Analytic Continuation of the Resolvent of the Laplacian and the Dynamical Zeta Function

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Abstract. Let \( s_0 < 0 \) be the abscissa of absolute convergence of the dynamical zeta function \( Z(s) \) for several disjoint strictly convex compact obstacles \( K_i \subset \mathbb{R}^N, i = 1, \ldots, k_0, k_0 \geq 3 \), and let 
\[
R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi, \quad \chi \in C_0^\infty(\mathbb{R}^N),
\]
be the cut-off resolvent of the Dirichlet Laplacian \( -\Delta_D \) in \( \Omega = \mathbb{R}^N \setminus \bigcup_{i=1}^{k_0} K_i \). We prove that there exists \( \sigma_1 < s_0 \) such that \( Z(s) \) is analytic for \( \Re(s) \geq \sigma_1 \) and the cut-off resolvent \( R_\chi(z) \) has an analytic continuation for \( \Im(z) < -\sigma_1, |\Re(z)| \geq C > 0 \).

1. Introduction

Let \( K \) be a subset of \( \mathbb{R}^N (N \geq 2) \) of the form \( K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}, \) where \( K_i \) are compact strictly convex disjoint domains in \( \mathbb{R}^N \) with \( C^\infty \) boundaries \( \Gamma_i = \partial K_i \) and \( k_0 \geq 3 \). Set \( \Omega = \mathbb{R}^N \setminus K \) and \( \Gamma = \partial K \). We assume that \( K \) satisfies the following (no-eclipse) condition:

(H) \[
\text{for every pair } K_i, K_j \text{ of different connected components of } K \text{ the convex hull of } K_i \cup K_j \text{ has no common points with any other connected component of } K.
\]

With this condition, the billiard flow \( \phi_t \) defined on the cosphere bundle \( S^*(\Omega) \) in the standard way is called an open billiard flow. It has singularities, however its restriction to the non-wandering set \( \Lambda \) has only simple discontinuities at reflection points. Moreover, \( \Lambda \) is compact, \( \phi_t \) is hyperbolic and transitive on \( \Lambda \), and it follows from [St1] that \( \phi_t \) is non-lattice and therefore by a result of Bowen [Bo], it is topologically weak-mixing on \( \Lambda \).

Given a periodic reflecting ray \( \gamma \subset \Omega \) with \( m_\gamma \) reflections, denote by \( d_\gamma \) the period (return time) of \( \gamma \), by \( T_\gamma \) the primitive period (length) of \( \gamma \) and by \( P_\gamma \) the linear Poincaré map associated to \( \gamma \). Denote by \( \Pi \) the set of all periodic rays in \( \Omega \) and let \( \lambda_{i,\gamma}, i = 1, \ldots, N-1 \), be the eigenvalues of \( P_\gamma \) with \( |\lambda_{i,\gamma}| > 1 \) (see [PS1]).

Let \( \mathcal{P} \) be the set of primitive periodic rays. Set
\[
\delta_\gamma = -\frac{1}{2} \log(\lambda_{1,\gamma} \ldots \lambda_{N-1,\gamma}), \quad \gamma \in \mathcal{P},
\]
\[
r_\gamma = \begin{cases} 0 & \text{if } m_\gamma \text{ is even}, \\ 1 & \text{if } m_\gamma \text{ is odd} \end{cases}
\]
and consider the dynamical zeta function
\[
Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{mr_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}.
\]

It is easy to show that there exists \( s_0 \in \mathbb{R} \) such that for \( \Re(s) > s_0 \) the series \( Z(s) \) is absolutely convergent and \( s_0 \) is minimal with this property. The number \( s_0 \) is called abscissa of absolute convergence. On the other hand, using symbolic dynamics and the results of [PP], we deduce that
Theorem 1. (I3) Assume \( s_0 < 0 \). Then for every \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) so that the cut-off resolvent \( R_\chi(z) \) is analytic for \( \text{Im}(z) < -(s_0 + \epsilon) \), \( \|\Re(s)\| \geq C_\epsilon \).
A similar result for a control problem has been established by Burq [B]. The proofs in [I3] and [B] are based on the construction of an asymptotic solution $U_M(x, s; k)$ with boundary data $m(x) = e^{ik\varphi(x)}h(x)$, $k \in \mathbb{R}$, $k \geq 1$, where $\varphi$ is a phase function ($\|\nabla \varphi\| = 1$) and $h \in C^\infty(\Gamma)$ has a small support. More precisely, $U_M(\cdot, s; k)$ is $C^\infty(\Omega)$-valued holomorphic function in

$$D_0 = \{s \in \mathbb{C} : \text{Re}(s) > s_0\},$$

and we have

$$\Delta - s^2)U_M(\cdot, s; k) = 0 \text{ for } x \in \hat{\Omega}, \text{ Re}(s) > s_0,$$

(1.1)

$$U_M(\cdot, s; k) \in L^2(\hat{\Omega}) \text{ if } \text{Re}(s) > 0,$$

(1.2)

$$U_M(x, s; k) = m(x, k) + r_M(x, s; k) \text{ on } \Gamma, \text{ Re}(s) > s_0,$$

(1.3)

where, for $r_M(x, s; k)$ and $s \in D_0$, $|\text{Im}(s + ik)| \leq 1$, we have the estimates

$$\|r_M(\cdot, s; k)\|_{C^p(\Gamma)} \leq C_p k^{-M+p} (\|\nabla \varphi\|_{C^{M+2+p}(\Gamma)} + 1) \|h\|_{C^{M+2+p}(\Gamma)}, \forall p \in \mathbb{N}. \tag{1.4}$$

The function $U_M(x, s; k)$ is given by a finite sum of series having the form

$$\sum_{n=0}^\infty \sum_{|j|=n+3, j_{n+2}=l} \sum_{q=0}^M e^{-s\varphi_j(x)} \sum_{\nu=0}^{2q} (a_{j,q,\nu}(x, s; k)(s + ik)^\nu)(ik)^{-q}, \tag{1.5}$$

where $j = (j_0, \ldots, j_{n+2})$ are configurations of length $|j| = n + 3$, $\varphi_j(x)$ are phase functions and the amplitudes $a_{j,q,\nu}(x, s; k)$ depend on $s \in \mathbb{C}$ and $k \in \mathbb{R}$ (see Sections 4 and 6 for the notation and more details). The main difficulty is to establish the summability of these series and to obtain suitable $C^p$ estimates of their traces on $\Gamma$ for $\text{Re}(s) > s_0$. The absolute convergence of $Z(s)$ makes it possible to study the absolute convergence of these series and to get crude estimates which lead to the properties (1.1)-(1.4) above. As one can see from its definition, the dynamical zeta function $Z(s)$ is determined by the periods of periodic rays and the corresponding Poincaré maps, and formally from $Z(s)$ we get almost no information about the dynamics of the rays in a neighborhood of the non-wandering set. On the other hand, the absolute convergence of $Z(s)$ is a strong condition which enables us to justify the existence of $U_M(x, s; k)$ establishing the absolute convergence of (1.5).

Passing to the case $\text{Re}(s) < s_0$, it is an interesting problem to examine the link between the analytic continuation of $R_\chi(z)$ for $\text{Im}(z) \geq -s_0$ and that of the dynamical zeta function $Z(s)$. Several years ago, Ikawa [I5] announced a result concerning a local analytic continuation of $R_\chi(z)$ in a neighborhood of a point $z_0 \in D_{\alpha,\epsilon}$, where

$$D_{\alpha,\epsilon} = \{s \in \mathbb{C} : \text{Im}(s) \leq -s_0 + |\text{Re}(s)|^{-\alpha}, |\text{Re}(z)| \geq C_\epsilon\}$$

assuming the following conditions:

(i) $Z(s)$ is analytic in a neighborhood of $iz_0$ and

$$|Z(iz_0)| \leq |z_0|^{1-\epsilon}, 0 < \epsilon < 1, \tag{1.6}$$

(ii) if $w(\eta) > 0$ is an eigenfunction of the Ruelle operator $L_{s_0}$ (see Section 4 for the notations $\Sigma^+_A, L_s, \tilde{R}(\xi, s)$) with eigenvalue 1, then the constants

$$M = \max_{\xi, \eta \in \Sigma^+_A} \frac{w(\xi)}{w(\eta)}, \quad m = \min_{\xi \in \Sigma^+_A} |\tilde{e}^{\tilde{R}(\xi, s_0)}|$$
satisfy the inequality $\frac{M}{m}\sqrt{\theta} < 1$ with a global constant $0 < \theta < 1$ depending of the dispersing properties of the billiard flow (see [13], [14], [St1]).

The constant $0 < \alpha < 1$ is chosen so that $a\log \frac{M}{m} < \alpha < 1$ with $a\log \frac{M}{m} < 1$ and $a|\log \theta| < 2$. Ikawa announced in [I5] that (ii) holds in the case of three balls centered at the vertices of an equilateral triangle, provided the radii of the balls are sufficiently small. In general the condition (ii) is rather restrictive. On the other hand, it is difficult to check the condition (i) if we have no precise information about the spectral properties of $L_s$ for $\Re(s)$ close to $s_0$. In [I5] there are no comments under when (i) holds and whether that happens at all. Following [I5], to obtain an analytic continuation of $R_\chi(z)$ in $D_{\alpha,\epsilon}$ for a start one has to establish the estimate (1.6) for all $z \in D_{\alpha,\epsilon}$. As we show in Section 6, this is related to the behavior of the iterations of the Ruelle operator $\check{L}_s = L_{-sf+\gamma}$ introduced in Section 4 for $s \in iD_{\alpha,\epsilon}$. It does not look like the tools required to do this were available at the time [I5] was written. To our knowledge a proof of the result announced in [I5] has not been published anywhere.

Starting with the work of Dolgopyat [D], there has been a considerable progress in the analysis of the spectral properties of the Ruelle transfer operators $\check{L}_s$ related to hyperbolic systems. The so called Dolgopyat type estimates for the norms of the iterations $\check{L}_s^n$ (see [D], [St2], [St3]) imply an estimate for the zeta function $Z(s)$ in a strip $s_0 - \epsilon \leq \Re(s) \leq s_0$, $\epsilon > 0$, see (4.3) in Sect. 4 below for details. On the other hand, it is important to note that the information given by the estimates of the iterations and the behavior of the spectrum of $\check{L}_s$ is richer than that related to the zeta function $Z(s)$. Assuming that the Dolgopyat type estimates (4.3) hold for the operator $\check{L}_s$, in this paper we prove the following main result:

**Theorem 2.** Let $s_0 < 0$. Suppose that the estimates (4.3) for the operator $\check{L}_s$ hold. Then there exist $\sigma_1 < s_0$ and $C > 0$ such that $Z(s)$ is analytic for $\Re(s) \geq \sigma_1$, the cut-off resolvent $R_\chi(z)$ is analytic in

$$\mathcal{U}_1 = \{z \in \mathbb{C} : \text{Im}(z) < -\sigma_1, |\Re(s)| \geq C\}$$

and satisfies an estimate of the form

$$|R_\chi(z)| \leq C_1(1 + |z|)^m, \ z \in \mathcal{U}_1. \quad (1.7)$$

The Dolgopyat type estimates (4.3) stated in Sect. 4 below always hold when $N = 2$ ([St2]), while for $N \geq 3$ it follows from some general results in [St3] that (4.3) hold under certain assumptions about the flow on $\Lambda$. These assumptions are listed in detail at the beginning of Appendix B below. It seems likely that most of these assumptions are either always satisfied or not really necessary in proving the estimates (4.3) for open billiard flows, however so far little work has been carried out in this direction.

The estimates (4.3) were originally obtained for the Ruelle operator $L_s$ related to a coding given by a Markov family of rectangles $R_i = [U_i, S_i], i = 1, \ldots, k$, where $U_i, S_i$ are subsets of local unstable and stable manifolds, respectively. For the proof of Theorem 2 we need Dolgopyat type estimates for the iterations of the Ruelle operator $\check{L}_s$ related to the symbolic coding defined by using the connected components of $K$. The link between the operators $\check{L}_s$ and $L_s$ and the estimates leading to (4.3) are discussed in Appendix B.

We should mention that our result implies the existence of an analytic continuation of $R_\chi(z)$ in a strip $0 \leq \text{Im}(s) \leq -\sigma_1, |\Re(s)| > C$, and we impose no restrictions on the eigenfunction $w(\eta)$ or on the behavior of $Z(s)$ for $\sigma_1 \leq \Re(s) \leq s_0$. The estimate (1.7) enables us to obtain a scattering
expansion with an exponential decay rate of the remainder for the solutions of the Dirichlet problem
\[
\begin{align*}
(\partial_t^2 - \Delta)u(t, x) &= 0, \quad x \in \Omega, \quad u|_{\mathbb{R} \times \Gamma} = 0, \\
u_{t=0} &= f \in C_0^\infty(\hat{\Omega}), \quad \partial_t u_{t=0} = g \in C_0^\infty(\hat{\Omega}).
\end{align*}
\] (1.8)

Set \( \mathcal{H} = \hat{H}(\Omega) \oplus L^2(\Omega), \mathcal{D}^m = (-\Delta_D + i)^{-m}\mathcal{H}, m \in \mathbb{N} \).

**Corollary 1.** Let \( N \) be odd and let \( \chi \in C_0^\infty(\mathbb{R}^n) \) be equal to 1 in a neighborhood of \( K \). Let \( u(t, x) \) be the solution of (1.8) with initial data \( (\chi_f, \chi g) \). Then under the assumptions of Theorem 2 there exists \( M \in \mathbb{N} \) such that for every \( \epsilon > 0 \) and for \( t > 0 \) sufficiently large we have
\[
\chi u(t, x) = \sum_{m(z_l) \leq -\sigma_1} \sum_{j=1}^{m(z_l)} w_{z_l,j}(x)e^{it\lambda_j t^{-1}} + E(t)(f, g),
\]
where
\[
\|E(t)(f, g)\|_{\mathcal{H}} \leq C e^{(\sigma_1 + \epsilon)t}\|(f, g)\|_{\mathcal{D}^M}.
\]
Here \( z_l \) are the resonances with \( \text{Im}(z_l) \leq -\sigma_1 \), \( m(z_l) \) are the multiplicities of \( z_l \) and \( w_{z_l,j} \) are related to the cut-off resonances states corresponding to \( z_l \).

A similar result was established by Ikawa [13] with \( \sigma_1 \) replaced by \( s_0 < 0 \). Recently, a local decay result for the solutions of the wave equation related to hyperbolic compact co-convex manifolds \( \Gamma \setminus \mathbb{H}^{n+1} \) was proved by C. Guillarmou and F. Naud [GN]. They obtain an exponentially decreasing remainder related to the Hausdorff dimension \( \delta \) of the limit set \( \Lambda(\Gamma) \). The number \( \delta \) corresponds to \( s_0 \) in our case and Corollary 1 shows that the optimal exponential decay for solutions to problems with hyperbolic geometry is not related to the absicissa of absolute convergence. It seems that this is the first general result of this kind for manifolds with variable curvature. For other results concerning scattering expansions for trapping obstacles the reader could consult [TZ] and the references given there.

The proof of Theorem 2 is long and technical. Below we discuss the main points. As in [I3], [I5], the idea is to construct an approximative solution \( U_M(x, s; k) \) which is analytic for \( \sigma_1 \leq \text{Re}(s) \leq s_0 \), \( |\text{Im}(s + ik)| \leq 1, k \geq 1 \) so that \( U_M(x, s; k) \) satisfies the conditions (1.1) - (1.3) with estimates for \( r_M(x, s; k) \) having higher order. The first approximation of \( U_M(x, s; k) \) becomes a finite sum over \( l = 1, \ldots, \kappa \) of series
\[
w_{0,l}(x, -is) = \sum_{n=0}^{\infty} \sum_{|j| = n+3, j_{n+2} = l} (-1)^{n+2} e^{-s \varphi_j(x)} a_j(x) = \sum_{n=0}^{\infty} U_{n+2,l}(x, -is),
\]
where \( j = (j_0, \ldots, j_n, j_{n+2}) \) are configurations of length \( |j| = n + 3 \), \( \varphi_j(x) \) are phase functions and \( a_j(x) \) are amplitudes determined by a recurrent procedure. The crucial point is to obtain an analytic continuation of \( w_{0,l}(x; s; k) \) from \( \text{Re}(s) > s_0 \) to a strip \( \sigma_1 \leq \text{Re}(s) \leq s_0 \) with \( \sigma_1 < s_0 \). To do this, the strategy is to establish suitable estimates for \( U_{n+2,l}(x, -is) \) and to apply a summation by packages. The structure of \( U_{n+2,l} \) is complicated since the phases \( \varphi_j(x) \) and the amplitudes \( a_j(x) \) are related to the dynamics of the reflecting rays having \( |j| \) reflexions and issued from the unstable manifold \( \{(x, \nabla \varphi(x)), x \in \text{supp} \ h \} \). It seems unlikely that an explicit relationship exists between \( U_{n+2,l}(x, -is) \) and the iterations \( L_{-s} f + g \) of the Ruelle operator \( L_{-s} f + g \) (see Sections 4 and 6). Consequently, one would not expect such relationship between \( \sum_{n=0}^{\infty} U_{n+2,l}(x, -is) \) and the zeta function \( Z(s) \). Thus, it appears the situation considered here is rather different to the case
of compact co-convex surfaces where one knows that the singularities of the Selberg zeta function coincide with the singularities of the corresponding Poincaré series which in turn is related to the resolvent of the Laplacian \(PPe\). It was observed by Ikawa [I5] that \(U_{n+2,l}(x, -is)\) can be compared with \(L^{n}_{s f + \tilde{g}} M_{n,s}(x) G_s v_0(\xi)\), where \(M_{n,s}(x)\) and \(G_s\) are suitable operators defined by means of billiard trajectories issued from appropriate unstable or stable manifolds, while \(v_0(\xi)\) is a function related to the boundary data \(m(x, s) = e^{-s\phi(x)}h(x)\). The precise definitions (with some small but essential differences\(^1\) from these in [I5]) are given in Section 4.

The first main step in the proof of the main result is Theorem 3 in Sections 4-5 below which provides an estimate of the form

\[
\|L^{n}_{s f + \tilde{g}} M_{n,s}(x) G_s v_0(\xi) - U_{n+2,l}(x, -is)\|_{CP(\Gamma)} \leq C_p(s, \varphi, h)(\theta + ca)^n, \forall p \in \mathbb{N}, \forall n \in \mathbb{N},
\]

where \(a = s_0 - \Re(s)\) and \(0 < \theta < 1\), \(C_p > 0\) are global constants. The assumption concerning the Dolgopyat estimates (4.3) of \(L_s\) is not required for the proof of Theorem 3. A statement similar to part (a) of Theorem 3 (corresponding to \(p = 0\)) was announced by Ikawa in [I5], however as far as we know no proof has ever been published. The proof of Theorem 3 is rather long and technical, however we consider it in detail since it is of fundamental importance for the considerations later on. Section 4 contains the proof of the case \(p = 0\), while Section 5 deals with \(p \geq 1\).

In Section 6 we obtain estimates for the traces \(w_{0,l}(x - is)|_{\Gamma}\) applying Theorem 3 and estimating the term \(\|L^{n}_{s f + \tilde{g}} M_{n,s}(x) G_s v_0(\xi)\|\). Here the Dolgopyat type estimates (4.3) for the iterations \(L^{n}_{s f + \tilde{g}}\) play a crucial role and we can justify the analyticity of \(w_{0,l}(x, -is)\) for \(\Re(s) \geq \sigma_0\) with \(\sigma_0 < s_0\). The estimates for \(w_{0,l}(x, -is)|_{\Gamma}\) for \(\sigma_0 \leq \Re(s) \leq s_0\) are different from those in the domain of absolute convergence \(\Re(s) > s_0\) and we must deal with two problems related to the analyticity with respect to \(s\) of our construction and to the control of the corresponding \(C^p(\Gamma)\) norms. The consequence of this procedure reflects on the first approximation \(W^{(0)}(x, -is; k)\) satisfying the conditions:

\[
\left\{
\begin{array}{l}
(\Delta_{x} - s^2)W^{(0)}(x, -is; k) = 0, \quad x \in \varOmega, \quad \sigma_0 \leq \Re(s) \leq 1, \\
W^{(0)}(x, -is; k) \in L^2(\varOmega) \quad \text{for} \quad \Re(s) > 0, \\
W^{(0)}(x, -is; k) - m(x, s) = (ik)^{-1}R_0(x, s; k) \quad \text{on} \quad \Gamma, \quad s \in \mathcal{D}_0
\end{array}
\right.
\]

with estimates

\[
\|R_0(x, s; k)\|_{CP(\Gamma)} \leq C_p k^{\beta + 3}, \quad 0 < \beta < 1, \quad \forall p \in \mathbb{N},
\]

where \(\mathcal{D}_0 = \{s \in \mathbb{C} : \sigma_0 \leq \Re(s) \leq 1, |\Im(s + ik)| \leq 1\}\). Finite lower order approximations \(W^{(j)}(x, s; k), j = 0, \ldots, M,\) are examined in Section 8 and we establish

\[
\sum_{j=0}^{M} W^{(j)}(x, s; k) - m(x, s) = (ik)^{-M}R_M(x, s; k), \quad x \in \Gamma, \quad s \in \mathcal{D}_0
\]

with polynomial estimates

\[
\|R_M(x, s; k)\|_{CP(\Gamma)} \leq Q_M k^{N(M)}, \quad s \in \mathcal{D}_0,
\]

\(^1\)In fact, it is difficult to see how the original definitions of the operators \(M_{n,s}\) and \(G_s\) in [I5] would work without the changes we have made in Sect. 4 below.
where $N(M)$ depends on $M$ and $(N(M) - M) \to \infty$ as $M \to \infty$. In this direction the situation is different from the absolutely convergent case treated in [I3], [B], where we have $N(M) = 0$ for $\Re(s) > s_0$. Since we need a finite number $M > (N - 1)/2$ lower order approximations, we choose $\sigma_1 < s_0$ close to $s_0$ to arrange an estimate
\[
\|R_M(x, s; k)\|_{L^\infty(\Gamma)} \leq C_M k^\alpha
\]
with $0 < \alpha < M - \frac{N+1}{2}$. The final step of our argument is in Section 9 where we solve an integral equation on the boundary $\Gamma$. To do this, we invert in $L^2(\Gamma)$ an operator $I + L(s; k)$ and we apply the last estimate to show that $L(s; k)$ has a small $L^2(\Gamma)$ norm for $k \geq k_0$.

2. Preliminaries

This section contains some basic facts about the dynamics of the billiard flow in the exterior $\Omega$ of $K$. Our main reference is [I3]; see also [B] and [PS1]. The notation follows mainly [I3].

Here and in the rest of the paper we assume that $K$ is as in Sect. 1. Denote by $A$ the $k_0 \times k_0$ matrix with entries $A(i, j) = 1$ if $i \neq j$ and $A(i, i) = 0$ for all $i$, and set
\[
\Sigma_A = \{\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0, \eta_1, \ldots, \eta_m, \ldots\} : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z}\},
\]
\[
\Sigma_A^+ = \{\eta_0, \eta_1, \ldots, \eta_m, \ldots\} : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \geq 0\},
\]
\[
\Sigma_A^- = \{\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0\} : 1 \leq \eta_j \leq k_0, \eta_j \in \mathbb{N}, \eta_{j-1} \neq \eta_j \text{ for all } j \leq 0\}.
\]

Let $\text{pr}_1 : S^*(\Omega) = \Omega \times S^{N-1} \to \Omega$ and $\text{pr}_2 : S^*(\Omega) \to S^{N-1}$ be the natural projections. Introduce the shift operator $\sigma : \Sigma_A \to \Sigma_A$ and $\sigma : \Sigma_A^+ \to \Sigma_A^- \text{ and } (\sigma(\xi))_i = \xi_{i+1}, i \in \mathbb{Z}, \xi \in \Sigma_A \text{ and } (\sigma(\xi))_i = \xi_{i+1}, i \in \mathbb{N}, \xi \in \Sigma_A^+$.

Fix a large ball $B_0$ containing $K$ in its interior. For any $x \in \Gamma = \partial K$ we will denote by $\nu(x)$ the outward unit normal to $\Gamma$ at $x$.

For any $\delta > 0$ and $V \subset \Omega$ denote by $S^*_V(V)$ the set of those $(x, u) \in S^*(\Omega)$ such that $x \in V$ and there exist $y \in \Gamma$ and $t \geq 0$ with $y + tu = x$, $y + su \in \mathbb{R}^N \setminus K$ for all $s \in (0, t)$ and $\langle u, \nu_T(y) \rangle \geq \delta$.

Remark 1. Notice that the condition (H) implies the existence of a constant $\delta_0 > 0$ depending only on the obstacle $K$ such that any $(x, u) \in S^*(\Omega)$ whose backward and forward billiard trajectories both have common points with $\Gamma$ belongs to $S^*_0(\Omega)$.

