Spectra of Ruelle transfer operators for Axiom A flows

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Abstract. For Axiom A flows on basic sets satisfying certain additional conditions we prove strong spectral estimates for Ruelle transfer operators similar to those of Dolgopyat [D2] for geodesic flows on compact surfaces (for general potentials) and transitive Anosov flows on compact manifolds with $C^1$ jointly non-integrable horocycle foliations (for the Sinai-Bowen-Ruelle potential). Here we deal with general potentials and on spaces of arbitrary dimension, although under some geometric and regularity conditions. As is now well known, such results have deep implications in some related areas, e.g. in studying analytic properties of Ruelle zeta functions and partial differential operators, closed orbit counting functions, and in other areas.

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1 Introduction

1.1 Introduction and main results

Let $\phi_t : M \rightarrow M$ be a $C^2$ Axiom A flow on a $C^2$ complete (not necessarily compact) Riemann manifold $M$ and let $\Lambda$ be a basic set for $\phi_t$. Let $\| \cdot \|$ be the norm on $T_xM$ determined by the Riemann metric on $M$ and let $E^u(x)$ and $E^s(x)$ ($x \in \Lambda$) be the tangent spaces to the strong unstable and stable manifolds $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$, respectively (see section 2). For any $x \in \Lambda$, $T > 0$ and $\delta \in (0,\epsilon]$ set

$$B^u_T(x,\delta) = \{ y \in W^u_\epsilon(x) : d(\phi_t(x),\phi_t(y)) \leq \delta, \ 0 \leq t \leq T \}.$$ 

We will say that $\phi_t$ has a regular distortion along unstable manifolds over the basic set $\Lambda$ if there exists a constant $\epsilon_0 > 0$ with the following properties:

(a) For any $0 < \delta \leq \epsilon \leq \epsilon_0$ there exists a constant $R = R(\delta,\epsilon) > 0$ such that

$$\mathrm{diam}(\Lambda \cap B^u_T(z,\epsilon)) \leq R \mathrm{diam}(\Lambda \cap B^u_T(z,\delta))$$

for any $z \in \Lambda$ and any $T > 0$.

(b) For any $\epsilon \in (0,\epsilon_0]$ and any $\rho \in (0,1)$ there exists $\delta \in (0,\epsilon]$ such that for any $z \in \Lambda$ and any $T > 0$ we have $\mathrm{diam}(\Lambda \cap B^u_T(z,\delta)) \leq \rho \mathrm{diam}(\Lambda \cap B^u_T(z,\epsilon))$.

Part (a) of the above condition resembles the Second Volume Lemma of Bowen and Ruelle [BR] about balls in Bowen’s metric; this time however we deal with diameters instead of volumes. In a separate paper [St3] we describe a rather general class of flows on basic sets satisfying this condition – see section 7 below for a brief account of these. There are reasons to believe that this condition may actually hold for all $C^2$ flows on basic sets – see the comments at the end of this subsection.

In this paper we deal with flows $\phi_t$ over basic sets $\Lambda$ having a regular distortion along unstable manifolds. Apart from that, in the main result below we impose an additional condition on $\phi_t$ and $\Lambda$, called the local non-integrability condition (LNIC); it is stated in section 2 below. It should be mentioned that this condition is rather weak and is always satisfied e.g. for contact flows that are either Anosov (i.e. $\Lambda = M$), or have one-dimensional stable and unstable manifolds (see section 7 below). One would expect that (LNIC) is satisfied in most interesting cases. For example, it was
shown in [St2] that open billiard flows (in any dimension) with $C^1$ (un)stable laminations over their non-wandering sets always satisfy (LNIC).

Let $R = \{R_i\}_{i=1}^k$ be a Markov family for $\phi_t$ over $\Lambda$ consisting of rectangles $R_i = [U_i, S_i]$, where $U_i$ (resp. $S_i$) are (admissible) subsets of $W^u(z_i) \cap \Lambda$ (resp. $W^s(z_i) \cap \Lambda$) for some $\epsilon > 0$ and $z_i \in \Lambda$ (cf. section 2 for details). Assuming that the local stable and unstable laminations over $\Lambda$ are Lipschitz, the first return time function $\tau : R = \bigcup_{i=1}^k R_i \to [0, \infty)$ and the standard Poincaré map $P : R \to R$ are essentially Lipschitz. Setting $U = \bigcup_{i=1}^k U_i$, the shift map $\sigma : U \to U$ is defined by $\sigma = \pi(U) \circ P$, where $\pi(U) : R \to U$ is the projection along the leaves of local stable manifolds. Let $\tilde{U}$ be the set of all $u \in U$ whose orbits do not have common points with the boundary of $R$ (see section 2). Given a Lipschitz real-valued function $f$ on $\tilde{U}$, set $g = g_f = f - P\tau$, where $Pf \in \mathbb{R}$ is the unique number such that the topological pressure $Pr(\sigma) = \pi(U)$ of $g$ with respect to $\sigma$ is zero (cf. e.g. [PP]). For $a, b \in \mathbb{R}$, one defines the Ruelle transfer operator $L_{g-(a+ib)\tau} : C^{\text{Lip}}(\tilde{U}) \to C^{\text{Lip}}(\tilde{U})$ in the usual way (cf. section 2). Here $C^{\text{Lip}}(\tilde{U})$ is the space of Lipschitz functions $g : \tilde{U} \to \mathbb{C}$. By $\text{Lip}(g)$ we denote the Lipschitz constant of $g$ and by $\|g\|_0$ the standard sup norm of $g$ on $\tilde{U}$.

