)$ for $x \in \Gamma, k \geq 1$, is an analytic function in $s$ for $\sigma_0 \leq \Re(s) \leq 1$. Moreover, we have upper bounds

$$\|R_0(x, s; k)\|_{\Gamma,p} \leq C_pk^{p+3+\delta_0}, \quad \forall p \in \mathbb{N}$$

which follows from the estimates for $\|U_{n+2,j}(x, -is)\|_{\Gamma_j,p}$ and (7.1). Of course, these bounds are not optimal, but they are sufficient for our argument.

In the next section we will construct lower order approximations.
8. Lower order terms of the asymptotic solution

We start with the observation that the terms with factor $(ik)^{-1}$ in the trace $W^{(0)}(x, -is; k)|_{\Gamma_i}$ on $\Gamma_i$ have the form

$$\sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s\varphi_j(x)} \sum_{\nu=0}^{2} \left[ a_{j,1,\nu}^{(j)}(x, s; k)|_{\Gamma_i} + \tilde{a}_{j,1,\nu}^{(j)}(x, s; k)|_{\Gamma_i} \right] (s + ik)^\nu.$$  

Here $a_{j,1,\nu}^{(j)}$ come from the trace of $w^{(0)}(x, -is; k)$, while $\tilde{a}_{j,1,\nu}^{(j)}$ come from the trace of $\tilde{w}^{(0)}(x, -is; k)$. Consider the boundary data

$$m_{1,j}^{(j,l)}(x, s; k) = (-1)^{n+2} e^{-s\varphi_j(x)} |_{\Gamma_i} \sum_{\nu=0}^{2} u_0(\nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j))) \; a_{j,1,\nu}^{(j)}(x, s; k)|_{\Gamma_i} (s + ik)^\nu.$$

and set

$$M_{1,j}^{(j,l)}(x, s; k) = \sum_{|j|=n+3, j_{n+2}=j} m_{1,j}^{(j,l)}(x, s; k) = e^{-s\varphi_j(x)} |_{\Gamma_i} \sum_{|j|=n+2, j_{n+2}=j} e^{s\varphi_j(x)} |_{\Gamma_i} m_{1,j}^{(j,l)}(x, s; k),$$

where, as in the previous section, $j$ is a configuration such that $|j| = n + 2$, $j_{n+2} = j$. Our aim is to apply the construction of Sections 4 and 6 to the oscillatory data $M_{1,n}^{(j,l)}(x, s; k)$. To do this, we need some estimates of the $C^p(\Gamma_h)$ norms of $M_{1,n}^{(j,l)}(x, s; k)$. Here and below we denote by $P^{(j,l)}$ some terms depending on the traces on $K_j$ and $K_l$. $j, l = 1, \ldots, \kappa_0$, while $j, j'$ will denote configurations.

The terms $M_{1,n}^{(j,l)}$ are obtained as the traces on $\Gamma_i$ of the terms with factor $(ik)^{-1}$ in the representation of

$$S_j(s) \left( e^{-s\varphi_j(x)} |_{\Gamma_i} m_{j,n}(x, s) \right),$$

where the boundary data

$$e^{-s\varphi_j(x)} |_{\Gamma_i} m_{j,n}(x, s)$$

was introduced in the previous section. For the amplitudes $a_{j,1,\nu}^{(j)}(x, s; k)$ we have $C^p(\Gamma_j)$ estimates $O(|\text{Im}(s)|^{p+3+\beta_0} \tilde{\theta}^n)$ since for the boundary data $e^{-s\varphi_j(x)} |_{\Gamma_i} m_{j,n}(x, s)$ we have estimates $O(|\text{Im}(s)|^{p+1+\beta_0} \tilde{\theta}^n)$. Thus we deduce

$$\|M_{1,n}^{(j,l)}(x, s; k)|_{\Gamma_j, (p)} \leq C_p |\text{Im}(s)|^{p+3+\beta_0} \tilde{\theta}^n, \forall n \in \mathbb{N} \tag{8.1}$$

with constants $C_p$ independent on $n \in \mathbb{N}$.