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. Denote by $X_1(z_0), X_2(z_0), \ldots, X_m(z_0), \ldots$ the successive reflection points (if any) of the forward trajectory $\gamma_+(z_0) = \{\text{pr}_1(\phi_t(z_0)) : 0 \leq t \leq \ell\}$. If $\gamma_+(z_0)$ is bounded (i.e. it has infinitely many reflection points), we will say that it has a forward itinerary $\eta = (\eta_1, \eta_2, \ldots)$ (or that it follows the configuration $\eta$) if $X_j(z_0) \in \partial K_{\eta_j}$ for all $j \geq 1$. Similarly, we will denote by $\gamma_-(z_0)$ the backward trajectory determined by $z_0$ and by $\ldots, X_m(z_0), \ldots, X_1(z_0), X_0(z_0)$ its backward reflection points (if any). For any $j \in \mathbb{Z}$ for which $X_j(z_0)$ exists denote by $\Xi_j(z_0)$ the direction of $\gamma(z_0) = \gamma_-(z_0) \cup \gamma_+(z_0)$ at $X_j(z_0)$, i.e. $\Xi_j(z_0) = \lim_{\ell \to 1} \text{pr}_2(\phi^1t_{\ell}(z_0))$, where $X_j(z_0) = \text{pr}_1(\phi^1t_{\ell}(z_0))$. Thus, $\phi^1t_j(z_0) = (X_j(z_0), \Xi_j(z_0))$. A finite string $J = (j_0, j_1, j_2, \ldots, j_m)$ of numbers $j_i = 1, 2, \ldots, k_0$ will be called an admissible configuration (of length $|J| = m + 1$) if $j_i \neq j_{i+1}$ for all $i = 0, 1, \ldots, m - 1$.

We will say that a billiard trajectory $\gamma$ with successive reflection points $x_0, x_1, \ldots, x_m$ follows the configuration $J$ if $x_i \in \Gamma_{j_i}$ for all $i = 0, 1, \ldots, m$.

A phase function on an open set $U$ in $\mathbb{R}^N$ is a smooth (C$^\infty$) function $\varphi : U \to \mathbb{R}$ such that $\|\nabla \varphi\| = 1$ everywhere in $U$. For $x \in U$ the level surface
\[
\mathcal{C}_\varphi(x) = \{y \in U : \varphi(y) = \varphi(x)\}
\]
has a unit normal field \( \pm \nabla \varphi(y) \).

**Remark 2.** It should be mentioned that in Sects. 2-5 the \( C^\infty \) smoothness assumption can be replaced by \( C^k \) for any \( k \geq 1 \).

The phase function \( \varphi \) defined on \( \mathcal{U} \) is said to satisfy the condition \( (P) \) on \( \Gamma_j \) if:

(i) the normal curvatures of \( C \) with respect to the normal field \( -\nabla \varphi \) are non-negative at every point of \( C \);

(ii) \( \mathcal{U}^+(\varphi) = \{ y + t \nabla \varphi(y) : t \geq 0, y \in \mathcal{U} \cap \Gamma_j \} \subset \bigcup_{i \neq j} K_i \).

A natural extension of \( \varphi \) on \( \mathcal{U}^+(\varphi) \) is obtained by setting \( \varphi(y + t \nabla \varphi(y)) = \varphi(y) + t \) for \( t \geq 0 \) and \( y \in \mathcal{U} \cap \Gamma_j \).

Given such a phase function \( \varphi \) and \( i \neq j \), denote by \( \mathcal{U}_i(\varphi) \) the set of all points \( x \) of the form \( x = X_1(y, \nabla \varphi(y)) + t \Xi_1(y, \nabla \varphi(y)) \), where \( y \in \mathcal{U} \cap \Gamma_j \) and \( t \geq 0 \) are such that \( X_1(y, \nabla \varphi(y)) \in \Gamma_i \), where

\[
\Gamma_{i,j} = \{ x \in \Gamma_i : \langle \nu(x), (y-x)/\|y-x\| \rangle \geq \delta_0 \text{ for all } y \in \Gamma_j \},
\]

Then setting \( \varphi_i(x) = \varphi(X_1(y, \nabla \varphi(y)) + t, x \) one gets a phase function \( \varphi_i \) satisfying the Condition \( (P) \) on \( \Gamma_i \). The operator sending \( \varphi \) to \( \varphi_i \) is denoted by \( \Phi_j \), i.e. \( \Phi_j(\varphi) = \varphi_i \).

Given an admissible configuration \( j = (j_0, j_1, \ldots, j_m) \) and a phase function \( \varphi \) satisfying the Condition \( (P) \) on \( \Gamma_{j_0} \), define

\[
\varphi_j = \Phi_{j_{m-1}} \circ \Phi_{j_{m-2}} \circ \ldots \circ \Phi_{j_1} \circ \Phi_{j_0}(\varphi).
\]

Notice that for any \( z \) in the domain \( \mathcal{U}_j(\varphi) \) of \( \varphi_j \) there exists \((x, u) \in S^*_+(\Gamma_{j_0})\) such that \( x \in \mathcal{U} \) and \( \gamma_+(x, u) \) follows the configuration \( j \), i.e. it has at least \( m \) reflection points and \( X_i(x, u) \in \Gamma_{j_i} \) for all \( i = 1, \ldots, m \), and \( z = X_m(x, u) + t \Xi_m(x, u) \) for some \( t \geq 0 \). Denote

\[
X^{-\ell}(z, \varphi_j) = X_{m-\ell}(x, u) \quad 0 \leq \ell \leq m.
\]

Several well-known facts about the dynamics of the billiard in \( \Omega \), phase functions and related objects will be frequently used throughout the paper and for convenience of the reader we state them here.

The following is a consequence of the hyperbolicity of the billiard flow in the exterior of \( K \) and can be derived from the works of Sinai on general dispersing billiards ([Si1], [Si2]) and from Ikawa’s papers on open billiards ([I3]; see also [B]). In this particular form it can be found in [Sj] (see also Ch. 10 in [PS1]).

**Proposition 1.** There exist global constants \( C > 0 \) and \( \alpha \in (0, 1) \) such that for any admissible configuration \( j = (j_0, j_1, \ldots, j_m) \) and any two billiard trajectories in \( \Omega \) with successive reflection points \( x_0, x_1, \ldots, x_m \) and \( y_0, y_1, \ldots, y_m \), both following the configuration \( j \), we have

\[
\|x_i - y_i\| \leq C (\alpha^i + \alpha^{m-i}) \quad 0 \leq i \leq m.
\]

Moreover, \( C \) and \( \alpha \) can be chosen so that if \((x_0, (x_1 - x_0)/\|x_1 - x_0\|)\) and \((y_0, (y_1 - y_0)/\|y_1 - y_0\|)\) belong to the same unstable manifold of the billiard flow, i.e. there exists a phase function \( \varphi \) satisfying the condition \( (P) \) on some open set \( \mathcal{U} \) containing \( x_0 \) and \( y_0 \) and such that \( \nabla \varphi(x_0) = (x_1 - x_0)/\|x_1 - x_0\| \) and \( \nabla \varphi(y_0) = (y_1 - y_0)/\|y_1 - y_0\| \), then \( \|x_i - y_i\| \leq C \alpha^{m-i} \) for \( 0 \leq i \leq m \).
Next, given a vector \( a = (a_1, \ldots, a_N) \in \mathbb{R}^N \), denote \( D_a = a_1 \frac{\partial}{\partial x_1} + \ldots + a_N \frac{\partial}{\partial x_N} \), and for any \( C^1 \) vector field \( f : U \to \mathbb{R}^N \) and any \( V \subset U \) set \( \| f \|_0(V) = \sup_{x \in V} \| f(x) \| \) and \( \| f \|_0 = \| f \|_0(U) \). Next, assuming \( f \) has continuous derivatives of all orders \( \leq p \) \((p \geq 1)\), set
\[
\| f \|_p(x) = \max_{a^{(1)}, \ldots, a^{(p)} \in S^{n-1}} \| (D_a^{(1)} \ldots D_a^{(p)} f)(x) \|, \quad \| f \|_p(V) = \sup_{x \in V} \| f \|_p(x), \quad \| f \|_p = \| f \|_p(U),
\]
\[
\| f \|_{(p)}(x) = \max_{0 \leq j \leq p} \| f \|_{(p)}(x), \quad \| f \|_{(p)}(V) = \sup_{x \in V} \| f \|_{(p)}(x), \quad \| f \|_{(p)} = \| f \|_{(p)}(U).
\]
Similarly, for \( x \in \Gamma \) and \( V \subset \Gamma \) set
\[
\| f \|_{\Gamma,(p)}(x) = \max_{a^{(1)}, \ldots, a^{(p)} \in S_x \Gamma} \| (D_a^{(1)} \ldots D_a^{(p)} f)(x) \|, \quad \| f \|_{\Gamma,(p)}(V) = \sup_{x \in V} \| f \|_{\Gamma,(p)}(x), \quad \| f \|_{\Gamma,(p)} = \| f \|_{\Gamma,(p)}(U),
\]
where \( S_x \Gamma \) is the unit sphere in the tangent plane \( T_x \Gamma \) to \( \Gamma \) at \( x \). Finally, set
\[
\| f \|_{\Lambda,(p)}(x) = \max_{0 \leq j \leq p} \| f \|_{\Gamma,(p)}(x), \quad \| f \|_{\Lambda,(p)}(V) = \sup_{x \in V} \| f \|_{\Lambda,(p)}(x), \quad \| f \|_{\Lambda,(p)} = \| f \|_{\Lambda,(p)}(U).
\]

**Remark 3.** It follows easily from the definitions that for any \( \delta > 0 \) and any integer \( p \geq 1 \) there exists a constant \( A_p = A_p(\delta, K) \) such that if \( \psi \) is a phase function which is at least \( C^{p+1} \)-smooth on some subset \( V \) of \( \Omega \) and \( x \in V \cap \Gamma \) with \( (x, \nabla \psi(x)) \in S_x \Gamma \), then \( \| \nabla \psi \|_p(x) \leq A_p \| \nabla \psi \|_{\Gamma,(p)}(x) \).

The following comprises Proposition 5.4 in [I1], Propositions 3.11 and 3.12 in [I3] and Lemma 4.1 in [I2] (see also the proof of the estimate (3.64) in [B]).

**Proposition 2.** For every integer \( p \geq 1 \) there exist global constants \( C_p > 0 \) and \( \alpha \in (0,1) \) such that for any admissible configuration \( j = (j_0, j_1, \ldots, j_m) \) and any phase functions \( \varphi \) and \( \psi \) satisfying the Condition (P) on \( \Gamma_{j_0} \) on some open set \( U \), we have
\[
\| \nabla \varphi_j \|_{(p)}(x) \leq C_p \| \nabla \varphi \|_{(p)}(U \cap B_0) \tag{2.1}
\]
for any \( x \in U_j(\varphi) \cap B_0 \), and
\[
\| \nabla \varphi_j - \nabla \psi_j \|_p(x) \leq C_p \alpha^m \| \nabla \varphi - \nabla \psi \|_p(U \cap B_0), \tag{2.2}
\]
\[
\| X^{-\ell}(\cdot, \nabla \varphi_j) - X^{-\ell}(\cdot, \nabla \psi_j) \|_{\Gamma,(p)}(x) \leq C_p \alpha^{m-\ell} \| \nabla \varphi - \nabla \psi \|_{(p)}(U \cap B_0) \tag{2.3}
\]
for any \( x \in U_j(\varphi) \cap U_j(\psi) \cap B_0 \) and \( 0 \leq \ell < m \). Finally, we can choose \( C_p > 0 \) so that
\[
\| X^{-\ell}(\cdot, \nabla \varphi_j) \|_{\Gamma,(p)}(x) \leq C_p \alpha^\ell \tag{2.4}
\]
for all \( x \in U_j(\varphi) \cap B_0 \) and \( 0 \leq \ell < m \).

Given \( x \) in the domain \( U \) of a phase function \( \varphi \), denote
\[
\Lambda_{\varphi}(x) = \left( \frac{G_{\varphi}(x)}{G_{\varphi}(X^{-1}(x, \nabla \varphi))} \right)^{1/(N-1)},
\]
where \( G_{\varphi}(y) \) is the Gauss curvature of \( C_{\varphi}(y) \) at \( y \). It follows from [I3] (or [B]) that there exist global constants \( 0 < \alpha_1 < \alpha < 1 \) such that
\[
0 < \alpha_1 \leq \Lambda_{\varphi}(y) \leq \alpha < 1 \tag{2.5}
\]
for any phase function \( \varphi \) and any \( y \in U(\varphi) \).
It follows from Lemma 3.1 in [I3] that there exist $\delta_1 > 0$ and $0 < d_0 < \frac{1}{2} \min_{i \neq j} \text{dist}(K_i, K_j)$ such that for any $i = 1, \ldots, k_0$, $x \in \Gamma_i$, and $x \in S' \cap \Gamma_\nu - 1$ with $0 \leq \langle \xi, \nu(x) \rangle \leq \delta_1$, the ray $\{x + t\xi : t \geq 0\}$ has no common points with $\bigcup_{j \neq i} B(K_j, d_0)$, where

$$B(A, d_0) = \{y + w : y \in A, w \in \mathbb{R}^N, ||w|| \leq d_0\}.$$ 

Fix an arbitrary $\chi_0 \in C_0^\infty(\mathbb{R})$ such that $\chi_0(t) = 0$ for $|t| < \delta_1/2$ and $\chi_0(t) = 1$ for $|t| \geq \delta_1$. Now for any $j = (j_0 = 1, j_1, \ldots, j_m)$ and any $x \in \mathcal{U}_j(\varphi)$, following [I3], define

$$(A_j(\varphi) h)(x) = \chi_0((\nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j)))) \Lambda_{\varphi,j}(x) h(X^{-m}(x, \nabla \varphi_j)),$$

where

$$\Lambda_{\varphi,j}(x) = \Lambda_{\varphi(j_1,\ldots,j_{m-1})}(x) \Lambda_{\varphi(j_1,\ldots,j_{m-1})}(X^{-1}(x, \nabla \varphi_j)) \cdots \Lambda_{\varphi}(X^{-m}(x, \nabla \varphi_j)) \in (0,1).$$

The following facts can be derived from [I1], [I3] (see also Proposition 5.1 in [B]).

**Proposition 3.** For every integer $p \geq 1$ there exists a global constant $C_p > 0$ such that for any admissible configuration $j = (j_0, j_1, \ldots, j_m)$ and any phase function $\varphi$ satisfying the Condition (P) on $\Gamma_{j_0}$ on some open set $\mathcal{U}$, we have $\|\Lambda_{\varphi,j}(x)\| \leq C_p \|\varphi\|_{(p)}(\mathcal{U} \cap B_0)$ for $x \in U_j(\varphi) \cap B_0$.

### 3. Stable and Unstable Manifolds for Open Billiards

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. For convenience we will assume that $x_0 \notin K$. Assume that the backward trajectory $\gamma_-(z_0)$ determined by $z_0$ is bounded, and let $\eta \in \Sigma_A$ be its itinerary.

Given $x \in \mathbb{R}^N$ and $\epsilon > 0$, by $B(x, \epsilon)$ we denote the open ball with center $x$ and radius $\epsilon$ in $\mathbb{R}^N$.

In this section we use some tools from [I3] to construct the local unstable manifold $W^u_{\text{loc}}(z_0)$ of $z_0$ in $S^*(\Omega)$ and show that it is Lipschitz in $z_0$ (and $\eta$). In a similar way one deals with local stable manifolds.

**Proposition 4.** There exists a constant $\epsilon_0 > 0$ such that for any $z_0 = (x_0, u_0) \in S^*_{\delta_0}(\Omega \cap B_0)$ whose backward trajectory $\gamma_-(z_0)$ has an infinite number of reflection points $X_j = X_j(z_0)$ $(j \geq 0)$ and $\eta \in \Sigma_A$ is its itinerary, the following hold:

(a) There exists a smooth ($C^\infty$) phase function $\psi = \psi_\eta$ satisfying the condition (P) on $\mathcal{U} = B(x_0, \epsilon_0) \cap \Omega$ such that $\psi(x_0) = 0$, $u_0 = \nabla \psi(x_0)$, and such that for any $x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)$ the billiard trajectory $\gamma_-(x, \nabla \psi(x))$ has an itinerary $\eta$ and therefore $d(\phi_t(x, \nabla \psi(x)), \phi_t(z_0)) \to 0$ as $t \to -\infty$. That is,

$$W^u_{\text{loc}}(z_0) = \{(x, \nabla \psi(x)) : x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)\}$$

is the local unstable manifold of $z_0$. Moreover, for any $p = 1, \ldots, k$ there exists a global constant $C_p > 0$ (independent of $z_0$ and $\eta$) such that

$$||\nabla \psi_\eta||_{(p)}(\mathcal{U}) \leq C_p. \tag{3.1}$$

(b) If $(y, v) \in S^*(\Omega \cap B_0)$ is such that $y \in C_\psi(x_0)$ and $\gamma_-(y, v)$ has the same itinerary $\eta$, then $v = \nabla \psi(y)$, i.e. $(y, v) \in W^u_{\text{loc}}(z_0)$.

(c) There exist a constant $\alpha \in (0,1)$ depending only on the obstacle $K$ and for every $p \geq 1$ a constant $C'_p > 0$ such that for any integer $r \geq 1$ and any $\zeta, \eta \in \Sigma_A$ with $\zeta_j = \eta_j$ for $-r \leq j \leq 0$, we have $||\nabla \psi_\zeta - \nabla \psi_\eta||_p(V) \leq C'_p \alpha^r$, where $V = \mathcal{U}(\psi_\eta) \cap \mathcal{U}(\psi_\zeta)$. 
Proof. (a) Take $\epsilon_0 > 0$ so small that whenever $(x, u) \in S^*_{\theta_0/2}(\Omega \cap B_0)$ and $(y, v) \in S^*(\Omega)$ is such that $||x - y|| < \epsilon_0$ and $||u - v|| < \epsilon_0$ we have $(y, v) \in S^*_{\theta_0}(\Omega)$. Then define $U$ as in part (a) above.

Set $d_m = ||X_{m+1} - X_m||$ and $u_m = \frac{X_m - X_{m+1}}{||X_m - X_{m+1}||} \in S^{n-1}$ ($m \geq 1$). Given any integer $m \geq 1$, consider the linear phase function $\psi^{(m)} = \psi^{(m, \eta)}$ in $\Omega$ such that $\nabla \psi^{(m)} \equiv u_m$ and $\psi^{(m)}(X_m) = -(d_m + d_{m+1} + \ldots + d_{-1})$. Then define

$$
\psi^{(m)}_m = \psi^{(m, \eta)} = \Phi_{\eta_{-1}} \circ \Phi_{\eta_{-2}} \circ \ldots \circ \Phi_{\eta_{-m+1}} \circ \Phi_{\eta_{-m+1}}(\psi^{(m)}).
$$

Clearly $\psi^{(m)}_m$ is a smooth phase function defined everywhere on $U$ (in fact, on a much larger subset of $\Omega$) with $\psi^{(m)}_m(X_0) = 0$. Moreover, it follows from Proposition 2.1 above that

$$
||\nabla \psi^{(m)}_m - \nabla \psi^{(m+1)}_m||_p(U) \leq \text{Const}_p \alpha^m, \quad m \geq 1
$$

(3.2)

for some global constant $\text{Const}_p > 0$ depending only on $K$ and $p$. Here we use the fact that $||\nabla \psi^{(m)} - \nabla \psi^{(m+1)}_m||_p \leq \text{Const}$, due to the special choice of the phase functions $\psi^{(m)}$ and $\psi^{(m+1)}$.

Since $\psi^{(m)}_m(X_0) = \psi^{(m+1)}_m(X_0) = 0$, it now follows that there exists a constant $C_p^m > 0$ such that

$$
||\psi^{(m)}_m(x) - \psi^{(m+1)}_m(x)|| \leq C_p^m \alpha^m
$$

for every $x \in U \cap B_0$. This implies that for every $x \in U$ there exists $\psi(x) = \lim_{m \to \infty} \psi^{(m)}_m(x)$. Now (3.2) shows that $\psi$ is $C^\infty$-smooth in $U$ and

$$
||\nabla \psi^{(m)}_m - \nabla \psi^{(m+1)}_m||_p(U) \leq \text{Const}_p \alpha^m, \quad m \geq 1
$$

(3.3)

In particular, $||\nabla \psi^{(m)}_m||_1 \equiv 1$ in $U$. Extending $\psi$ in a trivial way along straight line rays, we get a phase function $\psi$ satisfying the condition (P) in $U$.

Let us now show that $W = \{(x, \nabla \psi(x)) : x \in C_\psi(x_0) \cap U^+(\psi)\}$ is the local unstable manifold of $z_0$. Given $x \in C_\psi(x_0) \cap U^+(\psi)$ sufficiently close to $x_0$ and an arbitrary integer $r \geq 0$, consider the points $X^{-r}(x, \psi^{(m)}_m) \in \partial K_{\eta_{-r}}$ for $m \geq r$. By Proposition 1, there exist global constants $\text{Const} > 0$ and $\alpha \in (0, 1)$ such that $||X^{-r}(x, \psi^{(m)}_m) - X^{-r}(x, \psi^{(m')}_m)|| \leq \text{Const} \alpha^{m-r}$ for $m' \geq m > r$. Thus, there exists $X^{-r} = \lim_{m \to \infty} X^{-r}(x, \psi^{(m)}_m) \in \partial K_{\eta_{-r}}$ and

$$
||X^{-r}(x, \psi^{(m)}_m) - X^{-r}|| \leq \text{Const} \alpha^{m-r}, \quad m > r.
$$

(4.3)

It is now easy to see that $\{X^{-j}\}_{j=0}^\infty$ are the successive reflection points of a billiard trajectory in $\Omega$ and this is the trajectory $\gamma_-(x, \nabla \psi)$. The backward itinerary of the latter is obviously $\eta$. Moreover, (3.3) implies $d(\phi_t(x, \nabla \psi(x)), \phi_t(z_0)) \to 0$ as $t \to -\infty$, so $(x, \nabla \psi(x)) \in W^u_{\text{loc}}(z_0)$.

Finally, by (2.1), $||\psi^{(m)}_m||_p(U) \leq \text{Const}_p ||\psi^{(m)}_m||_p \leq \text{Const}_p$, and combining this with (3.3) gives (3.1).

(b) Let $(y, v) \in S^*(\Omega)$ be such that $y \in C_\psi(x_0)$ and $\gamma_-(y, v)$ has the same itinerary $\eta$. Define the phase functions $\varphi^{(m)}_m$ and $\varphi^{(m)}$ as in part (a) replacing the point $z_0 = (x_0, u_0)$ by $z = (y, v)$, and let $\varphi(x) = \lim_{m \to \infty} \varphi^{(m)}_m(x)$. Then by part (a), we have $W^u_{\text{loc}}(z) = \{(x, \nabla \varphi(x)) : x \in C_\varphi(y) \cap U^+(\phi)\}$.

On the other hand, it follows from Proposition 2 that there exist constants $\text{Const} > 0$ and $\alpha \in (0, 1)$ such that $||\nabla \psi^{(m)}_m - \nabla \varphi^{(m)}_m|| \leq \text{Const} \alpha^m$ for all $m \geq 0$, which implies $\varphi = \psi$. Thus, $v = \nabla \varphi(y) = \nabla \psi(y) \in W^u_{\text{loc}}(z_0)$.