We will say that the Ruelle transfer operators related to the function $f$ on $\tilde{U}$ are eventually contracting if for every $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq 1/a_0$, then for every integer $m > 0$ and every $h \in C^{\text{Lip}}(\tilde{U})$ we have

$$
\|L_{f-(P_f+a+ib)\tau}^m h\|_{\text{Lip,b}} \leq C \rho^m |b|^\epsilon \|h\|_{\text{Lip,b}},
$$

where the norm $\|\cdot\|_{\text{Lip,b}}$ on $C^{\text{Lip}}(\tilde{U})$ is defined by $\|h\|_{\text{Lip,b}} = \|h\|_0 + \frac{\text{Lip}(h)}{|b|}$. This implies in particular that the spectral radius of $L_{f-(P_f+a+ib)\tau}$ on $C^{\text{Lip}}(\tilde{U})$ does not exceed $\rho$.

Our main result in this paper is the following.

**Theorem 1.1.** Let $\phi_t : M \to M$ be a $C^2$ Axiom A flow on a $C^2$ complete Riemann manifold satisfying the condition (LNIC) and having a regular distortion along unstable manifolds over a basic set $\Lambda$. Assume in addition that the local holonomy maps along stable laminations through $\Lambda$ are uniformly Lipschitz. Then there exists a Markov family $R = \{R_i\}_{i=1}^k$ for $\phi_t$ over $\Lambda$ of arbitrary small size such that for any Lipschitz real-valued function $f$ on $\tilde{U}$ the Ruelle transfer operators related to $f$ are eventually contracting.

We refer the reader to section 2 below for the definition of holonomy maps. In general these are only Hölder continuous. It is known that uniform Lipschitzness of the local stable holonomy maps can be derived from certain bunching condition concerning the rates of expansion/contraction of the flow along local unstable/stable manifolds over $\Lambda$ (see [Ha1], [Ha2], [PSW]).

It should be mentioned that some kind of a non-integrability assumption is necessary for results like Theorem 1.1 (and Corollary 1.2). Indeed, Pollicott [Po] and Ruelle [R2] have constructed examples of mixing Axiom A flows with jointly integrable stable and unstable laminations for which the statements of Corollaries 1.4 and 1.5 below (and therefore that of Theorem 1.1 as well) do not hold. As one can see in section 2, (LNIC) is a rather weak non-integrability condition. Moreover, in the contact case it follows from another condition (see (ND) in section 6) which looks rather natural and is always satisfied for Anosov flows.

It follows from Theorem 7.1 below that a flow with one-dimensional unstable laminations over a basic set $\Lambda$ always has a regular distortion along unstable manifolds. The local stable holonomy maps are always Lipschitz (in fact $C^1$) if the stable laminations over $\Lambda$ are one-dimensional (see e.g. Theorem 1 and fact (2) on p. 647 in [Ha1]). Moreover, the flow always satisfies (LNIC) if it is Anosov with jointly non-integrable stable and unstable foliations, or it is contact (see Proposition 6.1 below for the latter). Thus, we get the following consequence of Theorem 1.1.
Corollary 1.2. Let $\phi_t : M \to M$ be a $C^2$ flow on a $C^2$ Riemann manifold $M$ with $\dim(M) = 3$, and let $\Lambda$ be a basic set for $\phi_t$ such that the stable and unstable laminations over $\Lambda$ are one-dimensional. Assume in addition one of the following: (i) the flow is contact, or (ii) the flow is Anosov and the stable and unstable laminations over $\Lambda$ are jointly non-integrable. Then there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ for $\phi_t$ over $\Lambda$ of arbitrary small size such that the Ruelle transfer operators related to any Lipschitz real-valued function $f$ on $\hat{U}$ are eventually contracting.

The above was first proved by Dolgopyat ([D1], [D2]) in the case of geodesic flows on compact surfaces. The second main result in [D2] concerns transitive Anosov flows on compact Riemann manifolds with $C^1$ jointly non-integrable local stable and unstable foliations. For such flows Dolgopyat proved that the conclusion of Theorem 1.1 holds for the Sinai-Bowen-Ruelle potential $f = \log \det(d\phi_t|_{E^u})$. Theorem 1.1 appears to be first result of this kind that works for any potential and in any dimension.$^1$

For contact Anosov flows in any dimension (LNIC) is always satisfied (see Proposition 6.1 below), so the following is also an immediate consequence of Theorem 1.1.

Corollary 1.3. Let $\phi_t : M \to M$ be a $C^2$ contact Anosov flow on a compact Riemann manifold having a regular distortion along unstable manifolds over $M$ and such that the local holonomy maps along stable laminations through $\Lambda$ are uniformly Lipschitz. Then there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ for $\phi_t$ of arbitrary small size such that the Ruelle transfer operators related to any Lipschitz real-valued function $f$ on $\hat{U}$ are eventually contracting.

Using a smoothing procedure as in [D2] (see also Corollary 3.3 in [St1]), an estimate similar to that in Theorem 1.1 holds for the Ruelle operator acting on the space $\mathcal{F}_\gamma(U)$ of Hölder continuous functions with respect to an appropriate norm $\|\cdot\|_{\gamma,b,U}$.