For fixed $j, l$ and fixed $n$ starting with the boundary data $M_{1,n}^{(j,l)}(x, s; k)$ we apply the construction of Sections 4 and 6 and we obtain a series

$$\sum_{m=0}^{\infty} V_{1,n,m}^{(j,l)}(x, s; k)$$

with

$$V_{1,n,m}^{(j,l)}(x, s; k) = \sum_{|j'|=m+3, j_{m+2}=l} (-1)^{m+2} e^{-s\varphi_{j'}(x)} l_{1,n,j'}^{(j,l)}(x, s; k),$$
where the phase functions \( \varphi_j(x) \) depend on the configurations \( j' \). Taking the summation over \( n \), we are going to study the series

\[
U^{(j,l)}_1(x, s; k) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V^{(j,l)}_{1,n,m}(x, s; k). \tag{8.2}
\]

We repeat the argument of the previous section, for \( \sigma_0 \leq \Re(s) \leq 1, |\Im(s + ik)| \leq 1 \) and applying (8.1) and Theorem 3, (b), we get the estimates

\[
\|V^{(j,l)}_{1,n,m}(x, s; k)\|_{\Gamma_1,p} \leq D_p k^{p+4+2\beta_0} \tilde{\vartheta}^{n+m}, \quad \forall n \in \mathbb{N}, \forall m \in \mathbb{N}
\]

with constants \( D_p \) independent on \( n, m \in \mathbb{N} \). Thus the double series defining \( U^{(j,h)}_1(x, s; k) \) is convergent and we can introduce

\[
\sum_{j,h=1}^{\kappa_0} U^{(j,h)}_1(x, s; k).
\]

In the same way, starting with boundary data \( \tilde{a}^{(j)}_{1,j}(x, s; k) \) related to the terms involving \( \tilde{a}^{(j)}_{j,1}(x, s; k) \), we define a function \( \tilde{U}^{(j,h)}_1(x, s; k) \). Consider the sum

\[
W^{(1)}(x, s; k) = -(ik)^{-1} \sum_{j,h=1}^{\kappa_0} \left( U^{(j,h)}_1(x, s; k) + \tilde{U}^{(j,h)}_1(x, s; k) \right)
\]

and notice that

\[
\|W^{(1)}(x, s; k)\|_{\Gamma_1,p} \leq Q'_{1,p}k^{p+6+2\beta_0}.
\]

From this construction it follows that for \( \sigma_0 \leq \Re(s) \leq 1, |s + ik| \leq 1 \), the sum

\[
W^{(0)}(x, s; k) + W^{(1)}(x, s; k)
\]

satisfies the condition (S) for \( \Re(s) \geq \sigma_0 \). Moreover, we have

\[
W^{(0)}(x, s; k) + W^{(1)}(x, s; k) - m(x, s; k) = (ik)^{-2}R_1(x, s; k), \quad x \in \Gamma \tag{8.3}
\]

and

\[
\|R_1(x, s; k)\|_{\Gamma_1,p} \leq Q_{1,p}k^{p+6+2\beta_0} \left( \|\nabla \varphi\|_{\Gamma_1,(p+3)} + 1 \right)h\|_{\Gamma_1,(p+3)}, \quad \forall p \in \mathbb{N}, \tag{8.4}
\]

while for \( \Re(s) \geq \sigma_0 + \delta, |s + ik| \leq 1 \), we get

\[
\|R_1(x, s; k)\|_{\Gamma_1,p} \leq Q_{1,p,\delta}k^{p} \left( \|\nabla \varphi\|_{\Gamma_1,(p+3)} + 1 \right)h\|_{\Gamma_1,(p+3)}, \quad \forall p \in \mathbb{N}. \tag{8.5}
\]

Repeating this procedure, we construct \( W^{(j)}(x, s; k) \) for \( 0 \leq j \leq M \) which are analytic functions for \( \sigma_0 \leq \Re(s) \) with values in \( C^\infty(\Omega) \). They satisfy the condition (S) for \( \sigma_0 \leq \Re(s) \leq 1 \) and we have

\[
\sum_{j=0}^{M} W^{(j)}(x, s; k) - m(s, k) = (ik)^{-M}R_M(x, s; k), \quad x \in \Gamma \tag{8.6}
\]

with polynomial estimates

\[
\|R_M(x, s; k)\|_{\Gamma_1,p} \leq Q_{M,p}k^{N(M)}, \tag{8.7}
\]

where \( Q_{M,p} \) depend on the norms of \( \nabla \varphi \) and \( h \). Thus we establish crude estimates with orders \( N(M) \) depending on \( M \) and it seems quite difficult to obtain more precise estimates for \( \sigma_0 \leq \Re(s) \leq 1 \). Of course, we have \( N(M) > M \) and \( (N(M) - M) \to \infty \) as \( M \to \infty \). For this reason in the domain \( \sigma_0 \leq \Re(s) \leq 1 \) we have no gains of the powers of \( k \). To obtain an approximation
we will consider an integral equation on the boundary and solve it for \( \sigma_0 < \sigma_1 \leq \Re(s) \leq 1 \) with \( \sigma_1 < s_0 \) dealing only with a finite number terms \( W^{(j)}(x, s; k) \), \( 0 \leq j \leq M \).