(c) Choose the constants $\alpha \in (0, 1)$ and $\text{Const}_p > 0$ ($p = 1, \ldots, k$) as in part (a). Let $\zeta, \eta \in \Sigma^A$ be such that $\zeta_j = \eta_j$ for all $-r \leq j \leq 0$ for some $r \geq 1$. Construct the phase functions $\psi^{(m, \eta)}_m$ and $\psi^{(m, \zeta)}_m (m \geq 1)$ as in part (a); then $\psi_\eta = \lim_{m \to \infty} \psi^{(m, \eta)}_m, \psi_\zeta = \lim_{m \to \infty} \psi^{(m, \zeta)}_m$. It follows from
Proposition 2 that \( \| \nabla \psi^{(r,\eta)} - \nabla \psi^{(r,\zeta)} \| \leq \text{Const}_p \alpha^r \). Combining this with (3.3) with \( m = r \) for \( \eta \) and then with \( \eta \) replaced by \( \zeta \), one gets
\[
\| \nabla \psi_{\eta} - \nabla \psi_{\zeta} \| \leq \| \nabla \psi_{\eta} - \nabla \psi^{(r,\eta)} \| + \| \nabla \psi^{(r,\eta)} - \nabla \psi^{(r,\zeta)} \| + \| \nabla \psi^{(r,\zeta)} - \nabla \psi_{\zeta} \| \leq \text{Const}_p \alpha^r .
\]
This proves the assertion. ■

4. Ruelle operator and asymptotic solutions

Given \( \xi \in \Sigma_A \), let \( \ldots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \ldots \) be the successive reflection points of the unique billiard trajectory in the exterior of \( K \) such that \( P_j(\xi) \in K_{\xi_j} \) for all \( j \in \mathbb{Z} \). Set \( f(\xi) = \| P_0(\xi) - P_1(\xi) \| \). Following [I3] (see also Section 3), one constructs a sequence \( \{ \varphi_{\xi,j} \}_{j=-\infty}^{\infty} \) of phase functions such that for each \( j, \varphi_{\xi,j} \) is defined and smooth in a neighborhood \( U_{\xi,j} \) of the segment \([P_j(\xi), P_{j+1}(\xi)]\) in \( \Omega \) and:

(i) \( \| \nabla \varphi_{\xi,j} \| = 1 \) on \( U_{\xi,j} \) and satisfies the condition (P) on \( U_{\xi,j} \);

(ii) \( \nabla \varphi_{\xi,j}(P_j(\xi)) = \frac{P_{j+1}(\xi) - P_j(\xi)}{\| P_{j+1}(\xi) - P_j(\xi) \|} \);

(iii) \( \varphi_{\xi,j} = \varphi_{\xi,j+1} \) on \( \Gamma_{\xi,j+1} \cap U_{\xi,j} \cap U_{\xi,j+1} \);

(iv) for each \( x \in U_{\xi,j} \) the surface \( C_{\xi,j}(x) = \{ y \in U_{\xi,j} : \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x) \} \) is strictly convex with respect to its normal field \( \nabla \varphi_{\xi,j} \).

More precisely, one can proceed as follows. Given \( \xi \in \Sigma_A \), let \( \xi^- = (\ldots, \xi_{-2}, \xi_{-1}, \xi_0) \) and let \( \psi_{\xi^-} \) be the phase function with \( \psi_{\xi^-}(P_0) = 0 \) and \( \nabla \psi_{\xi^-}(P_0) = (P_1 - P_0)/\| P_1 - P_0 \| \) constructed in Proposition 4(a). Set \( \varphi_{\xi_0} = \psi_{\xi^-} \) and \( \varphi_{\xi,j} = (\psi_{\xi^-})_{(\xi_0, \xi_1, \ldots, \xi_j)} \) for any \( j > 0 \). For \( j < 0 \), setting \( \xi(j) = (\ldots, \xi_{j-2}, \xi_{j-1}, \xi_j) \) and using again Proposition 4, we get a phase function \( \psi_{\xi(j)} \) with \( \psi_{\xi(j)}(P_j) = 0 \) and \( \nabla \psi_{\xi(j)}(P_j) = (P_{j+1} - P_j)/\| P_{j+1} - P_j \| \). By the uniqueness of the phase functions \( \psi_{\xi} \) (see Proposition 4(c)), it follows that there exists a constant \( c_j \) such that \( \psi_{\xi^-} = (\psi_{\xi(j)} + c_j)_{(\xi_0, \xi_1, \ldots, \xi_0)} \) (locally near the segment \([P_0, P_1])\). Setting \( \varphi_{\xi,j} = \psi_{\xi(j)} + c_j \), one obtains a phase function defined on some naturally determined (see the proof of Proposition 3 (a)) open set \( U_{\xi^-,-j} \) such that
\[
(\varphi_{\xi,j})_{(\xi_0, \xi_1, \ldots, \xi_{-1}, \xi_0)} = \psi_{\xi^-}, j < 0 .
\]
This completes the construction of the phase functions \( \varphi_{\xi,j} \).

Moreover, it follows from Proposition 2 that for any \( p = 1, \ldots, k \), there exists a global constant \( C_p > 0 \) such that
\[
\| \nabla \varphi_{\xi,j} \|_{(p)} \leq C_p
\]
for all \( \xi \in \Sigma_A \) and \( j \in \mathbb{Z} \).

Remark 4. Notice that the above construction can be carried out for \( j < 0 \) for any \( \xi \in \Sigma_A \) and any billiard trajectory \( \gamma \) in \( \Omega \) with reflection points \( \ldots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi) \) such that \( P_j(\xi) \in K_{\xi_j} \) for all \( j \leq 0 \). Then one defines a phase function \( \psi_{\xi^-} \) with \( \psi_{\xi^-}(P_0) = 0 \) as above, and using (4.1) one gets a sequence \( \{ \varphi_{\xi,j} \}_{j \leq 0} \) of phase functions such that for each \( j < 0 \), \( \varphi_{\xi,j} \) is defined and smooth in a neighborhood \( U_{\xi,j} \) of the segment \([P_j(\xi), P_{j+1}(\xi)]\) in \( \Omega \) and satisfies the conditions (i)-(iv). Moreover (4.2) holds for any \( p = 1, \ldots, k \) and any \( j \leq 0 \).
For any $y \in U_{\xi,j}$ denote by $G_{\xi,j}(y)$ the Gauss curvature of $C_{\xi,j}(x)$ at $y$. Now define $g : \Sigma_A \to \mathbb{R}$ by
\[
g(\xi) = \frac{1}{N - 1} \ln \frac{G_{\xi,1}(P_1(\xi))}{G_{\xi,0}(P_0(\xi))}.
\]
Given a function $F(\xi)$ defined on $\Sigma_A$, we set
\[
\var_n F = \sup \{|F(\xi) - F(\eta)| : \xi_i = \eta_i \text{ for } |i| \leq n\},
\]
and for $0 < \theta < 1$ we define $\|F\|_{\theta} = \sup_n \frac{\var_n F}{\theta^n}$, $\|F\|_{\infty} = \|F\|_{\infty} + \|F\|_\theta$ and introduce the space $\mathcal{F}_0(\Sigma_A) = \{F(\xi) : \|F\|_{\theta} < \infty\}$. The functions $f(\xi), g(\xi) \in \mathcal{F}_0(\Sigma_A)$ with a suitable $0 < \theta < 1$ (see [I4]). By Sinai's Lemma (see e.g. [PP]), there exist $\tilde{f}, \tilde{g} \in \mathcal{F}_0(\Sigma_A)$ depending on future coordinates only and $\chi_1, \chi_2 \in \mathcal{F}_0(\Sigma_A)$ such that
\[
f(\xi) = \tilde{f}(\xi) + \chi_1(\xi, \Gamma\xi) , \quad g(\xi) = \tilde{g}(\xi) + \chi_2(\xi, \Gamma\xi), \quad \xi \in \Sigma_A.
\]
As in the proof of Sinai's Lemma for any $k = 1, \ldots, k_0$ choose and fix an arbitrary sequence $n(\xi) = (\ldots, n_{-m}, \ldots, n_{-1}, n_0(\xi)) \in \Sigma_A$ with $n_0(\xi) \neq k$. Then for any $\xi \in \Sigma_A$ (or $\xi \in \Sigma_A^+ \cap \Sigma_A^+$) set
\[
e(\xi) = (\ldots, \eta_{-m}, \ldots, \eta_{-1}, \eta_0(\xi)) = (\xi_0, \xi_1, \ldots, \xi_m, \ldots) \in \Sigma_A.
\]
Then we have
\[
\chi_1(\xi) = \sum_{n=0}^{\infty} [f(\sigma^n(\xi)) - f(\sigma^n e(\xi))],
\]
and the function $\chi_2$ is defined similarly, replacing $f$ by $g$.

Setting $\chi(\xi, s) = -s\chi_1(\xi) + \chi_2(\xi)$, for the function $R(\xi, s) = -s f(\xi) + g(\xi) + i\pi$ we have
\[
R(\xi, s) = \tilde{R}(\xi, s) + \chi(\xi, s) - \chi(\sigma\xi, s) \quad \text{for } \xi \in \Sigma_A, \; s \in \mathbb{C},
\]
where $\tilde{R}(\xi, s) = -s \tilde{f}(\xi) + \tilde{g}(\xi) + i\pi$ depends on future coordinates of $\xi$ only (so it can be regarded as a function on $\Sigma_A^+ \times \mathbb{C}$). Below we need the Ruelle transfer operator $L_s : C(\Sigma_A^+) \to C(\Sigma_A^+)$ defined by
\[
L_s u(\xi) = \sum_{\eta \in \xi} e^{\bar{R}(\eta, s)} u(\eta)
\]
for any continuous (complex-valued) function $u$ on $\Sigma_A^+$ and any $\xi \in \Sigma_A$. Notice that
\[
L_s^n u(\xi) = (-1)^n \sum_{\eta \in \xi} e^{-s f(\eta) + \tilde{g}(\eta)} u(\eta) = (-1)^n L_{-s \tilde{f} + \tilde{g}}(\xi), \quad n \geq 0,
\]
hence $\|L^n\|_{\infty} = \|L^n_{-s \tilde{f} + \tilde{g}}\|_{\infty}$. Set $\tilde{L}_s = L_{-s \tilde{f} + \tilde{g}}$.

Define the map $\Phi : \Sigma_A \to \Lambda_{\partial K} = \Lambda \cap S^{\ast}_{\partial K}(\Omega)$ by
\[
\Phi(\xi) = (P_0(\xi), (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\|).
\]
Then $\Phi$ is a bijection such that $\Phi \circ \sigma = B \circ \Phi$, where $B : \Lambda_{\partial K} \to \Lambda_{\partial K}$ is the billiard ball map. Let $\pi : \Sigma_A \to \Sigma_A^+$ be the natural projection. Notice that for any function $v : \Sigma_A \to \mathbb{C}$ the function $v \circ \pi : \Sigma_A \to \mathbb{C}$ depends on future coordinates only, so $(v \circ \pi) \circ \Phi^{-1} : \Lambda_{\partial K} \to \mathbb{C}$ is constant on local unstable manifolds. Conversely, if $h : \Lambda_{\partial K} \to \mathbb{C}$ is constant on local unstable manifolds, then $v = h \circ \Phi : \Sigma_A \to \mathbb{C}$ depends on future coordinates only, so it can be regarded as a function on $\Sigma_A^+$. 

\[
\text{ANALYTIC CONTINUATION OF THE RESOLVENT 13}
\]
Denote by $C^\text{Lip}_u(\Lambda_{\partial K})$ the space of Lipschitz functions $h : \Lambda_{\partial K} \to \mathbb{C}$ that are constant on local unstable manifolds. For such $h$ let $\text{Lip}(h)$ denote the Lipschitz constant of $h$, and for $t \in \mathbb{R}, |t| \geq 1,$ define

$$\|h\|_{\text{Lip},t} = \|h\|_0 + \text{Lip}(h)\frac{|t|}{|t|}, \quad \|h\|_0 = \sup_{x \in \Lambda_{\partial K}} |h(x)|.$$ 

To estimate the norm of $\tilde{L}^n_s$, we will apply Dolgopyat type estimates ([D]) established in the case of open billiard flows in [St2] for $N = 2$ and in [St3] for $N \geq 3$ under certain assumptions (see Appendix B below). It follows from these results that there exist constants $s_0 < s_1$ and $0 < \rho < 1$ so that for $s = \tau + it$ with $\tau \geq s_0, |t| \geq 1$ and $n = p[\log |t|] + l, p \in \mathbb{N}, 0 \leq l \leq \lfloor \log |t| \rfloor - 1$, for any function $v \in C(\Sigma_A^+)$ of the form $v = h \circ \Phi$ for some $h \in C^\text{Lip}_u(\Lambda_{\partial K})$ we have

$$\|\tilde{L}^n_tv\|_\infty \leq C\rho^{p[\log |t|]}e^{\text{Pr}(-\tau\tilde{f} + \tilde{g})}\|h\|_{\text{Lip},t}.$$ 

(4.3)

We will deal with oscillatory data on $\Gamma_1$ (which can be replaced by any $\Gamma_j$) of the form

$$u_0(x; s) = e^{-is\varphi(x)}h(x), \quad x \in \Gamma_1, s \in \mathbb{C}.$$ 

(4.4)

Here $\varphi$ is a $C^\infty$ phase function satisfying the condition (P) on $\Gamma_1$ (see Sect. 2 above) and $h$ is a $C^\infty(\Gamma)$ function with small support on $\Gamma_1$. For every configuration $j = (j_0, j_1, \ldots, j_n), j_0 = 1,$ we can construct a function $u_j(x; s)$ following a recurrent procedure (see [I5]). We construct a sequence of phase functions $\varphi_j(x)$ and amplitudes $a_j(x)$ so that setting

$$u_j(x; s) = (-1)^|j|e^{-s\varphi_j(x)}a_j(x),$$

for the configurations $j$ and $j' = (j_0, j_1, \ldots, j_m, j_{m+1})$ we have $u_{j_0}(x; s) = u_0(x; s)$ on $\Gamma_1$ and $u_j(x; s) + u_j(x; s) = 0$ on $\Gamma_{j_{m+1}}$. The phase functions $\varphi_j$ are determined following the procedure in Section 2, while the amplitudes $a_j(x)$ are determined as the solutions of the transport equations and, using the notations of Section 2, we set $a_j(x) = (A_j(\varphi)h)(x)$.

Next, let $\mu = (\mu_0 = 1, \mu_1, \ldots) \in \Sigma_A^+$. It follows from [I3] that there exists a unique point $y(\mu) \in \Gamma_1$ such that the ray $\gamma(y(\mu))$ issued from a point $y(\mu)$ in direction $\nabla \varphi(y(\mu))$ follows the configuration $\mu$. Let $Q_0(\mu) = y(\mu), Q_1(\mu), \ldots$, be the consecutive reflection points of this ray. Define

$$f_i^+(\mu) = \|Q_i(\mu) - Q_{i+1}(\mu)\|, \quad g_i^+(\mu) = \frac{1}{N-1}\ln \frac{G_{\mu,i}^+(Q_{i+1}(\mu))}{G_{\mu,i}^+(Q_i(\mu))} < 0,$$

where $G_{\mu,i}^+(y)$ denotes the Gauss curvature of the surface

$$G_{\mu,i}^+(x) = \{z \in U_{(\mu_0, \mu_1, \ldots, \mu_i)}(\varphi) : \varphi(\mu_0, \mu_1, \ldots, \mu_i)(z) = \varphi(\mu_0, \mu_1, \ldots, \mu_i)(x)\}$$

at $y$.

Next, for $s \in \mathbb{C}$ and $\xi \in \Sigma_A^+$ with $\xi_0 = 1$, following [I5], set

$$\phi^+(\xi, s) = \sum_{n=0}^{\infty} (-s [f(\sigma^n e(\xi)) - f^+_n(\xi)] + [g(\sigma^n e(\xi)) - g^+_n(\xi)])$$

Formally, define $\phi^+(\xi, s) = 0$ when $\xi_0 \neq 1$, thus obtaining a function $\phi^+ : \Sigma_A^+ \times \mathbb{C} \to \mathbb{C}$. Now for any $s \in \mathbb{C}$ define the operator $G_s : C(\Sigma_A^+) \to C(\Sigma_A^+)$ by

$$G_s v(\xi) = \sum_{\sigma \eta = \xi} e^{-\phi^+(\sigma, s) - s f(\eta) + \tilde{g}(\eta)} v(\eta).$$

for any \( v \in C(\Sigma^+_A) \) and \( \xi \in \Sigma^+_A \). (Although similar, this is different from the corresponding definition in [15].)

We will replace the function \( v_0 \) from Sect. 4.2 in [I5] by \( \tilde{v} \in \mathcal{F}_\theta(\Sigma^+_A) \) defined by

\[
\tilde{v}_s(\xi) = e^{-s \varphi(Q_0(\xi))} h(Q_0(\xi))
\]

if \( \xi_0 = 1 \) and \( \tilde{v}_s(\xi) = 0 \) otherwise. Recall that \( h \) comes from the boundary data (4.4).

**Fix an arbitrary** \( \ell = 1, \ldots, k_0 \) **and an arbitrary point** \( x_0 \in \Gamma_\ell \). Define the function \( \phi^- (x_0; \cdot, \cdot) : \Sigma_A \times \mathbb{C} \to \mathbb{C} \) (depending on \( \ell \) as well) as follows. First, set \( \phi^- (x_0; \eta, s) = 0 \) if \( \eta_0 \neq \ell \). Next, assume that \( \eta \in \Sigma_A \) satisfies \( \eta_0 = \ell \). There exists a unique billiard trajectory in \( \Omega \) with successive reflection points \( P_i(x_0; \eta) \in \partial K_{\eta_i} \) \( (-\infty < i \leq 0) \) such that \( x_0 = \tilde{P}_{-1}(x_0; \eta) + t \nabla \varphi^- (\tilde{P}_{-1}(x_0; \eta)) \) for some \( t > 0 \). In general the segment \( [\tilde{P}_{-1}(x_0; \eta), x_0] \) may intersect the interior of \( K_\ell \). If this is the case, set again \( \phi^- (x_0; \eta, s) = 0 \). Otherwise, denote \( \tilde{P}_0(x_0; \eta) = x_0 \), and for any \( i < 0 \) set

\[
f^-_i (x_0; \eta) = \| \tilde{P}_{i+1}(x_0; \eta) - \tilde{P}_i(x_0; \eta) \|, \quad g^-_i (x_0; \eta) = \frac{1}{N-1} \ln \frac{G_{\eta,i}(\tilde{P}_{i+1}(x_0; \eta))}{G_{\eta,i}(\tilde{P}_i(x_0; \eta))},
\]

Then define

\[
\phi^- (x_0; \eta, s) = -s \sum_{i=-1}^{-\infty} [f^i(\sigma^i(\eta)) - f^-_i (x_0; \eta)] + \sum_{i=-1}^{\infty} [g(\sigma^i(\eta)) - g^-_i (x_0; \eta)]. \tag{4.5}
\]

We will show later that this series is absolutely convergent.

Next, similarly to [I5], define the operator \( \mathcal{M}_{n,s}(x_0) : C(\Sigma^+_A) \to C(\Sigma^+_A) \) (depending also on \( \ell \)) by

\[
(\mathcal{M}_{n,s}(x_0)v)(\xi) = \sum_{\sigma \eta = \xi} e^{-\phi^- (x_0; \sigma^{n+1} e(\eta), s) - \chi(\sigma^{n+1} e(\eta), s) - s \tilde{f}(\eta) - \tilde{g}(\eta)} v(\eta)
\]

for any \( v \in C(\Sigma^+_A) \) and \( \xi \in \Sigma^+_A \).

Let \( s_0 \in \mathbb{R} \) be the abscissa of absolute convergence of the dynamical zeta function (see Sect. 1) determined by \( \Pr (-s_0 \tilde{f} + \tilde{g}) = 0 \). Here \( \Pr (G) \) is the topological pressure of \( G \) defined by

\[
P(G) = \sup_{\mu \in \mathcal{M}} [h(\mu) + \int_{\Sigma_A^+} G d\mu],
\]

where \( \mathcal{M} \) is the set of all probabilistic measures on \( \Sigma_A^+ \) invariant with respect to \( \sigma \) and \( h(\mu) \) is the metric entropy of \( \mu \).

The part (a) in the following theorem is similar to (4.10) in [I5]:

**Theorem 3.** There exist global constants \( C > 0, c > 0 \) and \( \theta \in (0,1) \) depending only on \( K \) such that for any choice of \( \ell = 1, \ldots, k_0 \), \( x_0 \in \Gamma_\ell \) and \( a > 0 \) the following hold:

(a) For all integers \( n \geq 1 \), all \( \xi \in \Sigma^+_A \) with \( \xi_0 = \ell \) and all \( s \in \mathbb{C} \) with \( \Re(s) \geq s_0 - a \) we have

\[
|L^0_n \mathcal{M}_{n,s}(x_0) G_s \tilde{v}_s(\xi) - \sum_{|j| = n+3, j_3 = \ell} u_j(x_0; -i s)| \leq \quad C (\theta + ca)^n e^{C |\Re(s)| (1 + \|\varphi\|_{\Gamma_0} + \|\nabla \varphi\|_{\Gamma_0})} \left( |s| + \|\nabla \varphi\|_{\Gamma_0} \right) \|h\|_{\Gamma_0} + \|h\|_{\Gamma_1} \quad . \tag{4.6}
\]
(b) For any integers \( p \geq 1 \) and \( n \geq 1 \), any \( \xi \in \Sigma^+_A \) with \( \xi_0 = \ell \) and any \( s \in \mathbb{C} \) with \( \Re(s) \geq s_0 - a \) we have
\[
\left| \left( L^s_n \mathcal{M}_{n,s}(\cdot) \mathcal{G}_s \tilde{v}_s \right)(\xi) - \sum_{|j|=n+3,j_{n+2}=\ell} u_j(\cdot; -i s) \right|_{\Gamma,\rho} (x_0) \leq C(\theta + ca)^n e^{C[|\Re(s)| + (1 + \|\varphi\|_{\Gamma,0}) + \|\nabla \varphi\|_{\Gamma,1}]} \sum_{i=0}^{p} \left| s \|\nabla \varphi\|_{\Gamma,i} + \|\nabla \varphi\|_{\Gamma,i+1} \right|^{i+1} \|h\|_{\Gamma,\rho-i}. \tag{4.7}
\]

In this section we deal with part (a). The proof of part (b) is given in Section 5 below.

**Proof of Theorem 3(a).** Fix \( \ell, x_0 \in \Gamma_\ell \) and \( \xi \in \Sigma^+_A \) with \( \xi_0 = \ell \). Then for any \( s \in \mathbb{C} \) and \( n \geq 1 \), using Sect. 4.1 in [15], we get
\[
u_{(1,j_1,j_2,\ldots,j_{n+1},\ell)}(x_0, -i s) = (-1)^{n+2} e^{-s[\varphi(Q_0(j)) + f_0^+(x_0;j) + \ldots + f_{n+1}^+(x_0;j)]} \left( A_j(\varphi) h \right)(x_0), \tag{4.8}
\]
where \( f_i^+(x_0;j) = \|Q_i(x_0;j) - Q_{i+1}(x_0;j)\| \ (i = 0, 1, \ldots, n + 1) \), \( Q_i(x_0;j) \) being the reflection points of the billiard trajectory issued from a point \( y \in \Gamma_1 \) in direction \( \nabla \varphi(y) \) which follows the configuration \( j \) for its first \( n + 1 \) reflections and is such that \( Q_{n+2}(x_0;j) = x_0 \). We assume that \( x_0 \) and \( j \) are such that the segment \([Q_{n+1}(x_0;j), x_0]\) does not intersect the interior of \( K_\ell \). Then there is exactly one such trajectory.