Using Theorem 1.1 and an argument of Pollicott and Sharp [PoS1] one derives valuable information about the Ruelle zeta function $\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$, where $\gamma$ runs over the set of primitive closed orbits of $\phi_t : \Lambda \to \Lambda$ and $\ell(\gamma)$ is the least period of $\gamma$. In what follows $h_T$ denotes the topological entropy of $\phi_t$ on $\Lambda$.

Corollary 1.4. Under the assumptions in Theorem 1.1, Corollary 1.2 or Corollary 1.3, the zeta function $\zeta(s)$ of the flow $\phi_t : \Lambda \to \Lambda$ has an analytic and non-vanishing continuation in a half-plane $\Re(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, there exists $c \in (0,h_T)$ such that

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} = \text{li}(e^{h_T\lambda}) + O(e^{c\lambda})$$

as $\lambda \to \infty$, where $\text{li}(x) = \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}$ as $x \to \infty$.

As another consequence of Theorem 1.1 and the procedure described in [D2] one gets exponential decay of correlations for the flow $\phi_t : \Lambda \to \Lambda$.

Given $\alpha > 0$ denote by $\mathcal{F}_\alpha(\Lambda)$ the set of Hölder continuous functions with Hölder exponent $\alpha$ and by $\|h\|_\alpha$ the Hölder constant of $h \in \mathcal{F}_\alpha(\Lambda)$.

Corollary 1.5. Under the assumptions in Theorem 1.1, Corollary 1.2 or Corollary 1.3, let $F$ be a Hölder continuous function on $\Lambda$ and let $\nu_F$ be the Gibbs measure determined by $F$ on $\Lambda$.

$^1$Albeit under additional conditions but this appears to be inevitable.
Assume in addition that the manifold $M$ and the flow $\phi_t$ are $C^5$. For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that

$$\left| \int_{\Lambda} A(x)B(\phi_t(x)) \, dv_F(x) - \left( \int_{\Lambda} A(x) \, dv_F(x) \right) \left( \int_{\Lambda} B(x) \, dv_F(x) \right) \right| \leq Ce^{-ct}\|A\|_\alpha \|B\|_\alpha$$

for any two functions $A, B \in \mathcal{F}_\alpha(\Lambda)$.

There has been a considerable activity in recent times to establish exponential and other types of decay of correlations for various kinds of systems with some highly rated results of Chernov [Ch1], Dolgopyat [D1], [D2], Liverani [L1], [L2], Young [Y1], [Y2]. See also [BSC], [BaT], [T], [Ch2], [Ch3], [GL], [D3], [FMT], [Mel], [ChY] and the references there. In [L2] Liverani proves exponential decay of correlations for contact Anosov flows, and this appears to be the most general result of this kind so far. Recently Tsujii [T] obtained finer results for the same kind of flows. It should be stressed that in [L2] and [T] (and various other works; see the references there) spectral properties of a different kind of transfer operators are studied, namely the operators $\mathcal{L}_t g = g \circ \phi_{-t}$ ($t \in \mathbb{R}$) acting on functions $g$ on a compact manifold $M$, $\phi_t$ being a contact Anosov flow on $M$.

We should stress though that the main aim of this article is not to establish results on decay of correlations but rather to get strong spectral estimates for Ruelle transfer operators. These operators appear to be more difficult to deal with, since they are intimately related to geometric properties of the flow (and the basic set $\Lambda$ when Axiom A flows are considered). On the other hand, spectral results of the kind obtained in Theorem 1.1 appear to be much finer and to have a rather wide and deep range of applications.

In [St1] a modification of the method from [D2] was used to prove an analogue of Corollary 1.2 above for open billiard flows in the plane. Using similar tools, Naud [N] proved a similar result for geodesic flows on convex co-compact hyperbolic surfaces. The results in both [St1] and [N] are special cases of Corollary 1.2. Baladi and Vallée ([BaV]) obtained Dolgopyat type estimates for transfer operators in the case of a suspension of an interval map.

It has been well known since Dolgopyat’s paper [D2] that strong spectral estimates for Ruelle transfer operators as the ones described in Theorem 1.1 lead to deep results concerning zeta functions and related topics which are difficult to obtain by other means. For example, such estimates were fundamental in [PoS1], where the statement of Corollary 1.4 was proved for geodesic flows on compact surfaces of negative curvature. For the same kind of flows, fine and very interesting asymptotic estimates for pairs of closed geodesics were established in [PoS3], again by using the strong spectral estimates in [D2]. For Anosov flows with $C^1$ jointly non-integrable horocycle foliations full asymptotic expansions for counting functions similar to $\pi(\lambda)$ however with some homological constraints were obtained in [An] and [PoS2]. In [PeS2] Theorem 1.1 above was used to obtain results similar to these in [PoS3] about correlations for pairs of closed billiard trajectories for billiard flows in $\mathbb{R}^n \setminus K$, where $K$ is a finite disjoint union of strictly convex compact bodies with smooth boundaries satisfying the so called ‘no eclipse condition’ (and some additional conditions as well). For the same kind of models and using Theorem 1.1 again, a rather non-trivial result was established in [PeS1] about analytic continuation of the cut-off resolvent of the Dirichlet Laplacian in $\mathbb{R}^n \setminus K$, which appears to be the first of its kind in the field of quantum chaotic scattering. It is not clear at all how such a result could be proved without using the strong spectral estimates of the kind considered here. Finally, in a very recent preprint [PeS3], using Theorem 1.1 and under the assumptions in this theorem, a fine asymptotic was obtained for the number of closed trajectories in $\Lambda$ with primitive periods lying in exponentially shrinking intervals $(x - e^{-\delta x}, x + e^{-\delta x})$, $\delta > 0$, $x \to +\infty$.  