9. Integral Equation on the Boundary

In this section we assume that \( N \geq 3 \). The case \( N = 2 \) is simpler and can be treated by the same argument. We use the approximations \( W^{(j)}(x, s; k) \), \( j = 1, \ldots, M \) for suitable boundary data \( m(x, s) = e^{ik\varphi(x)}h(x) \), where \( M \) is fixed so that \( M > (N - 1)/2 \). Following the argument in the previous section, we have the estimates (8.6) and (8.7). Moreover, \( R_M(x, s; k) \) is analytic for \( \sigma_0 \leq \Re(s) \leq 1 \) and for \( \Re(s) \geq s_0 + \delta, |s + ik| \leq 1 \) the result of Ikawa [12] in the absolutely converging domain yields

\[
\|R_M(x, s; k)\|_{\Gamma, 0} \leq Q_{M, \delta}.
\]

By using Pragmen-Lindelöf theorem, (8.7) with \( p = 0 \) and the above estimate, we can find \( \sigma_0 < \sigma_1 < s_0 \) such for \( \sigma_1 \leq \Re(s) \leq 1 \), \( |s + ik| \leq 1 \), we have

\[
\|R_M(x, s; k)\|_{\Gamma, 0} \leq C_M k^\epsilon,
\]

where \( 0 < \epsilon < M - \frac{N-1}{2} \). Moreover, the constants \( C_M \) and \( \epsilon \) depend on the derivatives of \( \nabla \phi \) and \( h \).

Let \( G \subseteq \Gamma \) and let \( F \in L^2(\Gamma) \) with supp \( F \subset G \). Choose local coordinates in \( G \) of the form

\[
x(z) = (z, x_N(z)), z = (z_1, \ldots, z_{N-1}) \in J \subset \mathbb{R}^{N-1},
\]

and write

\[
F(x(z)) = (2\pi)^{-N+1} \int e^{i<z, \xi>} \hat{F}(\xi) d\xi
= (2\pi)^{-N+1} G(z) \int_{|\xi| \leq 1} e^{i<z, \xi>} \hat{F}(k\xi) k^{N-1} d\xi
+ (2\pi)^{-N+1} G(z) \int_{S^{N-2}} e^{ik\rho<z, \omega>} \hat{F}(k\rho\omega) k^{N-1} \rho^{N-2} d\rho d\omega = F_1 + F_2.
\]

Here \( G(z) \in C_0^\infty(\mathbb{R}^{N-1}) \), \( G(z) = 1 \) on supp \( F(x(z)) \) and

\[
\hat{F}(\xi) = \int e^{-i<z, \xi>} F(x(z)) dz.
\]

For oscillatory data \( m_1(z, \xi) = (2\pi)^{-N+1} G(z) e^{ik<z, \xi>} \) with \( |\xi| \leq 1 \), we apply the construction of Sections 4-7 and we construct an approximative solution \( V_1(x, s; k, \xi) \) which satisfies the condition \( (S) \) for \( \sigma_1 \leq \Re(s) \leq 1 \), \( |s + ik| \leq 1 \), and define an operator \( U_1(s; k) F \) with values in \( C^\infty(\Omega) \) so that the function \( U_1(s; k) F \) satisfies the condition \( (S) \) in \( \{ s \in \mathbb{C} : \sigma_1 \leq \Re(s) \leq 1, |s + ik| \leq 1 \} \). Moreover,

\[
U_1(s; k) F|_\Gamma = F_1 + k^{-M} \int_{|\xi| \leq 1} R_{1,M}(x, s; k, \xi) \hat{F}(k\xi) k^{N-1} d\xi
= F_1 + L_1(s; k) F
\]

with \( R_{1,M}(x, s; k, \xi) \) satisfying the estimate (9.1). It is clear that the constants \( C_M \) and \( \epsilon \) in (9.1) can be chosen to be uniform with respect to \( \xi, |\xi| \leq 1 \). It is easy to see that

\[
\|L_1(s; k) F\|_{L^2(\Gamma)}^2 \leq C_0 \left( \int_{|\xi| \leq 1} k^{-M+(N-1)/2+\epsilon} |\hat{F}(k\xi)| |k|^{(N-1)/2} d\xi \right)^2.
\]
\[ \leq C_0 k^{-2M+N-1+2\varepsilon} \int_{|\xi| \leq 1} d\xi \int_{\mathbb{R}^{N-1}} |\hat{F}(k\xi)|^2 k^{N-1} d\xi \leq C_1 k^{-2M+N-1+2\varepsilon} \|F\|^2_{L^2(\Gamma)} \]

with a constant \( C_1 > 0 \) depending only on the boundary \( \Gamma \).