On the other hand,
\[
(L^s_n \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s)(\xi) = (-1)^{n} \sum_{\sigma^{n+1} \mu = \xi} e^{-s f_0(\eta) + \tilde{g}_0(\eta)} (\mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s)(\eta)
\]
\[
= (-1)^{n} \sum_{\sigma^{n+1} \mu = \xi} e^{-s f_0(\eta) + \tilde{g}_0(\eta)} \sum_{\sigma^\ell = \eta} e^{-\phi^+(x_0;\sigma^{n+1} e(\zeta),s) - \chi(\sigma^{n+1} e(\zeta),s) - s f_0(\eta) + \tilde{g}_0(\eta)}
\]
\[
\times \sum_{\sigma^\mu = \zeta} e^{-\phi^+(\mu,s) + \chi(\mu,s) - s f_0(\eta) + \tilde{g}_0(\eta)} \tilde{v}_s(\mu)
\]
\[
= (-1)^{n} \sum_{\sigma^{n+2} \mu = \xi, \mu_0 = 1} e^{-s f_0(\mu) + \tilde{g}_0(\mu)} W^{(n+2)}(x_0; \mu, s), \tag{4.9}
\]
where the function \( W^{(n+2)}(x_0; \cdot, \cdot) = W^{(n+2)}_{1,\ell}(x_0; \cdot, \cdot) : \Sigma^+_A \times \mathbb{C} \rightarrow \mathbb{C} \) is defined by \( W^{(n+2)}(x_0; \mu, s) = 0 \) when \( \mu_0 \neq 1 \) or \( \mu_{n+2} \neq \ell \) and
\[
W^{(n+2)}(x_0; \mu, s) = e^{-\phi^-(x_0;\sigma^{n+1} e(\mu),s) - \chi(\sigma^{n+1} e(\mu),s) - \phi^+(\mu,s) + \chi(\mu,s) - s \varphi(Q_0(\mu))} h(Q_0(\mu)) \tag{4.10}
\]
whenever \( \mu_0 = 1 \) and \( \mu_{n+2} = \ell \). It follows from (4.9) that
\[
(L^s_n \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s)(\xi) = (-1)^{n} \left( L^{n+2}_{-s f_0 + \tilde{g}} W^{(n+2)}(x_0; \cdot, s) \right)(\xi). \tag{4.11}
\]

Clearly, in (4.9) the summation is over sequences
\[
\mu = (1, j_1, j_2, \ldots, j_{n+1}, \ell, \xi_1, \xi_2, \ldots) = (j, \xi), \tag{4.12}
\]
with \( \mu_{n+2} = \ell, \) where \( j = (1, j_1, j_2, \ldots, j_{n+1}, \ell). \)

Write for convenience
\[
W^{(n+2)}(x_0; \mu, s) = e^{w(x_0; \mu, s)} e^{-s \varphi(Q_0(\mu))} h(Q_0(\mu)), \tag{4.13}
\]
where
\[ w(x_0; \mu, s) = -\phi^-(x_0; \sigma^{n+1}e(\sigma\mu), s) - \chi(\sigma^{n+1}e(\sigma\mu), s) - \phi^+(\mu, s) + \chi(e(\mu), s). \] (4.14)

It follows from Propositions 1 and 2 that there exist global constants \( C > 0 \) and \( \alpha \in (0, 1) \) such that
\[ |f(\sigma^n e(\xi)) - f_n^+(\xi)| \leq C \alpha^n, \quad |g(\sigma^n e(\xi)) - g_n^+(\xi)| \leq C \|
abla \varphi\|_{\Gamma, (1)} \alpha^n \]
for all \( \xi \in \Sigma_A \) and all integers \( n \geq 1 \), so
\[ \phi^+(\mu, s) = (|s| + \|
abla \varphi\|_{\Gamma, (1)}) O(\alpha^n) + \sum_{i=0}^{n+1} (-s [f(\sigma^i e(\mu)) - f_i^+(\mu)] + [g(\sigma^i e(\mu)) - g_i^+(\mu)]). \]

Thus, using the definitions of \( \tilde{f}, \tilde{g} \) and \( \chi \) and the fact that \( \chi(\sigma^{n+2}e(\mu), s) = \chi(\sigma^{n+1}e(\sigma\mu), s) + |s| O(\alpha^n) \), we get
\[ -s [f_n^+(\mu) + f_1^+(\mu) + \ldots + f_{n+1}^+(\mu)] + [g_0^+(\mu) + g_1^+(\mu) + \ldots + g_{n+1}^+(\mu)] = (s + \|
abla \varphi\|_{\Gamma, (1)}) O(\alpha^n) - \phi^+(\mu, s) - s f_{n+2}(\mu) + \tilde{g}_{n+2}(\mu) + \chi(e(\mu), s) - \chi(\sigma^{n+1}e(\sigma\mu), s). \]

Now, fix for a moment \( n \geq 1 \) and \( \mu \) as in (4.12), and set \( \eta = \sigma^{n+1}e(\sigma(\mu)) \). Then we have
\[ \eta = \sigma^{n+1}e(\sigma(\mu)) = (...) \star, \mu_1, \mu_2, \ldots, \mu_{n+1}; \mu_{n+2} = \ell, \mu_{n+3}, \ldots, \] (4.15)
and as for \( \phi^+ \) one gets
\[ \phi^-(x_0; \eta, s) = (|s| + \|
abla \varphi\|_{\Gamma, (1)}) O(\alpha^n) - s \sum_{i=-1}^{-n-1} [f(\sigma^i \eta) - f_i^-(x_0; \eta)] + \sum_{i=-1}^{-n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)]. \]

From these estimates and (4.14) one derives that
\[ w(x_0; \mu, s) = s f_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - \phi^-(x_0; \eta, s) - s \sum_{i=0}^{n+1} f_i^+(\mu) + \sum_{i=0}^{n+1} g_i^+(\mu) + (|s| + \|
abla \varphi\|_{\Gamma, (1)}) O(\alpha^n) \]
\[ = s f_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s u(x_0; \mu) + v(x_0; \mu) + (|s| + \|
abla \varphi\|_{\Gamma, (1)}) O(\alpha^n), \] (4.16)
where
\[ u(x_0; \mu) = -\sum_{i=0}^{n+1} [f(\sigma^i \eta) - f_i^- (x_0; \eta)] + \sum_{i=0}^{n+1} f_i^+(\mu), \quad v(x_0; \mu) = -\sum_{i=-1}^{-n-1} [g(\sigma^i \eta) - g_i^- (x_0; \eta)] + \sum_{i=0}^{n+1} g_i^+(\mu). \]

We will show that
\[ \left| u(x_0; \mu) - \sum_{i=0}^{n+1} f_i^+(x_0; \mu) \right| \leq \text{Const} \alpha^n, \] (4.17)
and
\[ \left| e^{v(x_0; \mu)} h(Q_0(\mu)) - (A_j(\varphi) h)(x_0) \right| \leq \text{Const} \left( \|
abla \varphi\|_{\Gamma, (1)} \|h\|_{\Gamma,0} + \|h\|_{\Gamma, (1)} \right) \theta^n \] (4.18)
for some global constant \( \text{Const} > 0 \), where
\[ \theta = \sqrt{\alpha} \in (0, 1). \]

There exists a unique ray \( \gamma(y, \varphi) \) issued from a point \( y = y_n(x_0; \mu) \in \Gamma_1 \) in direction \( \nabla \varphi(y) \), following the configuration \( \mu \), and such that its \((n+2)\)nd reflection point is \( x_0 \). That is, if \( \tilde{Q}_1(x_0; \mu) \)
are the consecutive reflection points of $\gamma(y, \varphi)$, then $\tilde{Q}_{n+2}(x_0; \mu) = x_0$. Again it is essential to consider the case when the segment $[\tilde{Q}_{n+1}(x_0; \mu, x_0]$ does not intersect the interior of $K_k$.

Before we continue, let us make a few simple (however essential) remarks concerning the sequences of reflection points

\begin{equation}
Q_0(\mu) \in \Gamma_1 = \Gamma_{\mu_0}, Q_1(\mu) \in \Gamma_{\mu_1}, \ldots, Q_{n+1}(\mu) \in \Gamma_{\mu_{n+1}}, Q_{n+2}(\mu) \in \Gamma_{\mu_{n+2}} = \Gamma_{\ell}, \ldots, \tag{4.19}
\end{equation}

\begin{equation}
\tilde{Q}_0(x_0; \mu) \in \Gamma_1 = \Gamma_{\mu_0}, \tilde{Q}_1(x_0; \mu) \in \Gamma_{\mu_1}, \ldots, \tilde{Q}_{n+1}(x_0; \mu) \in \Gamma_{\mu_{n+1}}, \tilde{Q}_{n+2}(x_0; \mu) \in \Gamma_{\ell}, \ldots \tag{4.20}
\end{equation}

\begin{equation}
\ldots, P_{\eta-n-1}(\eta) \in \Gamma_{\eta-n-1} = \Gamma_{\mu_1}, \ldots, P_{-1}(\eta) \in \Gamma_{\eta-1} = \Gamma_{\mu_{n+1}}, P_0(\eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_{\ell}, \ldots, \tag{4.21}
\end{equation}

\begin{equation}
\ldots, \tilde{P}_{\eta-n-1}(x_0; \eta) \in \Gamma_{\eta-n-1} = \Gamma_{\mu_1}, \ldots, \tilde{P}_{-1}(x_0; \mu) \in \Gamma_{\eta-1} = \Gamma_{\mu_{n+1}}, \tilde{P}_0(x_0; \eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_{\ell}. \tag{4.22}
\end{equation}

It is clear that the sequences (4.19) and (4.20) ‘start’ from the same unstable manifold (determined by $\varphi = \text{const}$), therefore by Proposition 1 there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that

\begin{equation}
\|Q_i(\mu) - \tilde{Q}_i(x_0; \mu)\| \leq C \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2. \tag{4.23}
\end{equation}

Similarly, the right ends of sequences (4.21) and (4.22) determine points in the same unstable manifold, so these sequences ‘converge backwards’, i.e.

\begin{equation}
\|P_i(\eta) - \tilde{P}_i(x_0; \eta)\| \leq C \alpha^i \|, \quad i \leq 0. \tag{4.24}
\end{equation}

On the other hand, notice that the sequences (4.19) and (4.21) continue indefinitely to the right following the same patterns. Thus, these sequences ‘converge forwards’. More precisely, using Proposition 1 again, we have

\begin{equation}
\|Q_i(\mu) - P_{i-n-2}(\eta)\| \leq C \alpha^i, \quad 1 \leq i, \tag{4.25}
\end{equation}

Similarly, the sequences (4.20) and (4.22) ‘converge forwards’ to $\tilde{Q}_{n+2}(x_0; \mu) = \tilde{P}_0(x_0; \eta) = x_0$, namely

\begin{equation}
\|\tilde{Q}_i(x_0; \mu) - \tilde{P}_{i-n-2}(x_0; \eta)\| \leq C \alpha^i, \quad 1 \leq i \leq n + 2, \tag{4.26}
\end{equation}

It now follows from (4.2) and (4.24) that, taking a larger constant $C > 0$ if necessary,

\begin{equation}
|g(\sigma^i(\eta)) - g_i^-(x_0; \eta)| = \left| \frac{1}{N - 1} \ln \frac{G_{\eta,i}(P_{i+1}(\eta))}{G_{\eta,i}(P_i(\eta))} - \frac{1}{N - 1} \ln \frac{G_{\eta,i}(\tilde{P}_{i+1}(x_0; \eta))}{G_{\eta,i}(\tilde{P}_i(x_0; \eta))} \right| \leq C \alpha^i \tag{4.27}
\end{equation}

for all $i \leq 0$. In particular, the second series in (4.5) is absolutely convergent, and by (4.27) and Proposition 3, $|v(x_0; \mu)| \leq \text{Const}$ for some global constant $\text{Const} > 0$.

Next, setting

\begin{equation}
\tilde{a}_i(x_0; \mu) = \frac{1}{N - 1} \ln \left( \frac{G_{\mu,i}(\tilde{Q}_{i+1}(x_0; \mu))}{G_{\mu,i}(\tilde{Q}_i(x_0; \mu))} \right), \tag{4.28}
\end{equation}

and using (4.23) and Proposition 2, one gets

\begin{align*}
|\tilde{a}_i(x_0; \mu) - g_i^+(\mu)| &= \frac{1}{N - 1} \ln \left| \frac{G_{\mu,i}(\tilde{Q}_{i+1}(x_0; \mu))}{G_{\mu,i}(\tilde{Q}_i(x_0; \mu))} - \frac{G_{\mu,i}(Q_{i+1}(\mu))}{G_{\mu,i}(Q_i(x_0; \mu))} \right| \\
&\leq C \|\nabla \varphi\|_{\Gamma, (1)} \left( \|\tilde{Q}_i(x_0; \mu)\| + \|Q_{i+1}(x_0; \mu) - Q_i(\mu)\| + \|Q_{i+1}(x_0; \mu) - Q_i(\mu)\| \right) \\
&\leq C \|\nabla \varphi\|_{\Gamma, (1)} \alpha^{n+2-i}. \tag{4.29}
\end{align*}

for all $i = 0, 1, \ldots, n + 2$, where $C > 0$ is some sufficiently large global constant,
Next, notice that by construction \( \varphi_{n,i} = (\varphi_{n-2})_{(\mu_1, \ldots, \mu_{n+2+i})} + \text{const} \) for \(-n - 1 \leq i \leq -1\). Thus, by (2.2), (4.2) and (4.25), for all \(-n - 1 \leq i \leq -1\) we have

\[
|g_{n+2+i}(\mu) - g(\sigma^i \eta)| = \frac{1}{N - 1} \ln \frac{G^\varphi_{\mu,n+2+i}(Q_{n+2+i+1}(\mu))}{G^\varphi_{\mu,n+2+i}(Q_{n+2+i}(\mu))} - \ln \frac{G^\varphi_{\eta,n+i}(P_{n+1}(\eta))}{G^\varphi_{\eta,n+i}(P_1(\eta))}
\]

\[
\leq \text{Const} \left( \| \nabla \varphi_{(\mu_1, \ldots, \mu_{n+2+i})} - \nabla (\varphi_{n-2})_{(\mu_1, \ldots, \mu_{n+2+i})} \|_{\Gamma, (1)} + \| Q_{n+2+i}(\mu) - P_{n+1}(\eta) \| + \| Q_{n+2+i}(\mu) - P_1(\eta) \| \right)
\]

\[
\leq \text{Const} \| \nabla \varphi - \nabla (\varphi_{n-2}) \|_{\Gamma, (1)} \alpha^{n+2+i} + \text{Const} \alpha^{n+2+i}.
\]

(4.30)

In a similar way (4.26) implies

\[
|\bar{a}_{n+2+i}(x_0; \mu) - g(\sigma^i \eta)| \leq C \| \nabla \varphi \|_{\Gamma, (1)} \alpha^{n+2+i}, \quad -n - 1 \leq i \leq -1.
\]

(4.31)

To prove (4.18), notice that \( \chi_0((\nabla \tilde{\varphi}(x_0), \nu(\tilde{Q}_{n+1}(x_0; \mu))) = 1 \) (see Sect. 2 for the choice of the function \( \chi_0 \)), so \( (A_j \varphi) h)(x_0) = \Lambda_{\varphi, j}(x_0) h(Q_0(x_0; \mu)) \). The definition of \( \Lambda_{\varphi, j} \) gives

\[
\ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x_0; \mu)) = \sum_{i=0}^{n+1} \bar{a}_i(x_0; \mu).
\]

(4.32)

Next, assume for simplicity that \( n \) is odd (the other case is similar), and set \( m = (n + 1)/2 \). Using (4.27) – (4.31) we get

\[
\ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x_0; \mu)) - v(x_0; \mu) = \sum_{i=0}^{n+1} \bar{a}_i(x_0; \mu) + \sum_{i=-1}^{-n-1} [g(\sigma^i \eta) - g(\sigma^i \eta)] - \sum_{i=0}^{n+1} g_i^+(\mu)
\]

\[
= \sum_{i=-m-1}^{-n-1} [g(\sigma^i \eta) - g_i^-(x_0; \eta)] + \sum_{i=0}^{m} [\bar{a}_i(x_0; \mu) - g_i^+(\mu)]
\]

\[
+ \sum_{i=m+1}^{n+1} [\bar{a}_i(x_0; \mu) - g_i^-(x_0; \eta)] + \sum_{i=-1}^{-m} [g(\sigma^i \eta) - g_{n+2+i}^+(\mu)]
\]

\[
= O(\alpha^m) \| \nabla \varphi \|_{\Gamma, (1)} = O(\theta^n) \| \nabla \varphi \|_{\Gamma, (1)}.
\]

(4.33)

Since by (4.23),

\[
|h(Q_0(x_0; \mu)) - h(Q_0(\mu))| = \| h \|_{\Gamma, 1} O(\alpha^n),
\]

(4.34)

the above gives

\[
\left| e^{v(x_0; \mu)} h(Q_0(\mu)) - (A_j \varphi) h(x_0) \right| \leq \left| e^{v(x_0; \mu)} - e^{v(x_0; \mu)} \right| \| h(Q_0(\mu)) \|
\]

\[
+ \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x_0; \mu)) \| h(Q_0(\mu)) - h(Q_0(x_0; \mu)) \|
\]

\[
\leq e^{\max(v(x_0; \mu), \ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x_0; \mu))}) \| v(x_0; \mu) - \ln \Lambda_{\varphi, j}(\tilde{Q}_{n+2}(x_0; \mu)) \| h \|_{\Gamma, 0} + \| h \|_{\Gamma, (1)} O(\alpha^n)
\]

\[
\leq \text{Const} \left( \| \nabla \varphi \|_{\Gamma, (1)} h \|_{\Gamma, 0} + \| h \|_{\Gamma, (1)} \right) \theta^n,
\]

which proves (4.18).
Similarly to (4.27) one gets \(|f(\sigma^i(\eta)) - f_i^- (x_0; \eta)| \leq \text{Const } \alpha^{|i|}\), and also
\[ |f_i^+(\mu) - f_i^+(x_0; j)| = ||Q_i(\mu) - Q_{i+1}(\mu)|| - ||Q_i(x_0; j) - Q_{i+1}(x_0; j)|| \leq \text{Const } \alpha^{n+2-i}. \]
Combining these two estimates yields (4.17).

Next, using the notation from the beginning of this proof, notice that for any \(\mu\) as in (4.11) we have \(Q_i(x_0; j) = \tilde{Q}_i(x_0; \mu)\) for all \(i = 0, 1, \ldots, n + 2\) and therefore \(f_i^+(x_0; j) = ||\tilde{Q}_i(x_0; \mu) - \tilde{Q}_{i+1}(x_0; \mu)||\) for all \(i = 0, 1, \ldots, n + 1\). (This has been used already in the proof of (4.17).)

Define the function \(\tilde{W}^{(n+2)}(x_0; \cdot, \cdot) : \Sigma_A^+ \times \mathbb{C} \rightarrow \mathbb{C} \) by
\[
\tilde{W}^{(n+2)}(x_0; \mu, s) = 0 \text{ when } \mu_0 \neq 1 \text{ or } \mu_{n+2} \neq \ell \text{ and } \\
\tilde{W}^{(n+2)}(x_0; \mu, s) = e^{s \mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i}\tilde{Q}_i(x_0; \mu) \mathcal{H}_i \tilde{Q}_i(x_0; \mu) \\
\times \Lambda_\mathcal{G}(\tilde{Q}_i(x_0; \mu)) h(\tilde{Q}_0(x_0; \mu)),
\]
whenever \(\mu_0 = 1\) and \(\mu_{n+2} = \ell\), where \(j = j^{(n+2)}(\mu)\) is defined by (4.12).

Using (4.8), we can now write
\[
\sum_{|i|=n+3, j=1, j_{n+2}=\ell} u_j(x_0, -i s) = (-1)^n \sum_{\sigma\mu=\xi, \mu_0=1} e^{-s \varphi(\tilde{Q}_0(x_0; \mu))} \Lambda_{\mathcal{G}}(\tilde{Q}_i(x_0; \mu)) h(\tilde{Q}_0(x_0; \mu)) \\
= (-1)^n \sum_{\sigma\mu=\xi} e^{-s \mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i} \tilde{W}^{(n+2)}(x_0; \mu, s) = (-1)^n \left( L^{n+2}_{\mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i} \tilde{W}^{(n+2)}(x_0; \cdot, \cdot) \right)(\xi).
\]
This and (4.11) imply
\[
\left| (L^p_{\mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i})(\xi) - \sum_{|i|=n+3, j_{n+2}=\ell} u_j(x_0, -i s) \right| = \left| L^{n+2}_{\mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i} \left( W^{(n+2)}(x_0; \cdot, \cdot) - \tilde{W}^{(n+2)}(x_0; \cdot, \cdot) \right)(\xi) \right|.
\]

On the other hand, the so-called Basic Inequality for Ruelle transfer operators (see e.g. [PP]) gives that there exists a global constant \(C > 0\) (depending only on \(K\)) such that
\[
\left| L^p_{\mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i} u \right|_{\infty} \leq C e^{C |\mathcal{F}(x_0; \mu)|} e^{p \mathcal{F}(x_0; \mu)} \left| u \right|_{\infty}, \quad p \geq 0, \quad s \in \mathbb{C}, \quad (4.37)
\]
for any continuous function \(u : \Sigma_A^+ \rightarrow \mathbb{C}\).

Fix for a moment \(s \in \mathbb{C}\). We will now estimate \(\|W^{(n+2)}(x_0; \cdot, \cdot) - \tilde{W}^{(n+2)}(x_0; \cdot, \cdot)\|\) as a function of \(\mu \in \Sigma_A^+\). According to the definitions of \(W^{(n+2)}\) and \(\tilde{W}^{(n+2)}\), it is enough to consider \(\mu \in \Sigma_A^+\) with \(\mu_0 = 1\) and \(\mu_{n+2} = \ell\). For such \(\mu\), using (4.13), (4.16), (4.32), (4.33) and (4.35), we have
\[
\left| W^{(n+2)}(x_0; \mu, s) - \tilde{W}^{(n+2)}(x_0; \mu, s) \right| = e^{s \mathcal{F}_i + \mathcal{G}_i + \mathcal{H}_i} \tilde{Q}_i(x_0; \mu) \mathcal{H}_i \tilde{Q}_i(x_0; \mu) \\
\times \left| e^{s \varphi(\tilde{Q}_0(x_0; \mu)) - s \sum_{i=0}^{n+1} f_i^+(x_0; j) + \sum_{i=0}^{n+1} \mathcal{A}_i(x_0; \mu)} - h(\tilde{Q}_0(x_0; \mu)) \right|.
\]
To estimate (4.38), first notice that by (4.15) and Proposition 1,
\[
|f(\sigma^i \mu) - f(\sigma^{i-(n+2)} \eta)| \leq \text{Const } \alpha^i, \quad 0 \leq i \leq n + 2.
\]
Using this, (4.24), (4.26) and Proposition 1 again, one gets

\[ \left| \tilde{f}_{n+2}(\mu) - \sum_{i=0}^{n+1} f_i^+(x_0; \cdot) \right| \leq \text{Const} + \left| f_{n+2}(\mu) - \sum_{i=0}^{n+1} f_i^+(x_0; \cdot) \right| \leq \text{Const} + \sum_{i=0}^{n+1} |f(\sigma^i\mu) - f_i^+(x_0; \cdot)| \leq \text{Const}, \]

(4.39)

for some global constant Const > 0. Similarly, it follows from (4.15), (4.29) and (4.30) that

\[ \left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0; \cdot) \right| \leq \text{Const} \| \varphi \|_{\Gamma, (1)}. \]

(4.40)

Next, notice that

\[ |e^{(s+\|\nabla\varphi\|_{\Gamma, (1)} \theta^n)} - 1| \leq \text{Const} e^{\text{Const} (\text{Re}(s) + \|\nabla\varphi\|_{\Gamma, (1)}) (|s| + \|\nabla\varphi\|_{\Gamma, (1)}) \theta^n}. \]

Using the latter, (4.17), (4.18), (4.39) and (4.40) in (4.38) yields

\[ |W^{(n+2)}(x_0; \mu, s) - \tilde{W}^{(n+2)}(x_0; \mu, s)| \leq \text{Const} e^{\text{Const} (\text{Re}(s) + \|\nabla\varphi\|_{\Gamma, (1)}) e^{(s+\|\nabla\varphi\|_{\Gamma, (1)} \theta^n)} h(Q_0(\mu)) - h(\tilde{Q}_0(x_0; \mu))| \leq \text{Const} e^{\text{Const} (\text{Re}(s) + \|\nabla\varphi\|_{\Gamma, (1)}) \|h\|_{\Gamma, 0} + \|h\|_{\Gamma, (1)} \theta^n}. \]

Thus, choosing the global constant \( C > 0 \) sufficiently large, combining the above with (4.37) gives

\[ \left| \frac{L^{n+2}}{s-\tilde{f} + \tilde{g}} \left( W^{(n+2)}(x_0; \cdot, s) - \tilde{W}^{(n+2)}(x_0; \cdot, s) \right) \right| \leq C e^{\text{Const} (\text{Re}(s) + \|\nabla\varphi\|_{\Gamma, (1)}) e^{(s+\|\nabla\varphi\|_{\Gamma, (1)} \theta^n)} \|h\|_{\Gamma, 0} + \|h\|_{\Gamma, (1)} \theta^n}. \]

(4.41)

Next notice that

\[ \frac{d}{ds} Pr(-s \tilde{f} + \tilde{g}) = -\int_{\Sigma_A^+} \tilde{f} d\nu = -\int_{\Sigma_A^+} f d\nu = -c_0 < 0, \]

where \( \nu \) is the equilibrium state of \((-s_0 \tilde{f} + \tilde{g})\). Recall that \( Pr(-s_0 \tilde{f} + \tilde{g}) = 0 \), so \( Pr(-\text{Re}(s) \tilde{f} + \tilde{g}) < 1 \) for \( \text{Re}(s) > s_0 \). Now assume \( s_0 - a \leq \text{Re}(s) \) for some constant \( a > 0 \). Then

\[ e^{Pr(-\text{Re}(s) \tilde{f} + \tilde{g})} = 1 + c_0(s_0 - \text{Re}(s)) + O((\text{Re}(s) - s_0)^2) \leq 1 + c_1 a \]

for some constant \( c_1 > 0 \). Thus, \( e^{Pr(-\text{Re}(s) \tilde{f} + \tilde{g})} \theta \leq \theta + ca \) for some global constant \( c = c_1 \theta > 0 \). Combining this with (4.41), completes the proof of (4.6).
5. Estimates of the derivatives

In this section we prove Theorem 3(b). Throughout we assume that \( p \geq 1 \).

For any \( x \in \Gamma \), close to \( x_0 \) and any \( \eta \in \Sigma_A \) with \( \eta_0 = \ell \) define the points \( \widetilde{P}_j(x;\eta) \) and the functions \( f_i^\pm(x;\eta), g_i^\pm(x;\eta), \phi^\pm(x;\eta,s), \) etc., as in the beginning of Section 4 replacing the point \( x_0 \) by \( x \). We will assume that the segment \([\widetilde{P}_{-1}(x_0;\eta), x_0]\) has no common points with the interior of \( K_\ell \) and \( x \) is close enough to \( x_0 \) so that the same holds with \( x_0 \) replaced by \( x \).