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The main part of this article consists of sections 3-5 where Theorem 1.1 is proved. Section 2 contains some basic definitions and facts. The regular distortion along unstable manifolds has some natural consequences about diameters of cylinders defined by means of Markov families – these are derived in section 3. The essential part of the proof of Theorem 1.1 is in sections 4-5.

In section 6 we introduce a non-integrability condition for contact flows and show that, under some regularity assumption, it implies (LNIC). In section 7 we describe classes of flows over basic sets satisfying the conditions in Theorem 1.1 using the results in [St3]. It should be stressed that the central part of the arguments in [St3] is to prove a local version of regular distortion along unstable manifolds, where e.g. (1.1) is satisfied at a single point that is not yet clear how one can get a uniform global result over Λ from such local results.

The Appendix contains the proof of a technical lemma from section 5.

### 1.2 Comments on the proof of the main result

In the proof of Theorem 1.1 we use the general framework of Dolgopyat’s method from [D1], [D2] and its modification in [St1], however a significant new development is necessary. The main difficulty comes from the fact that very little is known about the (geometric) structure of the set Λ. For general potentials f the original method in [D2] was only applied to geodesic flows on surfaces. It requires more sophisticated arguments to deal with general potentials for flows on higher-dimensional spaces, especially when the flow is considered over a basic set (possibly with complicated fractal structure) rather than on a nice smooth compact manifold.

In this subsection we make some general remarks about the main points in the proof of Theorem 1.1. Given $f \in C^{\text{Lip}}(U)$, one wants to show that large powers of the Ruelle transfer operator $L_r = L_{f-(\nu_1+a+ib)r}^{-1}$ are contracting for small $a \in \mathbb{R}$ and large $b \in \mathbb{R}$. For any integer $N > 0$ we have

\[(1.2) \quad L_r^N h(x) = \sum_{\sigma^N(y)=x} e^{rN(y)} h(y),\]

where $r_N(y) = r(y) + r(\sigma(y)) + \ldots + r(\sigma^{N-1}(y))$. One of the main steps in [D2] is to define appropriately $C^1$ inverses $v_1, v_2 : U_0 \rightarrow U$ of $\sigma^N$, i.e. $\sigma^N(v_1(x)) = x$ for $x$ in some ‘small’ open subset $U_0$ of $U$. For these one ultimately shows that for large $N$ there exists $\lambda \in (0,1)$ such that

\[(1.3) \quad |e^{rN(v_1(x))} h(v_1(x)) + e^{rN(v_2(x))} h(v_2(x))| \leq \lambda \left[ |e^{rN(v_1(x))}| h(v_1(x)) + |e^{rN(v_2(x))}| h(v_2(x)) \right]\]

for small $|a|$ and large $|b|$ (of magnitude depending on $N$) and $h \in C^1(U)$ with $h > 0$, $||h|| \leq \text{Const}$, $||dh|| \leq \text{Const} \cdot |b|$. This leads to a similar estimate for the whole sum in (1.2). To make this cancellation mechanism\(^2\) work for general complex-valued functions $h$, one uses estimates involving $N_{f,h}$, where $\{N_f\}$ is a finite family of specially defined operators acting on positive functions in $C^1(U)$ with bounded logarithmic derivatives. One of the main features of Dolgopyat’s operators $N_f$ is that they are $L^2$-contractions with respect to the invariant Gibbs measure $\nu$ determined by $f - P\tau$ on $U$, and this is crucial for the proof of the main result.

The whole procedure is rather more complicated than the above, and we refer the reader to section 5 below for more details. It is worth mentioning though that it is the construction

\(^2\)Now the role of the 'bottom of the unstable spectrum' is played by the exponential of the least positive Lyapunov exponent.

\(^3\)To use the words of Liverani [L2] who is using a similar idea.
of the inverses \(v_1\) and \(v_2\) of \(\sigma^N\) and the proof of their main properties where the joint non-integrability of the stable and unstable families is used. In [D2] this results in finding \(C^1\) vector fields \(e_1(z), \ldots, e_n(z)\) defined in a small neighbourhood \(U_0\) of a point \(z_0 \in U\) so that for large \(N\),

\[
(1.4) \quad |\partial_{e_1}(\tau_N(v_1(u)) - \tau_N(v_2(u)))| \geq \epsilon \quad u \in U_0,
\]

\[
(1.5) \quad |\partial_{e_i}(\tau_N(v_1(u)) - \tau_N(v_2(u)))| << \epsilon \quad i > 1, u \in U_0.
\]

Notice that \(\epsilon^{1b\tau_N(y)}\) is the ‘tricky’ part of the exponential term \(\epsilon^{\tau_N(y)}\) in (1.2). With (1.4) and (1.5) one has that if \(u, u' \in U_0\) are \(C\epsilon\)-close however there is a \(c\epsilon\)-gap between their first coordinates (with respect to the vector field \(e_1\)) for some constants \(C > c > 0\), then

\[
(1.6) \quad \left| \tau_N(v_1(u)) - \tau_N(v_2(u)) - \tau_N(v_1(u') - \tau_N(v_2(u')) \right| \geq \text{const} \epsilon.
\]

This is what lies beneath the proof of (1.3).