To deal with the term \( F_2 \), we consider \( k\rho \) as a parameter and we apply the constructions in Section 4-7 for the oscillatory data \( m_2(z, \omega) = (2\pi)^{-N+1} G(z) e^{ik\rho z, \omega} \). Thus, we construct an approximate solution \( V_2(x, s; k\rho, \omega) \) with values in \( C^\infty(\Omega) \) satisfying the condition \((S)\) in \( \{ s \in \mathbb{C} : \sigma_1 \leq \text{Re}(s) \leq 1, |s + ik| \leq 1 \} \) and an operator \( U_2(s; k)F \) with values in \( C^\infty(\Omega) \) such that

\[ U_2(s; k)F|_\Gamma = F_2 \]

\[ + k^{-M} \int_1^\infty \int_{S^{N-2}} R_{2,M}(x, s; k\rho, \omega) \hat{F}(k\rho \omega) k^{N-1} \rho^{-2} d\rho d\omega \]

\[ = F_2 + L_2(s; k)F. \]

Moreover, applying this estimate for \( R_{2,M}(x, s; k\rho, \omega) \) uniformly with respect to \( \omega \in S^{N-2} \). Applying this estimate for \( R_{2,M}(x, s; k\rho, \omega) \), we get

\[ \|L_2(s; k)F_2\|_{L^2(\Gamma)}^2 \leq C_2 \left( k^{-M+(N-1)/2+\varepsilon} \int_1^\infty \int_{S^{N-2}} \rho^{-M+(N-2)/2+\varepsilon} |\hat{F}(k\rho \omega)|^{k^{N-1}} \rho^{(N-2)/2} d\rho d\omega \right)^2 \]

\[ \leq C_3 k^{-2M+N-1+2\varepsilon} \int_1^\infty \int_{S^{N-2}} \rho^{-2M+N-2+2\varepsilon} |\hat{F}(k\rho \omega)|^{k^{N-1}} \rho^{(N-2)/2} d\rho d\omega \]

\[ \leq C_4 k^{-2M+N-1+2\varepsilon} \|F\|^2_{L^2(\Gamma)} \]

with constants \( C_2, C_3, C_4 \) depending only on the boundary \( \Gamma \). Here we have used the fact that \(-2M + N - 2 + 2\varepsilon < -1\). Finally, we introduce

\[ U_G(s; k)F = U_1(s; k)F_1 + U_2(s; k)F_2 \]

and conclude that

\[ U_G(s; k)F|_\Gamma = F + L_1(s; k)F + L_2(s; k)F = F + L_G(s; k)F \]

with

\[ \|L_G(s; k)F\|_{L^2(\Gamma)} \leq B g k^{-M+(N-1)/2+\varepsilon} \|F\|_{L^2(\Omega)}. \]

By using a partition of unity on \( \Gamma \), we define an operator

\[ U(s; k) : L^2(\Gamma) \ni f \rightarrow U(s; k)f \in C^\infty(\bar{\Omega}) \]

and we deduce that \( U(s; k)f \) satisfies the condition \((S)\) for \( \sigma_1 \leq \text{Re}(s) \leq 1, |s + ik| \leq 1 \), while on the boundary we have the equation

\[ U(s; k)f|_\Gamma = f + L(s; k)f \]

with

\[ \|L(s; k)f\|_{L^2(\Gamma)} \leq B k^{-M+(N-1)/2+\varepsilon} \|f\|_{L^2(\Gamma)}. \]

For \( k \geq k_0 \) the operator \( I + L(s; k) \) is invertible and we define the operator

\[ R(s; k)f = U(s; k)(I - L(s; k))^{-1} f : L^2(\Gamma) \rightarrow C^\infty(\bar{\Omega}) \]

satisfying the condition \((S)\) for \( \sigma_1 \leq \text{Re}(s) \leq 1, |s + ik| \leq 1 \) and the condition

\[ R(s; k)f|_\Gamma = f. \]

This completes the proof of Theorem 2.
References


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