By Proposition 4 there exists a unique phase function \( \psi_\eta \) (also depending on \( x_0 \)) defined in a neighborhood \( U \) of \( x_0 \) in \( \Gamma_\ell \), such that \( \nabla \psi_\eta(x) = 0 \) and the backward trajectory \( \gamma_-(x, \nabla \psi_\eta(x)) \) of any point \( x \in U \) with \( \psi_\eta(x) = 0 \) has an itinerary \((\ldots, \eta_{-\ell}, \ldots, \eta_{-1}, \eta_0)\), (5.5) that is

\[
\nabla \psi_\eta(x) = \frac{\widetilde{P}_0(x;\eta) - \widetilde{P}_{-1}(x;\eta)}{\|\widetilde{P}_0(x;\eta) - \widetilde{P}_{-1}(x;\eta)\|}
\]

for any \( x \in C_\psi \cap U \). (Notice that in general \( \psi_\eta \) is different from the functions \( \varphi_{\eta,j} \) defined in the beginning of Sect. 4.) For any \( i \leq 0 \), denoting \( J = (\eta_i, \eta_{i+1}, \ldots, \eta_{-1}, \eta_0) \), we can write \( \psi_\eta = (\psi_{\eta,i})_J \) for some phase function \( \psi_{\eta,i} \) (defined on some naturally defined open subset \( V_{\eta,i} \) of \( \mathbb{R}^N \)) satisfying Ikawa’s condition (P) on \( \Gamma_{\eta_i} \). We then have \( \widetilde{P}_i(x;\eta) = X^{-i}(x, \nabla \psi_{\eta,i}) \). As in the beginning of Sect. 4 (see (4.22)) there one derives that there exists a global constant \( C_p > 0 \) such that

\[
\|\psi_{\eta,i}(\cdot;\eta)\|_{(p)}(V_{\eta,i} \cap B_0) \leq C_p \quad \text{for all } \eta \text{ and } i < 0. \quad (5.4)
\]

Using (2.4) in Proposition 2 with \( \varphi = \psi_{\eta,m} \) for some \( m \geq i \) and replacing \( C_p \) with a larger global constant if necessary, we get

\[
\|\widetilde{P}_i(\cdot;\eta)\|_{(p)}(x) \leq C_p \alpha^{|i|}, \quad j < 0. \quad (5.1)
\]

Similarly, for any \( \mu \in \Sigma_A \) with \( \mu_0 = 0 \) and \( \mu_{n+2} = k \) we have

\[
\|\tilde{Q}_i(\cdot;\eta)\|_{(p)}(x) \leq C_p \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2, \quad (5.2)
\]

and

\[
\|\tilde{Q}_i(\cdot;\mu) - \tilde{P}_{i-n-2}(\cdot;\eta)\|_{(p)}(x) \leq C_p \alpha^{|i|}, \quad 0 \leq i \leq n + 2. \quad (5.3)
\]

Next, recall the function \( \Lambda_\varphi \) from the beginning of this section. By Proposition 2,

\[
\|\nabla \varphi_J\|_{(p)} \leq C_p \|\nabla \varphi\|_{(p)} \quad (5.4)
\]

for any finite admissible configuration \( J \).

Since for any \( i < 0 \) we have \( \tilde{g}_i^\pm(x;\eta) = \ln \Lambda_\psi_{\eta,i}(\widetilde{P}_{i+1}(x;\eta)) \), it follows from (5.1), (5.2), (5.3) and Proposition 3 that for any \( p \geq 1 \) there exists a constant \( C'_p > 0 \) (independent of \( x, n, \eta \) and \( \mu \)) such that

\[
\|\tilde{g}_i^\pm(\cdot;\eta)\|_{(p)}(x) \leq C'_p \alpha^{|i|}, \quad i < 0. \quad (5.5)
\]

Similarly, according to (4.28) and Proposition 2, we can choose \( C'_p \) so that

\[
\|\tilde{a}_i(\cdot;\mu)\|_{(p)}(x) \leq C'_p \|\nabla \varphi\|_{(p)} \alpha^{n+2-i}, \quad 0 \leq i \leq n + 2, \quad (5.6)
\]

and as in the proof of (4.30),

\[
\|\tilde{a}_i(\cdot;\mu) - g_{i-n-2}(\cdot;\eta)\|_{(p)}(x) \leq C'_p \|\nabla \varphi\|_{(p+1)} \alpha^{|i|}, \quad 0 \leq i \leq n + 2. \quad (5.7)
\]

Next, given \( x \) as above, \( \mu \) and \( n \) with \( \mu_{n+2} = \ell \), define \( W^{(n+2)}(x;\mu,s) \) by (4.10), \( \eta \) by (4.15) and \( \tilde{W}^{(n+2)}(x;\mu,s) \) by (4.35) replacing \( x_0 \) by \( x \). We will estimate the derivatives of

\[
W^{(n+2)}(x;\mu,s) - \tilde{W}^{(n+2)}(x;\mu,s)
\]

with respect to \( x \).
First look at the first derivatives $D_b[W^{(n+2)}(\xi, s) - \widetilde{W}^{(n+2)}(\xi, s)](x)$, where $b \in S_\varepsilon \Gamma$. Writing $\phi^-(x; \eta, s) = -s \phi_1^-(x; \eta) + \phi_2^-(x; \eta)$, notice that for any $x, x' \in \Gamma_\varepsilon$ (close to $x_0$), we have

$$\phi_1^-(x; \eta) - \phi_1^-(x'; \eta) = -s \psi_\eta(x) + \psi_\eta(x')$$

so $D_b(\phi_1^-)(\cdot; \eta)) = D_b(\psi_\eta(x))$, and therefore by (4.35) and (4.15)

$$D_bw(\cdot; s, \mu)(x) = -s D_b\psi_\eta(x) + \sum_{i=-1}^{-\infty} D_b(g_i^-)(\cdot; \eta))(x) .$$

We will see later that the latter series is uniformly convergent.

Next, using the notation $j = (\mu_0, \mu_1, \mu_2, \ldots, \mu_{n+2})$ and

$$z(x; \mu, s) = s f_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s(\varphi_{\mu_0})_1(x),$$

it follows from (4.38) that

$$W^{(n+2)}(\xi, s) - \widetilde{W}^{(n+2)}(\xi, s)(x) = e^{w(x; \mu, s) - s\varphi(Q_0(\mu))} h(Q_0(\mu)) - e^{z(x; \mu, s)} \Lambda_{\varphi,j}(-\tilde{Q}_{n+2}(x; \mu)) h(Q_0(\mu))
\begin{equation}
= (I)(x) + (II)(x),
\end{equation}

where

$$(I)(x) = [e^{w(x; \mu, s) - s\varphi(Q_0(\mu))} - e^{z(x; \mu, s) + \ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x; \mu))}] h(Q_0(\mu)),$$

and

$$(II)(x) = e^{z(x; \mu, s)} \Lambda_{\varphi,j}(-\tilde{Q}_{n+2}(x; \mu)) [h(Q_0(\mu)) - h(Q_0(x; \mu))].$$

Let $O$ be a small compact connected neighbourhood of $x$ in $\Gamma$. Fix $\mu, s, n$ and $\eta$ with (4.15) temporarily, and set

$$\gamma(y) = w(y; \mu, s) - s\varphi(Q_0(\mu)), \quad \delta(y) = z(x; \mu, s) + \ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(x; \mu)), \quad y \in O.$$

To estimate $(I)$ first notice that by (4.16), (4.17), (4.27), (4.29), (4.32), (4.33), (4.39) and (4.40)

$$\|\gamma\|_0(O) = O(|s| + |s| \|\varphi\|_{r,0} + \|\nabla \varphi\|_{r,0} + 1),$$

and

$$\|e^\gamma\|_{r,0}(O) \leq \text{Const} e^{\text{Const} [\text{Re}(s)] (1 + \|\varphi\|_{r,0} + \|\nabla \varphi\|_{r,0})} .$$

It follows from (5.6) and (4.40) that $|\tilde{g}_{n+2}(\mu)| \leq \text{Const} \|\nabla \varphi\|_{r,0}$. Combining this with the definition of $z(x; \mu, s)$ and (4.39) implies

$$\|z(\cdot; \mu, s)\|_0(O) = O(|s| + |s| \|\varphi\|_{r,0} + \|\nabla \varphi\|_{r,0}) , \quad \|\delta\|_0(O) = O(|s| + |s| \|\varphi\|_{r,0} + \|\nabla \varphi\|_{r,0}) .$$

Next, we will estimate the derivatives of $\gamma$ and $\delta$. For any $q \geq 1$ and any $y \in O$, using (5.9), (3.1) and (5.5) we get

$$\|\gamma\|_{r,q}(y) = \|s \phi_1^- (\cdot; \eta) - \phi_2^- (\cdot; \eta)\|_{r,q}(y) \leq |s| \|\nabla \psi_\eta\|_{r,q}(y) + \sum_{i=-1}^{-\infty} \|g_i^- (\cdot; \eta)\|_{r,q}(y) \leq |s| \text{Const}_q + \text{Const}_q \sum_{i=-1}^{-\infty} \alpha^{i} \leq \text{Const}_q (|s| + 1),$$

(5.14)
where $\text{Const}_q$ denotes a positive global constant depending on $q$. Thus, for any $q \geq 0$,
\[ \|e^\gamma\|_{r,q}(O) \leq \text{Const}_q \|e^\gamma\|_{r,0}(O) \left( \max_{1 \leq i \leq q} \|\gamma\|_{r,i}(O) \right)^q \]
\[ \leq \text{Const}_q e^{\text{Const} \max \left\{ \|\Re(s)\|_{(1+\|\varphi\|_{r,0})}, \|\nabla \varphi\|_{r,(1)} \right\} (|s| + 1)^q} \cdot \]

Similarly, (5.4) gives
\[ \|z(\cdot, \mu, s)\|_{r,q}(y) = \|s (\varphi_{\mu_0})_j\|_{r,q}(y) \leq \text{Const}_q |s| \|\nabla \varphi\|_{r,(q)} \cdot \]
while (4.31) and (5.6) imply
\[ \| \ln \Lambda_{\varphi,j}(\tilde{Q}_{n+2}(\cdot; \mu))\|_{r,q}(y) \leq \sum_{i=0}^{n+1} \|\tilde{a}_i(\cdot; \mu)\|_{r,q}(y) \leq \text{Const}_q \|\nabla \varphi\|_{r,(q)} \cdot \]
for any $q \geq 0$, so
\[ \|\delta\|_{r,q}(y) \leq \text{Const}_q (|s| + 1) \|\nabla \varphi\|_{r,(q)} \cdot \]

The next step is to estimate the derivatives of $\gamma - \delta$. First notice that by Proposition 2 and (3.1) we have
\[ \|\nabla \psi_n - \nabla (\varphi_{\mu_0})_j\|_{r,q}(y) \leq \text{Const}_q \alpha^n \|\nabla \psi_n - \nabla \varphi_{\mu_0}\|_{r,(q)} \leq \text{Const}_q \alpha^n \|\nabla \varphi\|_{r,(q)} \cdot \]

Set again $m = \frac{n+1}{2}$, assuming for simplicity that $n$ is odd, and $\theta = \sqrt{\alpha} \in (0, 1)$. As in the proof of (4.18) above, for any $y \in \mathcal{O}$ and any $q \geq 1$, using (5.5), (5.6) and (5.7), we have
\[ \|\gamma - \delta\|_{r,q}(y) \leq \left| - s \psi_n + \sum_{i=-1}^{-\infty} g_i^{-} (\cdot; \eta) + s (\varphi_{\mu_0})_j - \sum_{i=0}^{n+1} \tilde{a}_i(\cdot; \mu) \right| \cdot \]
\[ \leq \|s\| \psi_n - (\varphi_{\mu_0})_j\|_{r,q}(y) + \sum_{i=-m-1}^{-\infty} \|g_i^{-} (\cdot; \eta)\|_{r,q}(y) \]
\[ + \sum_{i=0}^{m} \|\tilde{a}_i(\cdot; \mu)\|_{r,q}(y) + \sum_{i=m+1}^{n+1} \|\tilde{a}_i(\cdot; \mu) - g_i^{-n-2}(\cdot; \eta)\|_{r,q}(y) \]
\[ \leq \text{Const}_q (|s| \|\nabla \varphi\|_{r,(q)} + \|\nabla \varphi\|_{r,(q+1)}) \theta^n \cdot \]

From Sect. 4, a similar estimate holds for $q = 0$. Consequently,
\[ \|e^{\delta - \gamma}\|_{r,q}(O) \leq \text{Const}_q \|e^{\delta - \gamma}\|_{0}(O) \left( \max_{1 \leq i \leq q} \|\delta - \gamma\|_{r,i}(O) \right)^q \]
\[ \leq \text{Const}_q e^{\text{Const} \max \left\{ \|\Re(s)\|_{(1+\|\varphi\|_{r,0})}, \|\nabla \varphi\|_{r,(1)} \right\} (|s| + \|\nabla \varphi\|_{r,(1)})^q \theta^n} \cdot \]

Finally, as in the estimate just after (4.40), it follows that
\[ \|e^{\delta - \gamma - 1}\|_{0}(O) \leq \text{Const} e^{\text{Const} \max \left\{ \|\Re(s)\|_{(1+\|\varphi\|_{r,0})}, \|\nabla \varphi\|_{r,(1)} \right\} (|s| + \|\nabla \varphi\|_{r,(1)}) \theta^n} \cdot \]

The above, (5.12) and (5.14) imply that for any $q \geq 1$,
\[ \langle I \rangle \|r,q(O) \leq \text{Const}_q \|h\|_{0}(\Gamma) e^{\text{Const} \max \left\{ \|\Re(s)\|_{(1+\|\varphi\|_{r,0})}, \|\nabla \varphi\|_{r,(1)} \right\} (|s| + \|\nabla \varphi\|_{r,(1)})^q \theta^n} \cdot \]

Using similar estimates, for any $q \geq 1$ one gets
\[ \langle II \rangle \|r,q(O) \leq \text{Const}_q \alpha^n e^{\text{Const} \max \left\{ \|\Re(s)\|_{(1+\|\varphi\|_{r,0})}, \|\nabla \varphi\|_{r,(1)} \right\} \sum_{r=0}^{q-1} (|s| + 1)^{r+1} \|\nabla \varphi\|_{r,(r)} \theta^{r+1} \|h\|_{r,q-r}(O) \cdot \]
It now follows from (5.10) and the estimates for (I) and (II) found above that for any $p \geq 1$ we have
\[
\|W^{(n+2)}(\cdot; \mu, s) - \tilde{W}^{(n+2)}(\cdot; \mu, s)\|_{\Gamma,(p)}(x) \leq \text{Const}_p \theta^n e^{\text{Const} \|\Re(s)\| \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)}} \times \sum_{r=0}^{q} |s| \|\nabla\varphi\|_{\Gamma,(r)} + \|\nabla\varphi\|_{\Gamma,(r+1)}|^{r+1} \|h\|_{\Gamma,q-r}(O).
\]
Combining this with the argument from the end of Sect. 4 completes the proof of Theorem 3. ■

6. Estimates for $w_0(x, -is)$

Throughout this and the following sections we will use the notation
\[
E_p(s, \varphi, h) = \begin{cases} e^{C_p \|\Re(s)\| \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)}} \sum_{j=0}^{p} \left( |s| \|\nabla\varphi\|_{\Gamma,j} + \|\nabla\varphi\|_{\Gamma,j+1} \right)^{j+1} \|h\|_{\Gamma,p-j} & \text{if } p \geq 1, \\
C_0 e^{C_p \|\Re(s)\| \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)}} \left( |s| + \|\nabla\varphi\|_{\Gamma,(1)} \right) \|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)} & \text{if } p = 0,
\end{cases}
\]
where by $C_p$ we denote positive global constants depending on $p$ which may change from line to line.

First we will establish for $\sigma_0 \leq \Re(s) \leq 1$ the inequality
\[
\|L_{s}^{n} M_{\eta,s}(\cdot) - L_{s}^{n-1} M_{\eta-1,s}(\cdot) L_{s}\|_{\Gamma,p} \leq C_p E_p(s, \varphi, h) \theta^n,
\]
where $L_s = -L_{-s} f_{+\tilde{\theta}}$ and $\sigma_0 < s_0$. The precise choice of $\sigma_0$ depends on the estimates (4.3) and will be discussed below. For this purpose we write
\[
\left( L_{s}^{n} M_{\eta,s} - L_{s}^{n-1} M_{\eta-1,s} L_{s}\right) w(\xi) = -L_{s}^{n+1} \left[ V^{(n)}(x; s, \mu) - \tilde{V}^{(n)}(x; s, \mu) \right](\xi),
\]
where
\[
V^{(n)}(x; s, \mu) = \exp \left( -\phi^{-}(x; \sigma^{n+1} e(\mu), s) - \chi(\sigma^{n+1} e(\mu), s) \right) w(\mu),
\]
\[
\tilde{V}^{(n)}(x; s, \mu) = \exp \left( -\phi^{-}(x; \sigma^{n} e(\mu), s) - \chi(\sigma^{n} e(\mu), s) \right) w(\mu).
\]
The inequality (6.1) follows from the estimates
\[
\|\phi^{-}(x; \sigma^{n+1} e(\xi), s) - \phi^{-}(x; \sigma^{n} e(\xi), s)\|_{\Gamma,p} \leq C_p E_p(s, \varphi, h) \theta^n,
\]
\[
|\chi(\sigma^{n+1} e(\xi), s) - \chi(\sigma^{n} e(\xi), s)| \leq C(1 + |s|) \theta^n
\]
and the form of the operators $M_{\eta,s}(x)$. The estimate (6.3) is a consequence of the choice of $\chi_1, \chi_2$ and the fact that $f, g \in \mathcal{F}_\theta(\Sigma_A)$. To prove (6.2), notice that
\[
\sum_{i=-\infty}^{-1} \left[ f(\sigma^{n+1+i} e(\xi)) - f(\sigma^{n+i} e(\sigma(\xi))) \right] \leq C \theta^n,
\]
and similar estimates hold for the function $g$. The terms involving $f$ and $g$ are independent on $x$ and they are not important for the estimates of the derivatives. To deal with the terms depending on $x$, recall that
\[
\phi^{-}(x; \eta) = -s \phi_1^{-}(x; \eta) + \phi_2^{-}(x; \eta)
\]
with $D_b(\phi_1^{-}(\cdot; \eta))(x) = D_b(\psi_\eta(x))$. Here and below we use the notations of the previous section. On the other hand,
\[
\|\nabla \psi_{\sigma^{n+1} e(\mu)}(x) - \nabla \psi_{\sigma^{n} e(\sigma(\mu))}(x)\|_{\Gamma,p} \leq C_p \alpha^n.
\]
In fact, the backward trajectories $\gamma_{-(x, \nabla \psi_{\sigma_{n+1}e(\mu)}(x))}$ and $\gamma_{-(x, \nabla \psi_{\sigma_{n}e(\sigma(\mu))}(x))}$ follow an itinerary $(\mu_{n+1}, \mu_{n}, \ldots, \mu_{1})$ and we can apply Proposition 2. Now we repeat the argument used in the previous section for the estimate of $\|\gamma - \delta\|_{\Gamma, \rho}$. Set $m = n + 1$ and assume for simplicity that $n$ is odd. For fixed $n$ we set $\eta = \sigma^{n+1}e(\mu)$, $\tilde{\eta} = \sigma^{n}e(\sigma(\mu))$. The estimates of

$$\|\phi_{1}^{\ast}(x; \eta) - \phi_{1}^{\ast}(x; \tilde{\eta})\|_{\Gamma, \rho}$$

follows from (6.4). Next we write

$$\sum_{i=1}^{\infty} \left( g_{i}^{-}(x; \eta) - g_{i}^{-}(x; \tilde{\eta}) \right) = \sum_{i=-m-1}^{\infty} \left( g_{i}^{-}(x; \eta) - g_{i}^{-}(x; \tilde{\eta}) \right) + \sum_{i=m+1}^{n+1} \left( g_{i-n-2}^{-}(x; \eta) - \tilde{a}_{i}(x; \mu) \right) - \sum_{i=m+1}^{n+1} \left( g_{i-n-2}^{-}(x, \tilde{\eta}) - \tilde{a}_{i}(x; \mu) \right).$$

The $\|\cdot\|_{\Gamma, \rho}$ norms of the sums from $i = m + 1$ to $n + 1$ can be estimated as in Section 5 by using (5.7) since

$$\eta = \sigma^{n+1}e(\mu) = (\ldots, *, \mu_{0}, \mu_{1}, \ldots, \mu_{n+1} = \ell, \mu_{n+2}, \ldots),$$

$$\tilde{\eta} = \sigma^{n}e(\sigma(\mu)) = (\ldots, *, \mu_{1}, \ldots, \mu_{n+1} = \ell, \mu_{n+2}, \ldots),$$

and

$$\sum_{i=m+1}^{n+1} \| g_{i-n-2}^{-}(x; \eta) - \tilde{a}_{i}(x; \mu) \|_{\Gamma, \rho} \leq \sum_{i=m+1}^{n+1} \alpha^{i},$$

$$\sum_{i=m+1}^{n+1} \| g_{i-n-2}^{-}(x, \tilde{\eta}) - \tilde{a}_{i}(x; \mu) \|_{\Gamma, \rho} \leq \sum_{i=m+1}^{n+1} \alpha^{i}.$$

To estimate the sums from $i = -m - 1$ to $-\infty$ we apply (5.5) and this completes the proof of (6.1).

We have the representation

$$L_{n}^{s}M_{n, s} = \sum_{k=1}^{n} \left( L_{s}^{k}M_{k, s} - L_{s}^{k-1}M_{k-1, s}L_{s} \right) L_{s}^{n-k} \mathcal{M}_{0, s}L_{s}^{n},$$

and we get

$$\sum_{n=1}^{\infty} L_{n}^{s}M_{n, s}w = \sum_{k=1}^{n} \left[ \sum_{k=1}^{n} \left( L_{s}^{k}M_{k, s} - L_{s}^{k-1}M_{k-1, s}L_{s} \right) L_{s}^{n-k}w + \mathcal{M}_{0, s}L_{s}^{n}w \right].$$

Since $s_{0} \in \mathbb{R}$ is the abscissa of absolute convergence, for $\Re(s) > s_{0}$ we have $\Pr(-\Re(s) \tilde{f} + \tilde{g}) < 0$ and $\|L_{s}^{n}\| \leq 1, \forall n$. Consequently, the double sum at the right hand side is absolutely convergent for $\Re(s) > s_{0}$ and we may change the order of summation. Applying Fubini theorem, we are going to examine

$$\sum_{n=0}^{\infty} L_{n}^{s}M_{n, s}G_{s}v_{s} = \left( \mathcal{M}_{0, s} + \mathcal{R}_{s} \right) \sum_{n=0}^{\infty} L_{s}^{n}G_{s}v_{s},$$

where

$$\mathcal{R}_{s} = \sum_{k=1}^{\infty} \left( L_{s}^{k}M_{k, s} - L_{s}^{k-1}M_{k-1, s}L_{s} \right).$$
According to (6.1), the series defining $\mathcal{R}_s$ is absolutely convergent for $\sigma_0 \leq \Re(s) \leq 1$ and
\[ \|\mathcal{R}_s\|_{1,p} \leq C_p E_p(s, \varphi, h). \]
Consequently, the problem of the analytic continuation of the left hand side of (6.5) for $\Re(s) < s_0$ is reduced to that of the series $\sum_{n=0}^{\infty} L^n_s w_s, w_s = \mathcal{G}_s \tilde{v}_s$.