In [St1] a modification of the above was used to deal with open billiard flows in the plane. In this case the unstable manifolds are one-dimensional, so one has just one vector field \(e_1(u)\), however \(U\) is a Cantor set, so the construction of \(v_1(u)\) and \(v_2(u)\) is non-trivial. One of the main difficulties in [St1] was to show (using specific features of the model) that there exist constants \(c_2 > c_1 > 0\) such that for any \(\epsilon > 0\), \(U\) can be partitioned into intervals of lengths between \(c_1\epsilon\) and \(c_2\epsilon\) such that successive intervals with common points with \(\Lambda\) have a gap of a similar size between them\(^4\). Then the one-dimensionality of the unstable manifolds allows to construct the functions \(v_i(u)\) and prove (1.4). Moreover for any of the small intervals \(\Delta_1\) in the partition of \(U\) intersecting \(\Lambda\) one can find another interval \(\Delta_2\) intersecting \(\Lambda\) with a \(c_1\epsilon\)-gap between \(\Delta_1\) and \(\Delta_2\) such that (1.6) holds for all \(u \in \Delta_1\) and all \(u' \in \Delta_2\). The rest of Dolgopyat’s method was not so difficult to apply, although extra modifications were necessary to deal with the singularity of the Gibbs measures in this case.

In the present paper the dimension \(n\) of the unstable manifolds is arbitrary and the set \(\Lambda\) can be ‘anything from a Cantor set to a manifold’. Therefore to construct vector fields \(e_i(u)\) in a neighbourhood \(U_0\) of a point \(z_0 \in U\) with (1.4) and (1.5) is meaningless, unless we know that there are ‘many points’ of \(\Lambda\) in the direction of \(e_1(u)\) (or ‘very close’ to it) for a ‘large set’ of \(u\)’s in \(U_0\). To establish something like this however seems impossible without assuming anything in the spirit of the extra condition (LNIC) from section 2 below. With (LNIC), one constructs for large \(N > 0\) a pair of inverses \(v_1(u)\) and \(v_2(u)\) of \(\sigma^N\) such that the analogue of (1.4) holds on a neighbourhood of \(z\) with \(e_1\) replaced by unit vectors \(\eta\) in a cone whose axis is a parallel translate of \(\eta\). The above gives that if \(C_1\) and \(C_2\) are two cylinders in the vicinity of \(z\) of size \(\leq \delta\), for some small \(\delta > 0\), that can be ‘separated by a plane whose normal is close to \(\eta\)’ (one can make sense of this by using some parametrization of \(U\) near \(z\)), then (1.6) holds for all \(u \in C_1\) and all \(u' \in C_2\). However, since \(n > 1\) in general, just one direction \(\eta\) does not give enough opportunities to separate cylinders. So, one needs to construct a finite number of directions \(\eta_1, \ldots, \eta_{\ell_0}\) tangent to \(\Lambda\) (at various points close to the initial point \(z_0\)), and for each \(\ell = 1, \ldots, \ell_0\), a pair of inverses \(v_1^{(\ell)}(u)\) and \(v_2^{(\ell)}(u)\) of \(\sigma^N\) defined on some small open subset \(U_0\) of \(U\) such that ‘sufficiently many’ pairs of cylinders in \(U_0\) of size \(\leq \delta\) can be separated by planes each of them having a ‘normal close to some \(\eta_{\ell}'\’, and this can be done for all \(\delta\) in some interval \((0, \delta']\). This is the content of the main Lemma 4.2.

From Lemma 4.2 and the other constructions in section 4 one gets some analogue of (1.6) involving different pairs of inverses \(v_1^{(\ell)}(u)\) and \(v_2^{(\ell)}(u)\), and this turns out to be enough to implement an essentially modified analytic part of Dolgopyat’s method. This is done in section 5.

\(^4\)The same idea was later used in [N] to deal with limit sets of convex co-compact hyperbolic surfaces.
Significant new development is necessary due to the unknown structure of $\Lambda$. What saves a lot of potential extra problems is the fact that we work with a new metric $D$ on $U$ (or rather on the subset $\tilde{U}$ of $U$) defined by means of cylinders. It turns out that this metric fits well with the (modified) Dolgopyat operators $N_f$ and makes them work in a truly multidimensional situation. Moreover, with this new metric there is no need for the Gibbs measure $\nu$ to have the so called Federer property (see [D2]). In fact, it is not clear at all whether $\nu$ has this property in the cases we consider.

## 2 Preliminaries

Throughout this paper $M$ denotes a $C^2$ complete (not necessarily compact) Riemann manifold, and $\phi_t : M \rightarrow M$ ($t \in \mathbb{R}$) a $C^2$ flow on $M$. A $\phi_t$-invariant closed subset $\Lambda$ of $M$ is called hyperbolic if $\Lambda$ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ such that there exists a $d\phi_t$-invariant decomposition $T_xM = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of $T_xM$ ($x \in \Lambda$) into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at $x$, $|d\phi_t(u)| \leq C \lambda^t ||u||$ for all $u \in E^s(x)$ and $t \geq 0$, and $||d\phi_t(u)|| \leq C \lambda^{-t} ||u||$ for all $u \in E^u(x)$ and $t \leq 0$.