The analysis of $\sum_{n=0}^{\infty} L^n_s w_s$ is based on Dolgopyat type estimates (4.3) and we must show that $w_s = h_s \circ \Phi$ with some $h_s \in C_u^{\text{Lip}}(\Lambda_0K)$. This is proved in Appendix B, where we show that for $|\Re(s)| \leq a$ we have $\|w_s\|_{\text{Lip},t} \leq C_0$ with $C_0$ independent on $s$. Thus for $s = \tau + it$, $\sigma_0 \leq \tau \leq 1$, $|t| \geq 2$, we get
\[
\sum_{n=0}^{\infty} \|\tilde{L}_n w_s\|_\infty \leq \sum_{n=0}^{\infty} \sum_{p=0}^{[\log |t|] - 1} C \rho^{|\log |t||} e^{\text{Pr}(-\tau |\tilde{f} + \tilde{g})} \|w_s\|_{\text{Lip},t}
\leq \frac{CC_0}{1 - \rho^{[\log |t|]}} \sum_{n=0}^{[\log |t||} C \rho^{|\log |t||} e^{\text{Pr}(-\tau |\tilde{f} + \tilde{g})}
\leq C_1 \max \{|\log |t||, |t|^{\text{Pr}(-\tau |\tilde{f} + \tilde{g})}\}.
\]

On the other hand, for $\sigma_0$ sufficiently close to $s_0$ we have $\text{Pr}(-\sigma_0 \tilde{f} + \tilde{g}) = \beta_0 < 1$. Combining this with the estimate for $\mathcal{R}_s$, we conclude that for $\sigma_0 \leq \Re(s)$ and $|t| \geq 2$ we have
\[
\|\sum_{n=0}^{\infty} L^n s \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s\|_{\Gamma,0} \leq C_0 |t|^{|\beta_0|}.
\]

Introduce the function
\[
w_{0,j}(x, -is) = \sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2} = j} u_j(x, -is).
\]

The analysis in Section 5 of [I1] implies that the series defining $w_{0,j}(x, -is)$ is absolutely convergent for $x \in \Gamma_j$, $\Re(s) > s_0$ and we have
\[
\|w_{0,j}(x, -is)\|_{\Gamma_j,0} \leq C_{j,\delta}, \Re(s) \geq s_0 + \delta, \delta > 0.
\]

On the other hand, the analytic continuation of the series $\sum_{n=0}^{\infty} L^n_s \mathcal{M}_{n,s} \mathcal{G}_s$ established above and Theorem 3 (a) with sufficiently small $\varepsilon = s_0 - \Re(s) > 0$ guarantee an analytic continuation of $w_{0,j}(x, -is)$ for $x \in \Gamma_j$, $\Re(s) \geq \sigma_0$, $|\text{Im}(s)| \geq 2$ with $\sigma_0 = s_0 - \varepsilon$. Applying Theorem 3 (a) once more for $s = \sigma_0 + it$, we get the estimate
\[
\|w_{0,j}(x, -i\sigma_0 + t)\|_{\Gamma_j,0} \leq D_j |t|^{1+\beta_0}.
\]

The same argument works for all $\ell = 1, ..., \kappa_0$ and we get the same estimate for
\[
w_{0,\ell}(x, -is) = \sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2} = \ell} u_j(x, -is), x \in \Gamma_\ell.
\]

In particular, we obtain a $L^\infty(\Gamma)$ estimate for
\[
w_0(x, -is) = \sum_{\ell=1}^{\kappa_0} w_{0,\ell}(x, -is)
\]
and $\Re(s) \geq \sigma_0$. Clearly, we can choose $0 < \beta_0 < 1$ independent on $\ell = 1, \ldots, \kappa_0$.

Now we will obtain $C^p(\Gamma)$ estimates for $w_0(x, -is)$. To examine the regularity of the functions $w_{0,j}(x, -is)|\Gamma_j$ on $\Gamma_j$, set
\[ U_{n+2,j}(x, -is) = \sum_{|j|=n+3, j_{n+2}=j} u_j(x, -is). \]
We start with an estimate of the $C^p(\Gamma_j)$ norms of $U_{n+2,j}(x, -is)$. For this purpose, applying Theorem 3 (b) with $p \geq 1$, we must estimate the norms $\|L^s M_{n,s}(\cdot) G_{\delta} \bar{v}_s \|_{\Gamma_j, p}$. We write
\[ L^s M_{n,s} = M_{0,s}L^s + \sum_{k=1}^{m} \left( L^k_s \mathcal{M}_{k,s} - L^k_{s-1} \mathcal{M}_{k-1,s} L^s \right) L^{n-k}_s \]
\[ + \sum_{k=m+1}^{n} \left( L^k_s \mathcal{M}_{k,s} - L^k_{s-1} \mathcal{M}_{k-1,s} L^s \right) L^{n-k}_s = B_0 + B_1 + B_2, \]
where $m = \lfloor n/2 \rfloor$. For the term $B_0$, we use the estimate (4.3) with $0 < \rho < 1$ and we get
\[ \|B_0\|_{\Gamma_j, p} \leq C_p E_p(s, \varphi, h) \rho^n. \]
For the term $B_1$ we get
\[ \|B_1\|_{\Gamma_j, p} \leq C'_p E_p(s, \varphi, h) \sum_{k=1}^{m} \theta^k \rho^m \leq C''_p E_p(s, \varphi, h) \sqrt{\rho}^n. \]
Finally, for $B_2$ we obtain
\[ \|B_2\|_{\Gamma_j, p} \leq D_p E_p(s, \varphi, h) \sum_{k=m+1}^{n} \theta^k \leq D'_p E_p(s, \varphi, h) \sqrt{\rho}^{m+1}. \]
So changing $\theta$ by another global constant $0 < \tilde{\theta} < 1$, $\tilde{\theta} \geq \max \{\sqrt{\rho}, \sqrt{\tilde{\rho}}\}$, we arrange an estimate
\[ \|L^s M_{n,s} \|_{\Gamma_j, p} \leq B_p E_p(s, \varphi, h) \tilde{\theta}^n. \]
Thus with global constants $C_p$, $D_p$ we deduce
\[ \|U_{n+2,j}(x, -is)\|_{\Gamma_j, p} \leq C_p E_p(s, \varphi, h) (\theta^n + \tilde{\theta}^n) \leq D_p E_p(s, \varphi, h) \tilde{\theta}^n, \forall n \in \mathbb{N}. \]
Consequently, the series $w_{0,j}(x, -is)$ is convergent in $C^p(\Gamma_j)$ norm and for $\sigma_0 \leq \tau \leq s_0 + 1$ we have the estimates
\[ \|w_{0,j}(x, -\mathbf{i} \tau + t)\|_{\Gamma_j, p} \leq B_p E_p(s, \varphi, h), \ p \geq 1, \]
where the constants $B_p$ are independent on $j$. Summing over $\ell = 1, \ldots, \kappa_0$, we obtain the same estimate for $\|w_0(x, -\mathbf{i} \tau + t)\|_{\Gamma_j, p}$ and for $\Re(s) \geq \sigma_0$ the trace $w_0(x, -is)|\Gamma$ is an analytic function with values in $C^\infty(\Gamma)$.

It is interesting to observe that contracting the domain $\sigma_0 \leq \Re(s) \leq s_0 + 1$ we may obtain better bounds for the $C^p(\Gamma)$ norms. For example, we treat below the case $p = 0$ and the same argument works for $p \geq 1$. In the domain $\sigma_0 \leq \Re(s) \leq s_0 + \delta$, $\Im(s) \geq 2$, we apply the Phragmen-Lindelöf theorem (see 5.65 in [T]). Notice that when we decrease $\delta > 0$ the constant $C_{j,\delta}$ in (6.6) change but we have always the bound (6.6). Consequently, for $\sigma_0 \leq \tau \leq s_0 + \delta$ we deduce
\[ \|w_{0,j}(x, -\mathbf{i} \tau + t)\|_{\Gamma_j, 0} \leq B|\tau|^{\kappa(x)}, \ t \geq 2, \]
where $\kappa(x)$ is a linear function such that

$$\kappa(\sigma_0) = 1 + \beta_0, \ \kappa(s_0 + \delta) = 0.$$ 

It is clear that choosing $\delta > 0$ small enough, there exist $\sigma_0', \sigma_0 < \sigma_0' < s_0$ and $0 < \beta < 1$ so that for $\tau \geq \sigma_0'$ we have

$$\|w_{0,j}(x, -i\tau + t)\|_{\Gamma_j, 0} \leq A_j |t|^\beta, \ t \geq 2$$

and similarly we treat the case $t \leq -2$. Finally, for $\tau \geq \sigma_0'$, $|t| \geq 2$ we have

$$\|w_0(x, -i\tau + t)\|_{\Gamma, 0} \leq A|t|^\beta. \quad (6.7)$$

Here the constants $A_j$ depend on the norms of $\nabla \varphi$ and $h$ and summing over $\ell = 1, \ldots, \kappa_0$ for $\sigma_0 < \sigma_0' \leq \tau$ we get

$$\|w_0(x, -i\tau + t)\|_{\Gamma, 0} \leq A|t|^\beta. \quad (6.8)$$

**Remark 5.** In the following we will not use the estimate (6.8) but a similar argument based on Phragmen-Lindelöf theorem will be crucial in Section 9, where we need to control the behavior of the remainder $R_M(x, s; k)$ and its bounds when $|\text{Im}(s)| \to \infty$. On the other hand, (6.8) is related to the assumption of Ikawa (1.6) mentioned in the Introduction.

7. **The leading term $W^{(0)}(x, -is; k)$**

For our construction we need the following definition introduced by Ikawa in [I3].

**Definition 1.** Let $\omega \subset \mathbb{R}^N$ be an open set and let $\mathcal{D}$ be a domain in $\mathbb{C}$. We say that the function $U(x, s; k)$ satisfies the condition (S) in $(\omega, \mathcal{D})$ if the following hold:

(i) for each $k \in \mathbb{R}$, $U(\cdot, s; k)$ is a $C^\infty(\overline{\omega})$-valued holomorphic function in $\mathcal{D},$

(ii) $U(\cdot, s; k) \in L^2(\omega)$ for $\Re s > 0,$

(iii) $(\Delta - s^2)U(x, s; k) = 0$ in $\omega$ for every $s \in \mathcal{D}.$

Let $S_j(s)$ be the operator constructed in Section 4 of [I3]. To recall this construction, set $\Omega_j = \Omega \setminus K_j$ and consider boundary data $m(x; k) = e^{ik\psi(x)}b(x, s; k), \ k \in \mathbb{R}$, satisfying the condition (A) on $\Gamma_j$. The condition (A) means that there exists a phase function $\varphi$ satisfying the condition (P) in $\Gamma_j$ introduced in Section 2 such that

$$\varphi = \psi \text{ on } \text{supp } \bigcup_{s, k} g(\cdot, s; k)$$

and

$$\|b(x, s; k)\|_{\Gamma_j, p} \leq C_p, \ \forall k \geq 1, \forall p \in \mathbb{N}.$$
Then \( v(x, s; k) = S_j(s)m(x; k) \) is a \( C^\infty(\Omega) \)-valued entire function with the following properties:
\[
\begin{aligned}
(\Delta - s^2)v(x, s; k) &= 0, \quad x \in \overset{\circ}{\Omega}_j, \\
v(., s; k) &\in L^2(\Omega_j) \text{ if } \Re(s) > 0, \\
v(x, s; k) &= \sum_{q=0}^{M} \left( \sum_{\nu=0}^{2q} a_{q,\nu}(x)(s + ik)\nu \right) (\hat{\psi} - \psi(x, \nabla \hat{\psi})) + r_M(x, s; k), \\
v(x, s; k) &= m(x; k) + r_{1,M}(x, s; k) \text{ on } \Gamma_j.
\end{aligned}
\]

Moreover, in \( \Omega_j \) its derivatives remain bounded in \( \overset{\circ}{\Omega}_j \).

We write \( \hat{\psi} = \psi(x, \nabla \hat{\psi}) \) on \( \Gamma_j \).

Moreover, since the norms \( \|\nabla \sigma\|_{m(\Omega_j)} \leq A_{R, m}\|\nabla \sigma\|_{\Gamma_j, m+2q}, \quad m \in \mathbb{N}, \) \( \|r_M(x, s; k)\|_{m(\Omega_j)} \leq C_{R, m}e^{-\Re(s)(R+a+1)}k^{-M+m+2}\|\nabla \sigma\|_{\Gamma_j, m+2M}, \quad m \in \mathbb{N}. \) Similar estimates hold for \( r_{1,M}(x, s; k) \).

In the following we assume that \( \sigma_0 \leq \Re(s) \leq 1 \).

For boundary data \( m(x; k) = e^{-s\psi(x)}b(x, s; k) \) with \( \psi(x) \) satisfying the condition (\( P \)) on \( \Gamma_j \) such that
\[ \|b(., s; k)\|_{\Gamma_j, p} \leq C_p, \quad p \in \mathbb{N}, \]
we write
\[ m(x; k) = e^{i\psi(x)}\hat{b}(x, s; k) \]
with \( \hat{b}(x, s; k) = e^{-(s+ik)\psi(x)}b(x, s; k) \).

For \( \sigma_0 \leq \Re(s) \leq 1, \|\Im(s + ik)\| \leq 1 \) the amplitude \( \hat{b} \) and its derivatives remain bounded in \( C^p(\Gamma_j) \) norms. Consequently, we can apply the operator \( S_j(s) \) to the oscillatory term
\[
\sum_{|j|=n+3,j_{n+2}=j} (-1)^{n+2}e^{-s\varphi_j(x)}a_j(x) \bigg|_{\Gamma_j}
\]
\[
e^{-s\varphi_j(x)} \sum_{|j|=n+3,j_{n+2}=j} (-1)^{n+2}e^{-s\varphi_j(x)+s\varphi_j(x)}a_j(x) \bigg|_{\Gamma_j}
\]
\[
e^{-s\varphi_j(x)} \bigg|_{\Gamma_j} \left( e^{s\varphi_j(x)}U_{n+2,j}(x - is) \right) \bigg|_{\Gamma_j},
\]
where \( a_j(x) = (A_j(\varphi)h)h(x) \) and \( A_j(\varphi)h \) are introduced in Section 2, while \( j = (j_0, j_1, \ldots, j_{n+2}) \) is a configuration such that \( j_{n+2} = j \). The choice of \( j \) is not important for our argument. We may consider this term as the product of \( e^{-s\varphi_j(x)} \) with the amplitude
\[ m_{j,n}(x, s) = \left( e^{s\varphi_j(x)}U_{n+2,j}(x - is) \right) \bigg|_{\Gamma_j}. \]

Moreover, since the norms \( \|\nabla \varphi_j\|_{\Gamma_j,(p)} \) are uniformly bounded for \( p \in \mathbb{N} \), taking into account the estimates for \( \|U_{n+2,j}(x, -is)\|_{\Gamma_j,(p)} \) established in Section 6, we get
\[ \|m_{j,n}(., s)\|_{\Gamma_j,(p)} \leq C_p \|U_{n+2,j}(x, -is)\|_{\Gamma_j,(p)} \leq C_p \|\Im(s)\|^{p+1+\beta_0}\tilde{\eta}, \quad \forall n \in \mathbb{N}. \]

Let
\[ S_j(s)(e^{-s\varphi_j(x)}a_j|_{\Gamma_j}) = e^{-s\varphi_j(x)} \sum_{q=0}^{M} \sum_{\nu=0}^{2q} a_{j,q,\nu}^{(j)}(x)(s + ik)\nu (i k)^{-q} + r_{j,M}^{(j)}(x, s; k) \]
with $a_{j,0,0}^{(j)}(x) = a_j(x)$. On the other hand, we have the estimates

$$
\left\| S_j(s) \left( \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s \varphi_j(x)} a_j(x) \right) \right\|_{C^p(\Omega(R))} \leq B_{p,R} \left\| U_{n+2,j}(x, -is) \right\|_{\Gamma_j, (p+2)}
$$

with constants $B_{p,R}$ independent of $n$. Consequently, for $\sigma_0 \leq \Re(s) \leq 1$, $|\Im(s + ik)| \leq 1$, the series

$$
S_j(s) \left( w_{0,j}(x, -is)|_{\Gamma_j} \right) = S_j(s) \left( \sum_{n=0}^{\infty} \sum_{|j|=n+3, j_{n+2}=j} u_j(x, -is)|_{\Gamma_j} \right)
$$

is convergent in $C^p(\Omega(R))$. Taking the sum over $j = 1, \ldots, \kappa_0$, we conclude that the function

$$
w^{(0)}(x, -is; k) = \sum_{j=1}^{\kappa_0} S_j(s) \left( w_{0,j}(x, -is)|_{\Gamma_j} \right)
$$

satisfies the condition (S) in $(\tilde{\Omega}, \{ s \in \mathbb{C}; \Re(s) \geq \sigma_0, |\Im(s + ik)| \leq 1 \})$. Indeed, for each $k \in \mathbb{R}$ the properties $(i)$ and $(iii)$ of Definition 1 follow directly from the convergence of the series. The condition $(ii)$ follows easily since for $\Re(s) > 0$ we have absolutely convergent series and this case has been treated by Ikawa [I3].

Next consider the boundary data

$$
\tilde{\nu}(x; k) = e^{ik\psi(x)} \bar{g}(x; k)
$$

satisfying condition (E) on $\Gamma_j$ and the operator $\tilde{S}_j(s)$ introduced in Definition 4.6 and Proposition 4.7 in [I3]. This construction is easy since the rays involved in the construction leave a compact neighborhood of the obstacle. For more details we refer to [I3]. In particular, we consider the boundary data

$$
e^{-s\psi(x)} (\tilde{A}_j(\varphi) \hat{\nu})(x)
$$

with

$$
(\tilde{A}_j(\varphi) \hat{\nu})(x) = \left( 1 - u_0(\langle \nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j) \rangle) \right) \Lambda_{\varphi,j}(x) \hat{\nu}(X^{-m}(x, \nabla \varphi_j))
$$

and apply the operator $\tilde{S}_j(s)$ to the sum

$$
\tilde{U}_{n+2,j}(x, -is)|_{\Gamma_j} = \sum_{|j|=n+3, j_{n+2}=j} \tilde{a}_j(x, -is)|_{\Gamma_j} = \sum_{|j|=n+3, j_{n+2}=j} (-1)^{n+2} e^{-s \varphi_j(x)} \tilde{a}_j(x)|_{\Gamma_j},
$$

where $\tilde{a}_j(x) = (\tilde{A}_j(\varphi) \hat{\nu})(x)$. Repeating the above argument, we can justify the existence of

$$
\tilde{w}^{(0)}(x, -is; k) = \sum_{j=0}^{\kappa_0} \sum_{n=0}^{\infty} \tilde{S}_j(s) \left( \tilde{U}_{n+2,j}(x, -is)|_{\Gamma_j} \right)
$$

and show that $\tilde{w}^{(0)}(x, -is; k)$ satisfies the condition (S) in $(\Omega, \{ s \in \mathbb{C}; \Re(s) \geq \sigma_0, |\Im(s + ik)| \leq 1 \})$. Now introduce

$$
W^{(0)}(x, -is; k) = w^{(0)}(x, -is; k) + \tilde{w}^{(0)}(x, -is; k).
$$

The forms of $a_j(x)$ and $\tilde{a}_j(x)$ and the construction of Section 4 imply that

$$
W^{(0)}(x, -is; k)|_{\Gamma - m(x; k)} = (ik)^{-1} R_0(x, s; k).
$$
Our construction for \( \Re(s) > s_0 \) is the same as that of Ikawa [I3], so for \( \Re(s) > s_0 + \delta > s_0 \) we get the estimates

\[
\|R_0(x, s; k)\|_{\Gamma, p} \leq C_{p, \delta} k^p, \ p \in \mathbb{N},
\]

established in [I3]. Here the constants \( C_{p, \delta} \) depend on the norms of \( \nabla \varphi \) and \( h \). However the estimates of Section 6 for \( \sigma_0 \leq \Re(s) \leq 1, \ |\Im(s + ik)| \leq 1 \), do not imply (7.3). The reason is that we have sums of terms

\[
\sum_{|j|=n+3,j_n+2=j} e^{-s \varphi_j(x)} \sum_{\nu=0}^2 a_{j,1,\nu}(x, s; k)(s + ik)^{-1}
\]

and it is rather complicated to find an analogue of Theorem 3 in this situation, since the amplitudes \( a_{j,1,\nu} \) are determined by the transport equations and their forms are not as simple as those of \( A_j(\varphi)\hat{g}(x) \). Nevertheless, the analysis of Section 6 shows that \( R_0(x, s; k) \) for \( x \in \Gamma, k \geq 1 \), is an analytic function in \( s \) for \( \sigma_0 \leq \Re(s) \leq 1, \ |\Im(s + ik)| \leq 1 \). Moreover, we have upper bounds

\[
\|R_0(x, s; k)\|_{\Gamma, p} \leq C_p k^{p+3+j_0}, \ \forall p \in \mathbb{N}
\]

which follows from the estimates for \( \|U_{n+2,j}(x, -i s)\|_{\Gamma, j, p} \) and (7.1). Of course, these bounds are not optimal, but they are sufficient for our argument.

In the next section we will construct lower order approximations.

8. LOWER ORDER TERMS OF THE ASYMPTOTIC SOLUTION

We start with the observation that the terms with factor \( (ik)^{-1} \) in the trace \( W^{(0)}(x, -is; k)|_{\Gamma_1} \) on \( \Gamma_1 \) have the form

\[
\sum_{n=0}^{\infty} \sum_{|j|=n+3,j_n+2=j} (-1)^{n+2} e^{-s \varphi_j(x)} \sum_{\nu=0}^2 \left[ a_{j,1,\nu}(x, s; k)|_{\Gamma_1} + \tilde{a}_{j,1,\nu}(x, s; k)|_{\Gamma_1} \right] (s + ik)^\nu.
\]

Here \( a_{j,1,\nu} \) come from the trace of \( w^{(0)}(x, -i s; k) \), while \( \tilde{a}_{j,1,\nu} \) come from the trace of \( \tilde{w}^{(0)}(x, -i s; k) \). Consider the boundary data

\[
m_{1,j}^{(j,l)}(x, s; k) = (-1)^{n+2} e^{-s \varphi_j(x)}|_{\Gamma_1} \sum_{\nu=0}^2 u_0 \left( \left( \nabla \varphi_j(x), \nu(X^{-1}(x, \nabla \varphi_j(x))) \right) \right) a_{j,1,\nu}(x, s; k)|_{\Gamma_1} (s + ik)^\nu
\]

\[
= e^{-s \varphi_j(x)}|_{\Gamma_1} a_{j,1,\nu}(x, s; k)|_{\Gamma_1} (s + ik)^\nu,
\]

and set

\[
M_{1,n}^{(j,l)}(x, s; k) = \sum_{|j|=n+3,j_n+2=j} m_{1,j}^{(j,l)}(x, s; k)
\]

\[
= e^{-s \varphi_j(x)}|_{\Gamma_1} \sum_{|j|=n+2,j_n+2=j} e^{s \varphi_j(x)}|_{\Gamma_1} m_{1,j}^{(j,l)}(x, s; k),
\]

where, as in the previous section, \( j \) is a configuration such that \( |j| = n + 2, j_n+2 = j \). Our aim is to apply the construction of Sections 4 and 6 to the oscillatory data \( M_{1,n}^{(j,l)}(x, s; k) \). To do this, we need some estimates of the \( C^p(\Gamma_h) \) norms of \( M_{1,n}^{(j,l)}(x, s; k) \). Here and below we denote by \( P^{(j,l)} \) some terms depending on the traces on \( K_j \) and \( K_{l, j}, \ j, l = 1, \ldots, \kappa_0 \), while \( j, j' \) will denote configurations.
The terms $M^{(j,l)}_{1,n}$ are obtained as the traces on $\Gamma_l$ of the terms with factor $(ik)^{-1}$ in the representation of

$$S_j(s) \left( e^{-s\varphi_l(x)} |_{\Gamma_j} m_{j,n}(x,s) \right),$$

where the boundary data $e^{-s\varphi_l(x)} |_{\Gamma_j} m_{j,n}(x,s)$ was introduced in the previous section. For the amplitudes $a^{(j)}_{1,\nu}(x,s;k)$ we have $C^p(\Gamma_j)$ estimates $O(|\text{Im}(s)|^{p+3+\beta} \tilde{\theta}^n)$ since for the boundary data $e^{-s\varphi_j(x)} |_{\Gamma_j} m_{j,n}(x,s)$ we have estimates $O(|\text{Im}(s)|^{p+1+\beta} \tilde{\theta}^n)$. Thus we deduce

$$\|M^{(j,l)}_{1,n}(x,s;k)\|_{\Gamma_j,p} \leq C_p|\text{Im}(s)|^{p+3+\beta} \tilde{\theta}^n, \forall n \in \mathbb{N} \quad (8.1)$$

with constants $C_p$ independent on $n \in \mathbb{N}$.

For fixed $j,l$ and fixed $n$ starting with the boundary data $M^{(j,l)}_{1,n}(x,s;k)$ we apply the construction of Sections 4 and 6 and we obtain a series

$$\sum_{m=0}^{\infty} V^{(j,l)}_{1,n,m}(x,s;k)$$

with

$$V^{(j,l)}_{1,n,m}(x,s;k) = \sum_{|j'|=m+3,|n'+z|=l} (-1)^{m+2} e^{-s\varphi_{j'}(x)} b^{(j,l)}_{1,n,j'}(x,s;k),$$

where the phase functions $\varphi_{j'}(x)$ depend on the configurations $j'$. Taking the summation over $n$, we are going to study the series

$$U^{(j,l)}_{1}(x,s;k) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V^{(j,l)}_{1,n,m}(x,s;k). \quad (8.2)$$

We repeat the argument of the previous section, for $\sigma_0 \leq \text{Re}(s) \leq 1, |\text{Im}(s+ik)| \leq 1$ and applying (8.1) and Theorem 3, (b), we get the estimates

$$\|V^{(j,l)}_{1,n,m}(x,s;k)\|_{\Gamma_1,p} \leq D_p k^{p+4+2\beta} \tilde{\theta}^{n+m}, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}$$

with constants $D_p$ independent on $n,m \in \mathbb{N}$. Thus the double series defining $U^{(j,h)}_{1}(x,s;k)$ is convergent and we can introduce

$$\sum_{j,h=1}^{\kappa_0} U^{(j,h)}_{1}(x,s;k).$$

In the same way, starting with boundary data $m^{(j,h)}_{1,j}(x,s;k)$ related to the terms involving $\tilde{a}^{(j)}_{1,\nu}(x,s;k)$, we define a function $\tilde{U}^{(j,h)}_{1}(x,s;k)$. Consider the sum

$$W^{(1)}(x,s;k) = -(ik)^{-1} \sum_{j,h=1}^{\kappa_0} \left( U^{(j,h)}_{1}(x,s;k) + \tilde{U}^{(j,h)}_{1}(x,s;k) \right)$$

and notice that

$$\|W^{(1)}(x,s;k)\|_{\Gamma_1,p} \leq Q'_p k^{p+6+2\beta_0}. \quad (8.3)$$

From this construction it follows that for $\sigma_0 \leq \text{Re}(s) \leq 1, |\text{Im}(s+ik)| \leq 1$, the sum

$$W^{(0)}(x,s;k) + W^{(1)}(x,s;k)$$
satisfies the condition \( (S) \) for \( \Re(s) \geq \sigma_0 \). Moreover, we have
\[
W^{(0)}(x, s; k) + W^{(1)}(x, s; k) - m(x, k) = (ik)^{-2}R_1(x, s; k), \ x \in \Gamma
\]
and
\[
\|R_1(x, s; k)\|_{\Gamma,(p)} \leq Q_{1,p}k^{p+6+2j_0}\left(\|\nabla \varphi\|_{\Gamma,(p+3)} + 1\right)\|h\|_{\Gamma,(p+3)}, \ \forall p \in \mathbb{N},
\]
while for \( \Re(s) \geq s_0 + \delta, \ |s + ik| \leq 1 \), we get
\[
\|R_1(x, s; k)\|_{\Gamma,(p)} \leq Q_{1,p,\delta}k^p\left(\|\nabla \varphi\|_{\Gamma,(p+3)} + 1\right)\|h\|_{\Gamma,(p+3)}, \ \forall p \in \mathbb{N}.
\]

Repeating this procedure, we construct \( W^{(j)}(x, s; k) \) for \( 0 \leq j \leq M \) which are analytic functions for \( \sigma_0 \leq \Re(s) \) with values in \( C^\infty(\Omega) \). They satisfy the condition \( (S) \) for \( \sigma_0 \leq \Re(s) \leq 1, \ |\Im(s + ik)| \leq 1 \) and we have
\[
\sum_{j=0}^M W^{(j)}(x, s; k) - m(s, k) = (ik)^{-M}R_M(x, s; k), \ x \in \Gamma
\]
with polynomial estimates
\[
\|R_M(x, s; k)\|_{\Gamma,(p)} \leq Q_{M,p}k^{N(M)},
\]
where \( Q_{M,p} \) depend on the norms of \( \nabla \varphi \) and \( h \). Thus we establish crude estimates with orders \( N(M) \) depending on \( M \) and it seems quite difficult to obtain more precise estimates for \( \sigma_0 \leq \Re(s) \leq 1 \). Of course, we have \( N(M) > M \) and \( (N(M) - M) \to \infty \) as \( M \to \infty \). For this reason in the domain \( \sigma_0 \leq \Re(s) \leq 1 \) we have no gains of the powers of \( k \). To obtain an approximation we will consider an integral equation on the boundary and solve it for \( \sigma_0 < \sigma_1 \leq \Re(s) \leq 1 \) with \( \sigma_1 < \sigma_0 \) dealing only with a finite number terms \( W^{(j)}(x, s; k), 0 \leq j \leq M \).

9. INTEGRAL EQUATION ON THE BOUNDARY

In this section we assume that \( N \geq 3 \). The case \( N = 2 \) is simpler and can be treated by the same argument. We use the approximations \( W^{(j)}(x, s; k), j = 1, \ldots, M \) for suitable boundary data \( m(x, s) = e^{ik\varphi(x)}h(x) \), where \( M \) is fixed so that \( M > (N - 1)/2 \). Following the argument in the previous section, we have the estimates (8.6) and (8.7). Moreover, \( R_M(x, s; k) \) is analytic for \( \sigma_0 \leq \Re(s) \leq 1 \) and for \( \Re(s) \geq s_0 + \delta, \ |\Im(s + ik)| \leq 1 \) the result of Ikawa [12] in the absolutely converging domain yields
\[
\|R_M(x, s; k)\|_{\Gamma,0} \leq Q_{M,\delta}.
\]
By using Pragmen-Lindelöf theorem, (8.7) with \( p = 0 \) and the above estimate, we can find \( \sigma_0 < \sigma_1 < \sigma_0 \) such for
\[
s \in D_1 = \{s \in C : \sigma_1 \leq \Re(s) \leq 1, \ |\Im(s + ik)| \leq 1\},
\]
we have
\[
\|R_M(x, s; k)\|_{\Gamma,0} \leq C_Mk^{\alpha},
\]
where \( 0 < \alpha < M \frac{N-1}{2} \). Moreover, the constants \( C_M \) and \( \alpha \) depend on the derivatives of \( \nabla \varphi \) and \( h \).

Let \( \mathcal{G} \subset \Gamma_j \) and let \( F \in L^2(\Gamma_j) \) with \( \text{supp} \ F \subset \mathcal{G} \). Choose local coordinates in \( \mathcal{G} \) of the form
\[
x(x') = (x', h(x')), \ x' = (x_1, \ldots, x_{N-1}) \in J \subset \mathbb{R}^{N-1},
\]
and write
\[
F(x'') = (2\pi)^{-N+1}\int e^{ix'\cdot \xi'} \hat{F}(\xi')d\xi'
\]
\[(2\pi)^{-N+1}G(x') \int_{|\xi'| \leq \epsilon} e^{ik<x',\xi'>} \hat{F}(k\xi')k^{N-1}d\xi' + (2\pi)^{-N+1}G(x')\epsilon \int_{|\omega| = \epsilon} e^{ikp<x',\omega>} \hat{F}(k\rho\omega)k^{N-1}\rho^{N-2}d\rho d\omega = F_1 + F_2.\]

Here \(G(x') \in C_0^\infty(\mathbb{R}^{N-1}), G(x') = 1\) on \(\text{supp} F(x(x'))\),

\[\hat{F}(\xi') = \int e^{-ik<x',\xi'>} F(x(x')) dx'.\]

and \(0 < \epsilon < 1\) will be chosen sufficiently small according to the construction of a phase function \(\varphi(x; \xi')\) satisfying the condition (P) (see Appendix A for more details). Notice that the choice of \(\epsilon\) is uniform with respect to \(\Gamma_j\) and \(\epsilon\) is independent on the parameters \(k \geq 1\) and \(\rho\). To apply the construction of Sections 4-7, consider the function

\[\psi(x'; \xi') = \psi(x', h(x'); \xi') = < x', \xi' >, \quad x' \in J, \quad |\xi'| < \epsilon\]

depending on \(\xi'\) and determined on for \(x = (x', h(x')) \in U \subset \Gamma_j\) for fixed \(j = 1, \ldots, \kappa_0\). We construct a phase function \(\varphi(x; \xi')\) defined in a neighborhood of \(\Gamma_j\) such that

\(i\) \(\varphi(x; \xi') = \psi(x; \xi')\) on \(U \cap \Gamma_j\),

\(ii\) \(\frac{\partial \varphi}{\partial y}(x; \xi') \geq \delta > 0\) on \(\Gamma_j\),

\(iii\) the principal curvatures of \(G_\varphi(x; \xi') = \{y : \varphi(y; \xi') = \varphi(x; \xi')\}\) with respect to the normal field \(-\nabla \varphi\) are positive for all \(x \in \Gamma_j\),

\(iv\) the phase \(\varphi(x; \xi')\) satisfies the condition (P) on \(\Gamma_j\).

For sake of completeness we discuss in Appendix A the construction of \(\varphi(x; \xi')\). Next for the oscillatory data \(m_1(x; \xi') = (2\pi)^{-N+1}G(x')e^{ik\varphi(x; \xi')}\) we construct an approximative solution \(V_1(x, s; k, \xi')\) which satisfies the condition (S) in \(D_1\), and define an operator \(U_1(s; k)F\) with values in \(C^\infty(\Omega)\) so that the function \(U_1(s; k)F\) satisfies the condition (S) in \(D_1\).

Moreover,

\[U_1(s; k)F|_\Gamma = F_1 + k^{-M} \int_{|\xi'| \leq \epsilon} R_{1,M}(x, s; k, \xi') \hat{F}(k\xi')k^{N-1}d\xi' + F_1 + L_1(s; k)F\]

with \(R_{1,M}(x, s; k, \xi')\) satisfying the estimate (9.1). It is clear that the constants \(C_M\) and \(\alpha\) in (9.1) can be chosen to be uniform with respect to \(\xi', |\xi'| \leq \epsilon\). It is easy to see that

\[\|L_1(s; k)F\|^2_{L^2(\Gamma)} \leq C_0 \left( \int_{|\xi'| \leq \epsilon} k^{-M+(N-1)/2+\alpha} |\hat{F}(k\xi')|^2 k^{(N-1)/2}d\xi' \right)^2 \leq C_0 k^{-2M+N-1+2\alpha} \int_{R^{N-1}} |\hat{F}(k\xi')|^2 k^{N-1}d\xi' \leq C_1 k^{-2M+N-1+2\alpha} \|F\|^2_{L^2(\Gamma)}\]

with a constant \(C_1 > 0\) depending only on the boundary \(\Gamma\).

To deal with the term \(F_2\), we consider \(kp\) as a parameter. The phase function

\[\Psi(x', h(x'); \omega) = < x', \omega >, \quad x' \in J, \quad |\omega| = \epsilon\]

depends on the parameter \(\omega\) and as above we can construct phase functions \(\varphi(x; \omega)\) satisfying the conditions (i) - (iv). We apply the constructions in Section 4-7 for the oscillatory data
Moreover, we arrange the estimate (9.1) for $C_{m_{36}}V$. PETKOV AND L. STOYANOV

$$U \parallel V$$

and we deduce that

$$B \parallel V$$

By using a partition of unity on $\Gamma$, we define an operator

$$(\cdot, (\cdot))$$

Applying this estimate for $S$ with $\chi$ cut-off resolvent

$$(\cdot, (\cdot))$$

Thus

$$R = R_1(s, k)F$$

Theorem 2.

$$L_2(\Omega)$$

such that

$$U_2(s; k)F|_\Gamma = F_2$$

Moreover, we arrange the estimate (9.1) for $R_2, M(x, s; k, \rho, \omega)$ uniformly with respect to $|\omega| = \epsilon$. Applying this estimate for $R_2, M(x, s; k, \rho, \omega)$, we get

$$\|L_2(s; k)F_2\|_{L^2(\Gamma)}^2 \leq C_3 \left( k^{-M + (N-1)/2 + \alpha} \int_1^\infty \int_{|\omega| = \epsilon} \rho^{-M + (N-2)/2 + \alpha |\hat{F}(k\rho\omega)|^2 k^{N-1} \rho^{-2} d\rho d\omega \right)^2$$

$$\leq C_4 k^{-2M + N - 1 + 2\alpha} \|F\|_{L^2(\Gamma)}^2$$

with constants $C_2, C_3, C_4$ depending only on the boundary $\Gamma$. Here we have used the fact that $-2M + N - 2 + 2\alpha < -1$. Finally, we introduce

$$U_\varphi(s; k)F = U_1(s; k)F_1 + U_2(s; k)F_2$$

and conclude that

$$U_\varphi(s; k)F|_\Gamma = F + L_1(s; k)F + L_2(s; k)F = F + L_\varphi(s; k)F$$

with

$$\|L_\varphi(s; k)F\|_{L^2(\Gamma)} \leq B_\varphi k^{-M + (N-1)/2 + \alpha} \|F\|_{L^2(\Omega)}.$$

By using a partition of unity on $\Gamma$, we define an operator

$$U(s; k) : L^2(\Omega) \ni f \longrightarrow U(s; k)f \in C^\infty(\Omega)$$

and we deduce that $U(s; k)f$ satisfies the condition (S) in $\mathcal{D}_1$, while on the boundary $\Gamma$ we have the equation

$$U(s; k)f|_\Gamma = f + L(s; k)f$$

with

$$\|L(s; k)f\|_{L^2(\Gamma)} \leq Bk^{-M + (N-1)/2 + \alpha} \|f\|_{L^2(\Gamma)}.$$

For $k \geq k_0$ the operator $I + L(s; k) : L^2(\Gamma) \rightarrow L^2(\Gamma)$ is invertible and we define the operator

$$R(s; k)f = U(s; k)(I + L(s; k))^{-1}f : L^2(\Gamma) \longrightarrow C^\infty(\bar{\Omega})$$

satisfying the condition (S) for $s \in \mathcal{D}_1$ and the condition

$$R(s; k)f|_\Gamma = f.$$
To obtain Corollary 1 we establish the estimate
\[ \|R_{\chi}(z)\|_{H^M(\Omega') \to L^2(\Omega')} \leq C(1 + |z|^{m-M}) \],
where \( m \in \mathbb{N} \) is the integer in (1.7) and \( M \in \mathbb{N} \). Taking \( M > m \), the proof goes repeating that in the non-trapping case (see Theorem 1 in [TZ]).

10. APPENDIX A

As in the previous section consider a local representation \( x_N = h(x') \) of the boundary \( \Gamma_j \) with \( x' = (x_1, \ldots, x_{N-1}) \in J \subset \mathbb{R}^{N-1} \). We wish to construct a phase function \( \varphi(x; \xi') \) such that
\[ \varphi(x', h(x'); \xi') = \langle x', \xi' \rangle, \quad x = (x', h(x')) \in U, \xi' = (\xi_1, \ldots, \xi_{N-1}), \]
\( U \) being a small neighborhood of a fixed point \( x_0 \in \Gamma_j \) so that \( \varphi(x; x') \) satisfies the conditions (i) – (iv) of Section 9. For simplicity we will avoid the notation \( \xi' \) in \( \varphi(x; \xi') \). We will assume that \( |\xi'| \leq \epsilon \), where \( \epsilon > 0 \) will be chosen sufficiently small. It is convenient to consider a little more general problem with boundary data related to a smooth function \( \chi(x') \) such that \( |\nabla_{x'} \chi(x')| \leq \delta < 1 \) for \( x' \in \text{supp} \chi(x') = W \subset J \). We will construct a phase function \( \varphi(x) \) such that
\[ \varphi(x', h(x')) = \chi(x'), \quad x' \in J. \] (10.1)
This condition means that for \( x = (x', h(x')) \in \Gamma_j \) with \( x' \notin \text{supp} \chi \) we have \( \varphi(x) = 0 \).

From the boundary condition (10.1) we will determine the derivatives of \( \varphi \) on the boundary \( \Gamma_j \).

Set \( \varphi_{x'} = (\varphi_{x_1}, \ldots, \varphi_{x_{N-1}}), h_{x'} = (h_{x_1}, \ldots, h_{x_{N-1}}), \chi_{x'} = (\chi_{x_1}, \ldots, \chi_{x_{N-1}}) \). We have \( \varphi_{x'} + \varphi_{xN} h_{x'} = \chi_{x'} \) and setting \( \varphi_{xN} = \sqrt{1 - |\varphi_{x'}|^2} \), we are going to solve the system
\[ \varphi_{x'} + \sqrt{1 - |\varphi_{x'}|^2} h_{x'} = \chi_{x'}. \]

Since \( \sqrt{1 - |\varphi_{x'}|^2} h_{x_j} = \chi_{x_j} - \varphi_{x_j}, j = 1, \ldots, N - 1 \), we have
\[ (1 - |\varphi_{x'}|^2)|h_{x'}|^2 = |\chi_{x'}|^2 + |\varphi_{x'}|^2 - 2\langle \chi_{x'}, \varphi_{x'} \rangle. \]

On the other hand,
\[ 2\langle \chi_{x'}, \varphi_{x'} \rangle + 2\sqrt{1 - |\varphi_{x'}|^2}\langle h_{x'}, \chi_{x'} \rangle = 2|\chi_{x'}|^2 \]
and we obtain the equation
\[ (1 + |h_{x'}|^2)(1 - |\varphi_{x'}|^2) - 2\langle h_{x'}, \chi_{x'} \rangle \sqrt{1 - |\varphi_{x'}|^2} + |\chi_{x'}|^2 - 1 = 0. \]
Consequently, for \( \varphi_{xN} = \sqrt{1 - |\varphi_{x'}|^2} \) we obtain
\[ \varphi_{xN}(x', h(x')) = \frac{1}{1 + |h_{x'}|^2} \left( (h_{x'}, \chi_{x'}) + \sqrt{((h_{x'}, \chi_{x'}))^2 + (1 - |\chi_{x'}|^2)(1 + |h_{x'}|^2)} \right). \]

Now it is easy to see that we have the condition
\[ \langle \nabla \varphi(x), \nu(x) \rangle \geq \delta_0 > 0, \; x = (x', h(x')) \in U. \] (10.2)
In fact in local coordinates \( (x', h(x')) \) the outward normal to \( \Gamma_j \) is given by
\[ \nu(x) = \frac{1}{\sqrt{1 + |h_{x'}|^2}} (-h_{x'}, 1) \]
and we get
\[ \langle \nabla \varphi(x), \nu(x) \rangle = \frac{1}{\sqrt{1 + |h_{x'}|^2}} \left[ \varphi_{xN} - \langle h_{x'}, \chi_{x'} \rangle + \sqrt{1 - |\varphi_{x'}|^2}|h_{x'}|^2 \right] \]
\[
\begin{align*}
\frac{1}{\sqrt{1 + |h_x'|^2}} \left[ (1 + |h_x'|^2)\varphi_{xN} - \langle h_x', \chi_x' \rangle \right] \geq \sqrt{1 - |\chi_x'|^2} \geq \sqrt{1 - \delta^2} = \delta_0 > 0.
\end{align*}
\]

Moreover, for \( \delta \) sufficiently small, \( \delta_0 \) is close to 1. By using (10.2), we can solve locally the eiconal equation \( |\nabla \varphi(x)| = 1 \) with initial data
\[
\varphi(x', h(x')) = \chi(x'),
\]
\[
\nabla_x \varphi(x', h(x')) = (\varphi_{x'}(x', h(x'))), \varphi_{xN}(x', h(x'))), (x', h(x')) \in U.
\]
For \( x' \) such that \( \chi(x') = 0 \) we get
\[
\varphi_{xN}(x', h(x')) = \frac{1}{\sqrt{1 + |h_x'|^2}}, \varphi_{x'}(x', h(x')) = -\frac{h_x'}{\sqrt{1 + |h_x'|^2}}
\]
so \( \nabla_x \varphi(x', h(x')) \) coincides with the unit outward normal \( \nu(x', h(x')) \) to \( \Gamma_j \) at \( x = (x', h(x')) \). Thus if \( x = z + t\nu(z), t \geq 0 \) with \( z = (x', h(x')) \in \Gamma_j \) and \( x' \in J \setminus W \), we have \( \varphi(x) = t \) and \( \varphi(x) \) coincides with the phase function \( \Psi(x) \) defined \textbf{globally} in a neighborhood of \( \Gamma_j \) and having boundary data \( \Psi(x) = 0, \forall x \in \Gamma_j \). Consequently, we may consider \( \Psi(x) \) as a continuation of \( \varphi(x) \) and so \( \varphi(x) \) is defined \textbf{globally} in a small open neighborhood of \( \Gamma_j \).

To deal with the local boundary condition \( \langle x', \xi' \rangle \), we choose a smooth cut-off function \( \alpha(x'), 0 \leq \alpha(x') \leq 1, |\nabla_x \alpha(x')| \leq C \) such that \( \alpha(x') = 1 \) if \( (x', h(x')) \in U \) and set
\[
\chi(x') = \alpha(x') \langle x', \xi' \rangle.
\]
Then
\[
|\chi_{x'}| \leq C_1 |\xi'| \leq \delta, \quad C_1 \leq C(\text{diam } \Gamma_j) + 1
\]
if \( |\xi'| \leq \epsilon = \delta/C_1 \). For \( |\xi'| \) small by partition of unity on \( \Gamma_j \) we may choose the constants \( \delta < 1 \) and \( \epsilon < 1 \) introduced above to be uniform with respect to the choice of the local coordinates in \( \Gamma_j \).

Next we will show that the principal curvatures of the wave front
\[
G_{\varphi}(x_0) = \{ y \in \mathbb{R}^N : \varphi(y) = \varphi(x_0) \}, \quad x_0 \in \Gamma_j
\]
are strictly positive for every \( x_0 \in \Gamma_j \). Since the phase \( \varphi(x) \) coincides with \( \Psi(x) \) outside some open subset, we will treat only the subset of the wave front set \( G_{\varphi}(x_0) \) lying in a small neighborhood of a point \( x_0 \in U \). Assume that the local coordinates \( x_N = h(x') \) are such that \( h(x_0') = 0, h_{x'}(x_0') = 0 \). Consider a parameterization
\[
G_{\varphi}(x_0) = \{ \omega(\sigma), \quad \sigma \in \bar{U}, \omega(0) = x_0, \omega(\sigma) = (\omega_1(\sigma), ..., \omega_N(\sigma)) \}.
\]
Moreover, we can assume that the matrix
\[
\Omega' = \left( \frac{\partial \omega_k(0)}{\partial \sigma_j} \right)_{k,j=1}^{N-1}
\]
is invertible. Then
\[
\nabla \varphi(\sigma) = \nabla \varphi(x'(\sigma), h(x'(\sigma))) = \left( \varphi_{x'}(x'(\sigma), h(x'(\sigma))), \varphi_{xN}(x'(\sigma), h(x'(\sigma))) \right),
\]
where \( x'(\sigma) \) is determined by
\[
\omega(\sigma) = (x'(\sigma), h(x'(\sigma))) + l(\sigma)\nabla \varphi(x'(\sigma), h(x'(\sigma))), l(\sigma) \in \mathbb{R}, \quad l(0) = 0.
\]
For \( \sigma = 0 \) we have
\[
\frac{\partial \varphi_{xN}}{\partial \sigma_j}(0) = - \sum_k h_{x_k}(x_0') \frac{\partial x_k'}{\partial \sigma_j}(0) \varphi_{xN}(x_0) + O(\epsilon), \quad i,j = 1, ..., N - 1.
\]
for some constants $\alpha, \epsilon > 0$ the principal curvatures of $\Gamma_j$ j.

This yields $X' = \Omega' + O(\epsilon)$ and we obtain

$$\left( \frac{\partial \nabla \varphi}{\partial \sigma_1}, \ldots, \frac{\partial \nabla \varphi}{\partial \sigma_{N-1}} \right)(0) = \left( \frac{\partial \omega}{\partial \sigma_1}, \ldots, \frac{\partial \omega}{\partial \sigma_{N-1}} \right)(0) \left[ (\Omega')^{-1} H \sqrt{1 - |\xi|^2} + O(\epsilon) \right].$$

The principal curvatures of $\Gamma_j$ at $x_0$ are the eigenvalues of the matrix $H$, while the principal curvatures of $G_\varphi(x_0)$ are the eigenvalues of the matrix

$$(\Omega')^{-1} H \sqrt{1 - |\xi|^2} + O(\epsilon).$$

Thus for small $\epsilon > 0$ the principal curvatures of $G_\varphi(x_0)$ are positive. To see that the phase $\varphi(x)$ satisfies the condition (P) on $\Gamma_j$, notice that for small $x_0 > 0$ and $x_0 \in \Gamma_j$ the wave front set $C(x_0) = G_\varphi(x_0)$ is a strictly convex surface. For this purpose we must take $||\langle x', \xi' \rangle|| \leq \eta$, where $\eta > 0$ is small enough and $\eta$ depends only on the diameters of $K_i$, $i = 1, \ldots, \kappa_0$. Now the rays issued from $y \in C(x_0)$ in direction $\nabla \varphi(y)$ cover the complement of $C(x_0)$ and (P) is satisfied.

11. Appendix B

Here we first state the assumptions about the billiard flow and the set $\Lambda$ under which the results in [St3] imply the Dolgopyat type estimates (4.3). We then explain how to apply these in the situation described in Sect. 6 above.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$W^s_\epsilon(x) = \{ y \in S^*(\Omega) : d(\varphi_t(x), \varphi_t(y)) \leq \epsilon \text{ for all } t \geq 0, d(\varphi_t(x), \varphi_t(y)) \to_{t \to 0} 0 \},$

$W^u_\epsilon(x) = \{ y \in S^*(\Omega) : d(\varphi_t(x), \varphi_t(y)) \leq \epsilon \text{ for all } t \leq 0, d(\varphi_t(x), \varphi_t(y)) \to_{t \to 0} 0 \}$

be the (strong) stable and unstable manifolds of size $\epsilon$. Then $E^u(x) = T_x W^u_\epsilon(x)$ and $E^s(x) = T_x W^s_\epsilon(x)$.