The flow $\phi_t$ is called an Axiom $A$ flow on $M$ if the non-wandering set of $\phi_t$ is a disjoint union of a finite set consisting of fixed hyperbolic points and a compact hyperbolic subset containing no fixed points in which the periodic points are dense (see e.g. [KH]). A non-empty compact $\phi_t$-invariant hyperbolic subset $\Lambda$ of $M$ which is not a single closed orbit is called a basic set for $\phi_t$ if $\phi_t$ is transitive on $\Lambda$ and $\Lambda$ is locally maximal, i.e. there exists an open neighbourhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \phi_t(V)$. When $M$ is compact and $M$ itself is a basic set, $\phi_t$ is called an Anosov flow.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

\[ W^s_\epsilon(x) = \{ y \in M : d(\phi_t(x),\phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, \ 0 \rightarrow t \rightarrow -\infty \} , \]

\[ W^u_\epsilon(x) = \{ y \in M : d(\phi_t(x),\phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, \ 0 \rightarrow t \rightarrow -\infty \} \]

be the (strong) stable and unstable manifolds of size $\epsilon$. Then $E^u(x) = T_xW^u_\epsilon(x)$ and $E^s(x) = T_xW^s_\epsilon(x)$. Given $\delta > 0$, set $E^u(x; \delta) = \{ u \in E^u(x) : ||u|| \leq \delta \}$; $E^s(x; \delta)$ is defined similarly.

It follows from the hyperbolicity of $\Lambda$ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x,y) < \epsilon_1$, then $W^u_{\epsilon_0}(x)$ and $\phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ intersect at exactly one point $[x,y] \in \Lambda$ (cf. [KH]). That is, there exists a unique $t \in [-\epsilon_0,\epsilon_0]$ such that $\phi_t([x,y]) \in W^u_{\epsilon_0}(y)$. Setting $\Delta(x,y) = t$, defines the so called temporal distance function ([KB],[Ch1], [D2]) which will be used significantly throughout this paper (see Figure 1 on p. 32). For $x, y \in \Lambda$ with $d(x,y) < \epsilon_1$, define $\pi_y(x) = [x,y] = W^s_\epsilon(x) \cap \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$. Thus, for a fixed $y \in \Lambda$, $\pi_y : W \rightarrow \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ is the projection along local stable manifolds defined on a small open neighbourhood $W$ of $y$ in $\Lambda$. Choosing $\epsilon_1 \in (0,\epsilon_0)$ sufficiently small, the restriction $\pi_y : \phi_{[-\epsilon_1,\epsilon_1]}(W^u_{\epsilon_0}(x)) \rightarrow \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y))$ is called a local stable holonomy map. Combining such a map with a shift along the flow we get another local stable holonomy map $H^u_{\epsilon_1} : W^u_{\epsilon_1}(x) \cap \Lambda \rightarrow W^u_{\epsilon_0}(y) \cap \Lambda$. In a similar way one defines local holonomy maps along unstable laminations.

Given $z \in \Lambda$, let $\exp^u_z : E^u(z;\epsilon_0) \rightarrow W^u_{\epsilon_0}(z)$ and $\exp^s_z : E^s(z;\epsilon_0) \rightarrow W^s_{\epsilon_0}(z)$ be the corresponding exponential maps. A vector $\eta \in E^u(z) \setminus \{0\}$ will be called tangent to $\Lambda$ at $z$ if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp^u_z(t_m v^{(m)}) \in \Lambda \cap W^u_{\epsilon_0}(z)$ for

\[ n \in \mathbb{N} \]
all \( m, v^{(m)} \to \eta \) and \( t_m \to 0 \) as \( m \to \infty \). It is easy to see that a vector \( \eta \in E^u(z) \setminus \{0\} \) is tangent to \( \Lambda \) at \( z \) if there exists a \( C^1 \) curve \( z(t) \) \((0 \leq t \leq a)\) in \( W^u_\epsilon(z) \) for some \( a > 0 \) with \( z(0) = z \) and \( \dot{z}(0) = \eta \) such that \( z(t) \in \Lambda \) for arbitrarily small \( t > 0 \).

The following is the local non-integrability condition for \( \phi_t \) and \( \Lambda \) mentioned in section 1.

(LNIC): There exist \( z_0 \in \Lambda, \epsilon_0 > 0 \) and \( \theta_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0] \), any \( \tilde{z} \in \Lambda \cap W^u_\epsilon(z_0) \) and any tangent vector \( \eta \in E^u(\tilde{z}) \) to \( \Lambda \) at \( \tilde{z} \) with \( \|\eta\| = 1 \) there exist \( \tilde{z} \in \Lambda \cap W^u_\epsilon(\tilde{z}), \tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z}) \) with \( \tilde{y}_1 \neq \tilde{y}_2 \), \( \delta = \|\tilde{z} - \tilde{y}_1 - \tilde{y}_2\| > 0 \) and \( \epsilon' = \epsilon'(\tilde{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon] \) such that

\[
|\Delta(\exp^\epsilon_\eta(v), \pi_{\tilde{y}_1}(z)) - \Delta(\exp^\epsilon_\eta(v), \pi_{\tilde{y}_2}(z))| \geq \delta \|v\|
\]

for all \( z \in W^u_\epsilon(\tilde{z}) \cap \Lambda \) and \( v \in E^u(z; \epsilon') \) with \( \exp^\epsilon_\eta(v) \in \Lambda \) and \( \langle v, \pi_{\tilde{y}} \eta \rangle \geq \theta_0 \), where \( \eta \) is the parallel translate of \( \eta \) along the geodesic in \( W^u_\epsilon(z_0) \) from \( z \) to \( \tilde{z} \). (See Figure 2 on p. 33.)

It should be mentioned that if in (LNIC) one requires \( \tilde{y}_2 = \tilde{z} \), this would replace (2.1) by

\[
|\Delta(\exp^\epsilon_\eta(v), \pi_{\tilde{y}}(z))| \geq \delta \|v\|
\]

with \( \tilde{y} = \tilde{y}_1 \), which is still a rather general non-integrability condition. However in its present form (LNIC) is a substantially weaker condition. It is easy to see that the uniform non-integrability condition (UNI) of Chernov [Ch1] and Dolgopyat [D2] implies (LNIC)\(^6\).

We will say that \( A \) is an admissible subset of \( W^u_\epsilon(z) \cap \Lambda \) \((z \in \Lambda)\) if \( A \) coincides with the closure of its interior in \( W^u_\epsilon(z) \cap \Lambda \). Admissible subsets of \( W^u_\epsilon(z) \cap \Lambda \) are defined similarly. Following [D2], a subset \( R \) of \( \Lambda \) will be called a rectangle if it has the form \( R = [U, S] = \{[x, y] : x \in U, y \in S\} \), where \( U \) and \( S \) are admissible subsets of \( W^u_\epsilon(z) \cap \Lambda \) and \( W^s_\epsilon(z) \cap \Lambda \), respectively, for some \( z \in \Lambda \).

In what follows we will denote by \( \text{Int}^u(U) \) the interior of \( U \) in the set \( W^u_\epsilon(z) \cap \Lambda \). In a similar way we define \( \text{Int}^s(S) \), and then set \( \text{Int}(R) = \text{Int}^u(U), \text{Int}^s(S) \).

Given \( \xi = [x, y] \in R \), let \( W^u_\epsilon(R)(\xi) = [U, y] = \{[x', y'] : x' \in U \} \) and \( W^s_\epsilon(R)(\xi) = [x, S] = \{[x, y'] : y' \in S \} \subset W^u_\epsilon(x) \).

The interiors of these sets in the corresponding leaves are defined by \( \text{Int}^u(W^u_\epsilon(R)(\xi)) = \text{Int}^u(U), y \) and \( \text{Int}^s(W^u_\epsilon(R)(\xi)) = \text{Int}^s(S) \).

Let \( \mathcal{R} = \{R_i\}_{i=1}^k \) be a family of rectangles with \( R_i = [U_i, S_i], U_i \subset W^u_\epsilon(z_i) \cap \Lambda \) and \( S_i \subset W^s_\epsilon(z_i) \cap \Lambda \), respectively, for some \( z_i \in \Lambda \). Let \( \mathcal{R} \) be called complete if there exists \( T > 0 \) such that for every \( x \in \Lambda \), \( \phi_t(x) \in R \) for some \( t \in (0, T) \). Given a complete family \( \mathcal{R} \), the related Poincaré map \( \mathcal{P} : R \to R \) is defined by \( \mathcal{P}(x) = \phi_{\tau(x)}(x) \in R \), where \( \tau(x) > 0 \) is the smallest positive time with \( \phi_{\tau(x)}(x) \in R \). The function \( \tau \) is called the first return time associated with \( \mathcal{R} \). A complete family \( \mathcal{R} = \{R_i\}_{i=1}^k \) of rectangles in \( \Lambda \) is called a Markov family of size \( \chi > 0 \) for the flow \( \phi_t \) if \( \text{diam}(R_i) < \chi \) for all \( i \) and: (a) for any \( i \neq j \) and any \( x \in \text{Int}(R_i) \cap \partial^{-1}(\text{Int}(R_j)) \) we have \( \mathcal{P}(\text{Int}^u(W^u_\epsilon(R_i)(x))) \subset \text{Int}^u(W^u_\epsilon(\mathcal{P}(x))) \) and \( \mathcal{P}(\text{Int}^u(W^u_\epsilon(R_i)(x))) \supset \text{Int}^u(W^u_\epsilon(R_j)(\mathcal{P}(x))) \); (b) for any \( i \neq j \) at least one of the sets \( R_i \cap \phi_{[0,\chi]}(R_j) \) and \( R_j \cap \phi_{[0,\chi]}(R_i) \) is empty.

The existence of a Markov family \( \mathcal{R} \) of an arbitrarily small size \( \chi > 0 \) for \( \phi_t \) follows from the construction of Bowen [B] (cf. also Ratner [Ra]).