The following pinching condition is one of the assumptions mentioned above:

(P): There exist constants $C > 0$ and $\alpha > 0$ such that for every $x \in \Lambda$ we have

$$\frac{1}{C} e^{\alpha x t} \|u\| \leq \|d\varphi_t(x) \cdot u\| \leq C e^{\beta x t} \|u\|, \quad u \in E^u(x), t > 0,$$

for some constants $\alpha_x, \beta_x > 0$ depending on $x$ but independent of $u$ with $\alpha \leq \alpha_x \leq \beta_x$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in \Lambda$.

Notice that when $N = 2$ this condition is always satisfied. For $N \geq 3$, (P) follows from certain estimates on the eccentricity of the connected components $K_j$ of $K$. According to general regularity results ([PSW]), (P) implies that $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ are Lipschitz in $x \in \Lambda$. It appears that in the proof of the estimates (4.3), in the case of open billiard flows (and some geodesic flows), one should be able to replace the condition (P) by just assuming Lipschitzness of the stable and unstable laminations – this will be the subject of some future work.
Next, consider the following non-flatness condition:

\((\text{NF}):\) For every \(x \in \Lambda\) there exists \(\epsilon_x > 0\) such that there is no \(C^1\) submanifold \(X\) of positive codimension with \(\Lambda \cap W^u_{\epsilon_x}(x) \subset X\).

Clearly this condition is always satisfied if \(N = 2\), while for \(N \geq 3\) it is at least generic. In the proof of the main result in [St3] this condition plays a technical role, and one would expect that a future refinement of the proof would remove this condition.

Next, we need some definitions from [St3]. Given \(z \in \Lambda\), let \(\exp^u_z : E^u(z) \to W^u_{\epsilon_0}(z)\) and \(\exp^s_z : E^s(z) \to W^s_{\epsilon_0}(z)\) be the corresponding exponential maps. A vector \(b \in E^u(z) \setminus \{0\}\) will be called tangent to \(\Lambda\) at \(z\) if there exist infinite sequences \(\{v^{(m)}\} \subset E^u(z)\) and \(\{t_m\} \subset \mathbb{R} \setminus \{0\}\) such that \(\exp^u_z(t_m v^{(m)}) \in \Lambda \cap W^u_{\epsilon_0}(z)\) for all \(m\), \(v^{(m)} \to b\) and \(t_m \to 0\) as \(m \to \infty\). It is easy to see that a vector \(b \in E^u(z) \setminus \{0\}\) is tangent to \(\Lambda\) at \(z\) if there exists a \(C^1\) curve \(z(t)\) \((0 \leq t \leq a)\) in \(W^u_{\epsilon_0}(z)\) for some \(a > 0\) with \(z(0) = z\) and \(\dot{z}(0) = b\), and \(z(t) \in \Lambda\) for arbitrarily small \(t > 0\). In a similar way one defines tangent vectors to \(\Lambda\) in \(E^s(z)\).

Denote by \(d\omega\) the standard symplectic form on \(T^*(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{R}^N\). The following condition says that \(d\omega\) is in some sense non-degenerate on the ‘tangent space’ of \(\Lambda\) near some its points:

\((\text{ND}):\) There exist \(z_0 \in \Lambda, \epsilon > 0\) and \(\mu_0 > 0\) such that for any \(\tilde{z} \in \Lambda \cap W^u_{\epsilon_0}(z_0)\) and any unit vector \(b \in E^u(\tilde{z})\) tangent to \(\Lambda\) at \(\tilde{z}\) there exist \(\tilde{z} \in \Lambda \cap W^u_{\epsilon_0}(z_0)\) arbitrarily close to \(\tilde{z}\) and a unit vector \(a \in E^s(\tilde{z})\) tangent to \(\Lambda\) at \(\tilde{z}\) with \(|d\omega(a, b)| \geq \mu_0|\).

Clearly when \(N = 2\) this condition is always satisfied. In fact, it seems very likely (and there is some evidence supporting it) that this conditions is always satisfied for open billiard flows.

The following is an immediate consequence of the main result in [St3].

**Theorem B.1** Assume that the billiard flow \(\varphi_t\) over \(\Lambda\) satisfies the conditions (P), (NF) and (ND). Then the estimates (4.3) hold.

In the remaining part of this section we show how to apply the Dolgopyat type estimates from [St3] to obtain the estimates of \(\|L^*_{s0}G_{s0}\|_{T, 0}\) required in Sect. 6. The problem is that the operator \(L_s\) acts on \(C(\Sigma^x_0)\), that is, it is related to the coding of billiard trajectories by means of the components of \(K\), while the Dolgopyat type estimates apply to Ruelle transfer operators defined by means of Markov families and acting on functions that are Lipschitz with respect to the standard metric in the phase space. Here, following [PS2], we describe how the two types of Ruelle transfer operators relate, and show that the function \(G_{s0}\), when transformed in a natural way to a function on a (unstable leaf of a) Markov rectangle, is Lipschitz. This and the assumptions about \(K\) made in Sect. 1 make it possible to apply the estimates from [St3].

Given \(E \subset \Lambda\) we will denote by \(\text{Int}_E(E)\) and \(\partial_E E\) the interior and the boundary of the subset \(E\) of \(\Lambda\) in the topology of \(\Lambda\), and by \(\text{diam}(E)\) the diameter of \(E\). Following [D], a subset \(R\) of \(\Lambda\) will be called a rectangle if it has the form \(R = [U, S] = \{[x, y] : x \in U, y \in S\}\), where \(U\) and \(S\) are subsets of \(W^u_{\epsilon_0}(z) \cap \Lambda\) and \(W^s_{\epsilon_0}(z) \cap \Lambda\), respectively, for some \(z \in \Lambda\) that coincide with the closures of their interiors in \(W^u_{\epsilon_0}(z) \cap \Lambda\) and \(W^s_{\epsilon_0}(z) \cap \Lambda\). Here \(W^s_{\epsilon_0}(z)\) and \(W^u_{\epsilon_0}(z)\) are the strong stable and unstable manifolds of size \(\epsilon > 0\) at \(z \in \Lambda\) (which are parts of \(W^s_{\text{loc}}(z)\) and \(W^u_{\text{loc}}(z)\) constructed in Sect. 3 above).

Let \(R = \{R_i\}_{i=1}^k\) be a Markov family of rectangles \(R_i = [U_i, S_i]\) for \(\Lambda\) (see e.g. [KH], [D] or [St3] for the definition). Set \(R = \bigcup_{i=1}^k R_i\), denote by \(\mathcal{P} : R \to R\) the corresponding Poincaré map, and by \(\tau\) the first return time associated with \(\mathcal{R}\). Then \(\mathcal{P}(x) = \varphi_{\tau(x)}(x) \in R\) for any \(x \in R\). Notice
that \( \tau \) is constant on each stable fiber of each \( R_i \). We will assume that the size \( \chi = \max_i \text{diam}(R_i) \) of the Markov family \( \mathcal{R} = \{ R_i \}_{i=1}^k \) is sufficiently small so that each rectangle \( R_i \) is between two boundary components \( \Gamma_i \) and \( \Gamma_i^\prime \) of \( K \), that is for any \( x \in R_i \), the first backward reflection point of the billiard trajectory \( \gamma \) determined by \( x \) belongs to \( \Gamma_i \), while the first forward reflection point of \( \gamma \) belongs to \( \Gamma_i^\prime \).

Moreover, using the fact that the intersection of \( \Lambda \) with each cross-section to the flow \( \varphi_t \) is a Cantor set, we may assume that the Markov family \( \mathcal{R} \) is chosen in such a way that

(i) for any \( i = 1, \ldots, k \) we have \( \partial \Lambda U_i = \emptyset \).

Finally, partitioning every \( R_i \) into finally many smaller rectangles if necessary and removing some ‘unnecessary’ rectangles from the family formed in this way, we may assume that

(ii) for every \( x \in R \) the billiard trajectory of \( x \) from \( x \) to \( \mathcal{P}(x) \) makes exactly one reflection.

From now on we will assume that \( \mathcal{R} = \{ R_i \}_{i=1}^k \) is a fixed Markov family for \( \varphi_t \) of size \( \chi < \varepsilon_0/2 \) satisfying the above conditions (i) and (ii). Set

\[
U = \bigcup_{i=1}^k U_i.
\]

The shift map \( \tilde{\sigma} : U \to U \) is given by \( \tilde{\sigma} = \pi(U) \circ \mathcal{P} \), where \( \pi(U) : R \to U \) is the projection along stable leaves.

Let \( \mathcal{A}(A_{ij})_{i,j=1}^k \) be the matrix given by \( A_{ij} = 1 \) if \( \mathcal{P}(R_i) \cap R_j \neq \emptyset \) and \( A_{ij} = 0 \) otherwise. Consider the symbol space

\[
\Sigma_A = \{(i_j)_{j=-\infty}^\infty : 1 \leq i_j \leq k, A_{ij, i_{j+1}} = 1 \text{ for all } j \},
\]

with the product topology and the shift map \( \sigma : \Sigma_A \to \Sigma_A \) given by \( \sigma((i_j)) = ((i_j')) \), where \( i_j' = i_{j+1} \) for all \( j \). As in [B] one defines a natural map \( \Psi : \Sigma_A \to R \). Namely, given any \( i = (i_j)_{j=-\infty}^\infty \in \Sigma_A \) there is exactly one point \( x \in R_{i_0} \) such that \( \mathcal{P}^j(x) \in R_{i_j} \) for all integers \( j \). We then set \( \Psi((i_j)) = x \). One checks that \( \Psi \circ \sigma = \mathcal{P} \circ \Psi \) on \( R \). It follows from the condition (i) above that the map \( \pi \) is a bijection.

In a similar way one deals with the one-sided subshift of finite type

\[
\Sigma_A^+ = \{(i_j)_{j=0}^\infty : 1 \leq i_j \leq k, A_{ij, i_{j+1}} = 1 \text{ for all } j \geq 0 \},
\]

where the shift map \( \sigma : \Sigma_A^+ \to \Sigma_A^+ \) is defined in the same way. There exists a unique map \( \psi : \Sigma_A^+ \to U \) such that \( \psi \circ \sigma = \pi(U) \circ \Psi \), where \( \pi : \Sigma_A \to \Sigma_A^+ \) is the natural projection.

Notice that the root function \( \tau : \Sigma_A \to [0, \infty) \) defined by \( \tau(\xi) = \tau(\Psi(\xi)) \) depends only on the forward coordinates of \( \xi \in \Sigma_A \). Indeed, if \( \xi_+ = \eta_+ \), where \( \xi_j = (\xi_j)_{j=0}^\infty \), then for \( x = \Psi(\xi) \) and \( y = \Psi(\eta) \) we have \( x, y \in R_i \) for \( i = \xi_0 = \eta_0 \) and \( \mathcal{P}^j(x) \) and \( \mathcal{P}^j(y) \) belong to the same \( R_{i_j} \) for all \( j \geq 0 \). This implies that \( \tau(x) = \tau(y) \). Thus, \( \tau(x) = \tau(y) \). So, we can define a root function \( \tau : \Sigma_A^+ \to [0, \infty) \) such that \( \tau \circ \pi = \tau \circ \Psi \).

Let \( B(\Sigma_A^+) \) be the space of bounded functions \( g : \Sigma_A^+ \to \mathbb{C} \) with its standard norm \( ||g||_0 = \sup_{\xi \in \Sigma_A^+} |g(\xi)| \). Given a function \( g \in B(\Sigma_A^+) \), the Ruelle transfer operator \( \mathcal{L}_g : B(\Sigma_A^+) \to B(\Sigma_A^+) \) is defined by \( (\mathcal{L}_g h)(\eta) = \sum_{\sigma(\eta) = \xi} e^{g(\eta)} h(\eta) \).

We can also use coding of the flow over \( \Lambda \) by using the boundary components of \( K \). This is described in Sect. 4 above. We will use the notation from there, notably \( f(\xi), g(\xi), \eta^{(k)} \) for any \( k = 1, \ldots, k_0, \epsilon(\xi), \chi_f = \chi_1, \chi_2 = \chi_2, \tilde{f}(\xi) \) and \( \tilde{g}(\xi) \). Define the map \( \Phi : \Sigma_A \to \Lambda_{\partial K} = \Lambda \cap S_A^*(\Omega) \) by \( \Phi(\xi) = (P_0(\xi), (P_1(\xi) - P_0(\xi))/||P_1(\xi) - P_0(\xi)||) \). Then \( \Phi \) is a bijection such that \( \Phi \circ \sigma = B \circ \Phi \),
where $B : \Lambda_{\partial K} \rightarrow \Lambda_{\partial K}$ is the billiard ball map. As before, given any function $G \in B(\Sigma^+_A)$, the Ruelle transfer operator $L_G : B(\Sigma^+_A) \rightarrow B(\Sigma^+_A)$ is defined by $(L_G H)(\eta) = \sum_{\sigma(\eta) = \xi} e^{G(\xi)} H(\eta)$.

Let $V_0$ be the set of those $(p, u) \in S^*(\Omega)$ such that $p = q + t u$ and $(p, u) = \varphi_t(q, u)$ for some $(q, u) \in S^*_{\partial K}(\Omega)$ with $\langle u, \nu(q) \rangle > 0$ and some $t \geq 0$. Setting $\omega(p, u) = (q, u)$, we get a smooth map $\omega : V_0 \rightarrow S^*_{\partial K}(\Omega)$ defined on an open subset $V_0$ of $S^*(\Omega)$ containing $\Lambda$.

Consider the bijection $S = \Phi^{-1} \circ \omega \circ \Psi : \Sigma_A \rightarrow \Sigma_A$. Its restriction to $\Sigma^+_A$ defines a bijection $S : \Sigma^+_A \rightarrow \Sigma^+_A$. Moreover $S \circ \sigma = \sigma \circ S$. Define the function $g' : \Sigma_A \rightarrow \mathbb{R}$ by $g'(\tilde{i}) = g(S(\tilde{i}))$.

Next, for any $i = 1, \ldots, k$ choose $\tilde{z}^{(i)} = (\ldots, j_m^{(i)}, \ldots, j_1^{(i)})$ such that $(\tilde{z}^{(i)}, i) \in \Sigma^+_A$. It is convenient to make this choice in such a way that $\tilde{z}^{(i)}$ corresponds to the local unstable manifold $U_i \subset \Lambda \cap W^u(z_i)$, i.e. the backward itinerary of every $z \in U_i$ coincides with $\tilde{z}^{(i)}$. Now for any $\tilde{i} = (i_0, i_1, \ldots) \in \Sigma^+_A$ (or $\tilde{i} \in \Sigma_A$) set $\hat{e}(\tilde{i}) = (\tilde{z}^{(i_0)}, i_0, i_1, \ldots) \in \Sigma_A$. According to the choice of $\tilde{z}^{(i_0)}$, we then have $\Psi(\hat{e}(\tilde{i})) = \psi(\tilde{i}) \in U_{i_0}$. (Notice that without the above special choice we would only have that $\Psi(\hat{e}(\tilde{i}))$ and $\psi(\tilde{i}) \in U_{i_0}$ lie on the same stable leaf in $R_{i_0}$.) Next, define $\hat{\chi}_g(\tilde{i}) = \sum_{n=0}^{\infty} \left[ g'(\sigma^n(\tilde{i})) - g'(\sigma^n \hat{e}(\tilde{i})) \right]$ for $\tilde{i} \in \Sigma_A$. As before, the function $\hat{g} : \Sigma_A \rightarrow \mathbb{R}$ given by $\hat{g}(\tilde{i}) = g'(\tilde{i}) - \hat{\chi}_g(\tilde{i})$ depends on future coordinates only, so it can be regarded as a function on $\Sigma^+_A$.

We will now describe a natural relationship between the operators $L_G : B(\Sigma^+_A) \rightarrow B(\Sigma^+_A)$ and $L_g : B(\Sigma^+_A) \rightarrow B(\Sigma^+_A)$ with $g$ appropriately defined by means of $G$.

First define $\Gamma : B(\Sigma_A) \rightarrow B(\Sigma_A)$ by $\Gamma(V) = V \circ \Phi^{-1} \circ \omega \circ \Psi = V \circ S$. Since by property (ii) of the Markov family, $\omega : R \rightarrow \Lambda_{\partial K}$ is a bijection, it follows that $\Gamma$ is a bijection and $\Gamma^{-1}(v) = v \circ \Psi^{-1} \circ \omega^{-1} \circ \Phi$. Moreover $\Gamma$ induces a bijection $\Gamma : B(\Sigma^+_A) \rightarrow B(\Sigma^+_A)$. Indeed, assume that $v \in B(\Sigma_A)$ depends on future coordinates only. Then $v \circ \Phi^{-1}$ is constant on local stable manifolds in $S^*_A(\Omega)$. Hence $v \circ \Phi^{-1} \circ \omega$ is constant on local stable manifolds on $R$, and therefore $\Gamma(v) = v \circ \Phi^{-1} \circ \omega \circ \Psi$ depends on future coordinates only.

Next, let $v, w \in B(\Sigma^+_A)$ and let $V = \Gamma(v)$, $W = \Gamma(w)$. Given $\tilde{i}, \tilde{j} \in \Sigma^+_A$ with $\sigma(\tilde{j}) = \tilde{i}$, setting $\xi = S(\tilde{i})$ and $\eta = S(\tilde{j})$, we have $\sigma(\eta) = \xi$. Thus,

$$L_W V(\tilde{i}) = \sum_{\sigma(\tilde{j}) = \tilde{i}} e^W(\tilde{j}) V(\tilde{j}) = \sum_{\sigma(\tilde{j}) = \tilde{i}} e^{\hat{g}(\tilde{j})} w(S(\tilde{j})) = L_w v(\xi)$$

for all $\tilde{i} \in \Sigma^+_A$. This shows that $(L_w v) \circ S = L_{\Gamma(w)} \Gamma(v)$.

The following proposition is derived from [PS2].

**Proposition 5.** There exist Lipschitz functions $\delta_1, \delta_2 : U \rightarrow \mathbb{R}$ such that setting $\hat{\delta}_s(\tilde{i}) = e^{s \delta_1(\psi(\tilde{i})) + \delta_2(\psi(\tilde{i}))}$ for $\tilde{i} \in \Sigma^+_A$ and $s \in \mathbb{C}$ we have

$$(L^n_{-s \hat{f} + \hat{g}} u)(S(\tilde{i})) = \frac{1}{\delta_s(\tilde{i})} \cdot L^n_{-s r + \hat{g}} \left( \hat{\delta}_s \cdot (u \circ S) \right)(\tilde{i}), \quad \tilde{i} \in \Sigma^+_A, s \in \mathbb{C},$$

for any $u \in C(\Sigma^+_A)$ and any integer $n \geq 1$. 
For the needs of Sect. 6 above, we need to estimate \( \|L_n^{s} G_s \tilde{v}_s\|_{\Gamma_0}. \) Recall that the operator \( G_s \) defined in Sect. 4 above. For any integer \( n \geq 0 \) we have

\[
L_n^{s} G_s v^{(\xi)} = \sum_{\sigma^{s} = \xi} \sum_{\tau = \eta} e^{-s f_\sigma(\eta) + g_\tau(\eta)} e^{-\varphi^+(\zeta, s) - s f(\zeta) + g(\zeta)} v(\zeta)
\]

Thus, it is enough to estimate \( \|L_n^{s} G_s v^{(\xi)}\| \). As in Sects. 4-6 above, we will consider these operators over \( \Gamma_1 \).

Given \( s \in \mathbb{C} \), consider the functions \( w_s : U_1 \to \mathbb{R} \) and \( \hat{w}_s : \Sigma^+ A \to \mathbb{R} \) defined by

\[
w_s(x) = w_s(\psi(I)) = \hat{w}_s(\hat{I}) = e^{-\varphi^+(\xi, s) \hat{v}_s(\xi)},
\]

for \( x = \psi(I) \in U_1, I \in \Sigma^+ A \). In order to use the Dolgopyat type estimates from [St3], we have to show that \( w_s \) is Lipschitz on \( U_1 \). We will deal in details with

\[
w_s^{(1)}(x) = e^s \sum_{n=0}^{\infty} e^{-s n \varphi(\xi)} h(Q_n(\xi)) = e^s w_s^{(1)}(x) w_s^{(2)}(x).
\]

Fix an arbitrary point \( y_1 \in \Lambda \) such that \( \eta^{(1)} \in \Sigma_\Lambda \) corresponds to the local unstable manifold \( W^u_{\text{loc}}(y_1) \), i.e. the backward itinerary of every \( z \in W^u_{\text{loc}}(y_1) \cap V_0 \) coincides with \( \eta^{(1)} \). Under the assumptions about \( K \) made in Sect. 1, the map \( \mathcal{H}_1 : U_1 \to W^u_{\text{loc}}(y_1) \) defined by \( \mathcal{H}_1(x) = \varphi_{\Delta, y_1}(x, y_1) \) is Lipschitz. Here \( \Delta \) is the so called temporal distance function (see e.g. [D] or Sect. 2 in [St3]). In more details, given \( x \in U_1 \) there exist \( z \in W^u_{\text{loc}}(x) \) and \( t \in \mathbb{R} \) such that \( \varphi_{t}(z) \in W^u_{\text{loc}}(y_1) \); then by definition \( \Delta(x, y_1) = t \) and \( z = [x, y_1] \), so \( \mathcal{H}_1(x) = \varphi_{t}(z) \).

Next, consider the \( N \)-dimensional submanifold \( X = \{ q, q + t \varphi_\gamma(q) : q \in \Gamma_1, 0 < t \} \) of \( S^\gamma(\mathbb{R}^N) \) and the (stable) holonomy map \( \mathcal{H} : W^u_{\text{loc}}(y_1) \cap \Lambda \to X \) defined by \( \mathcal{H}(y) = W^s(y) \cap X \). Since \( \varphi_{\gamma} \) satisfies Ikawa’s condition \( (P) \), it is easy to see that \( W^s(y) \) is transversal to \( X \), so \( \mathcal{H}(y) = W^s(y) \cap X \) is well-defined for \( y \in W^s_{\text{loc}}(y_1) \cap \Lambda \). Moreover, using again the assumptions about \( K \), it follows from some general arguments in [PSW] that the stable (and unstable) holonomy maps for the billiard flow \( \varphi_{\gamma} \) are Lipschitz. In particular, \( \mathcal{H} \) is Lipschitz.

We can now write down \( w_s^{(1)}(x) \) using the maps \( \mathcal{H} \) and \( \mathcal{H}_1 \) as follows. Given \( x \in U_1 \), we have \( x = \psi(I) \) for some \( \hat{I} \in \Sigma^+_A \), with \( i_0 = 1 \). Setting \( \xi = \mathcal{S}(\hat{I}), \) we then have \( \xi_0 = 1 \). For any integer \( m > 1 \) consider

\[
B_m = \sum_{n=0}^{m-1} [f(\sigma^n e(\xi)) - f^n_+(\xi)] - \varphi(Q_0(\xi)).
\]

Setting \( y = \mathcal{H}_1(x) \in W^u_{\text{loc}}(y_1) \) and \( z = \mathcal{H}(y) \), we have that \( z \in W^s_{\text{loc}}(y) \), and moreover \( \omega(z) = (Q_0(\xi), \nabla \varphi(Q_0(\xi))). \) Thus, \( Q_0(\xi) = \text{pr}_1(\omega(z)) = \text{pr}_1(\omega(\mathcal{H}(\mathcal{H}_1(x)))) \) is Lipschitz in \( x \in U_1 \). Next, set \( \epsilon(u) = \|\text{pr}_1(u) - \text{pr}_1(\omega(u))\| \); then \( u = \varphi_{\epsilon(u)}(\omega(y)) \) and \( \epsilon(u) \) is a smooth function on an open subset of \( S^\gamma(\Omega) \) (where \( \omega \) is defined and takes values in \( S^\gamma(\Omega) \)). For \( B_m \) we have

\[
B_m = O(\theta^m) + \epsilon(y) - \epsilon(z) - \varphi(\omega(z)) = O(\theta^m) + \epsilon(y) - \varphi(z),
\]

and letting \( m \to \infty \) we get

\[
w_s^{(1)}(x) = e^{\left[ \epsilon(y) - \varphi(z) \right]} h(\omega(z)) = e^{\left[ \epsilon(\mathcal{H}_1(x)) - \varphi(\mathcal{H}(\mathcal{H}_1(x))) \right]} h(\omega(\mathcal{H}(\mathcal{H}_1(x))))
\]
so $w_s^{(1)}(x)$ is Lipschitz in $x \in U_1$. Moreover, for $x \in U_1$ and bounded $\Re(s)$ we obtain an uniform bound for the Lipschitz norm of $w_s^{(1)}(x)$. The same argument works for $w_s^{(2)}(x)$.

References


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