From now on we will assume that \( \mathcal{R} = \{R_i\}_{i=1}^k \) is a fixed Markov family for \( \phi_t \) of size \( \chi < \epsilon_0/2 < 1 \). Set \( U = \bigcup_{i=1}^k U_i \) and \( \text{Int}^u(U) = \bigcup_{i=1}^k \text{Int}^u(U_i) \). The shift map \( \sigma : U \to U \) is given by \( \sigma = \pi(U) \circ \mathcal{P}, \) where \( \pi(U) : R \to U \) is the projection along stable leaves. Notice that \( \tau \) is constant on each stable leaf \( W^s_\epsilon(R_i)(x) \). For any integer \( m \geq 1 \) and any function \( h : U \to \mathbb{C} \) define \( h_m : U \to \mathbb{C} \) by \( h_m(u) = h(u) + h(\sigma(u)) + \ldots + h(\sigma^{m-1}(u)) \).\(^7\)

Denote by \( \tilde{U} \) the core of \( U \), i.e. the subset of those \( x \in U \) such that \( \mathcal{P}^m(x) \in \text{Int}(R) = \bigcup_{i=1}^k \text{Int}(R_i) \) for all \( m \in \mathbb{Z} \). It is well-known (see [B]) that \( \tilde{U} \) is a residual subset of \( U \) and has full measure with respect to any Gibbs measure on \( U \). Clearly in general \( \tau \) is not continuous on \( U \), however \( \tau \) is essentially Lipschitz on \( U \) in the sense that there exists a constant \( L > 0 \) such that if

\(^6\)In fact, Chernov and Dolgopyat used (UNI) only for Anosov flows on 3-dimensional manifolds. It is quite clear that when \( \dim E^u(x) > 1 \) \((x \in \Lambda)\), (LNIC) is a much weaker condition than (UNI).

\(^7\)
Given an essentially Lipschitz function $g : \hat{U} \to C$ we will denote by $\text{Lip}^\circ(g)$ the smallest constant $L \geq 0$ such that $|g(x) - g(y)| \leq L \, d(x, y)$ for all $x, y \in \hat{U}$ with $x, y \in U_i \cap \sigma^{-1}(U_j)$ for some $i, j$.

Let $B(\hat{U})$ be the space of bounded functions $g : \hat{U} \to C$ with its standard norm $\|g\|_0 = \sup_{x \in \hat{U}} |g(x)|$. Given a function $g \in B(\hat{U})$, the Ruelle transfer operator $L_g : B(\hat{U}) \to B(\hat{U})$ is defined by $(L_g h)(u) = \sum_{\sigma(v) = u} e^{\theta(v)} h(v)$. If $g \in B(\hat{U})$ is essentially Lipschitz on $\hat{U}$, then $L_g$ preserves the space $\text{CLip}(\hat{U})$ of Lipschitz functions $g : \hat{U} \to C$.

In what follows we will assume that $f$ is a fixed real-valued function in $\text{CLip}(\hat{U})$. Let $P = P_f$ be the unique real number so that $\text{Pr}_\sigma(f - P \tau) = 0$, where $\text{Pr}_\sigma(h)$ is the topological pressure of $h$ with respect to the shift map $\sigma$ (see e.g. [PP]). Set $g = g_f = f - P \tau$.

By Ruelle-Perron-Frobenius’ Theorem (see e.g. chapter 2 in [PP]) for any real number $a$ with $|a|$ sufficiently small, as an operator on $\text{CLip}(\hat{U})$, $L_f(\cdot-P+a)\tau$ has a largest eigenvalue $\lambda_a$ and there exists a (unique) regular probability measure $\hat{\nu}_a$ on $\hat{U}$ with $L^*_f(\cdot-P+a)\tau \hat{\nu}_a = \lambda_a \hat{\nu}_a$, i.e. $\int L_f(\cdot-P+a)\tau H d\hat{\nu}_a = \lambda_a \int H d\hat{\nu}_a$ for any $H \in \text{C}(\hat{U})$. Fix a corresponding (positive) eigenfunction $h_a \in \text{CLip}(\hat{U})$ such that $\int h_a \, d\hat{\nu}_a = 1$. Then $d\nu = h_0 d\hat{\nu}_0$ defines a $\sigma$-invariant probability measure $\nu$ on $\hat{U}$. Since $\text{Pr}_\sigma(f - P \tau) = 0$, it follows from the main properties of pressure (cf. e.g. chapter 3 in [PP]) that $|\text{Pr}_\sigma(f - (P + a)\tau)| \leq |\tau|_0 |a|$. Moreover, for small $|a|$ the maximal eigenvalue $\lambda_a$ and the eigenfunction $h_a$ are Lipschitz in $a$, so there exist constants $a'_0 > 0$ and $C_0 > 0$ such that $|h_a - h_0| \leq C_0 |a|$ on $\hat{U}$ and $|\lambda_a - 1| \leq C_0 |a|$ for $|a| \leq a'_0$.

For $a, b \in \mathbb{R}$, $|a| \leq a'_0$, as in [D2], consider the function

$$f^{(a)}(u) = f(u) - (P + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a$$

and the operators

$$L_{ab} = L_{f^{(a)}-1\sigma^{tr}} : \text{CLip}(\hat{U}) \to \text{CLip}(\hat{U}) \quad , \quad \mathcal{M}_a = L_{f^{(a)}} : \text{CLip}(\hat{U}) \to \text{CLip}(\hat{U}) \ .$$

One checks that $\mathcal{M}_a 1 = 1$ and $|(L_{ab}^m h)(u)| \leq (\mathcal{M}_a^m |h|)(u)$ for all $u \in \hat{U}$, $h \in \text{CLip}(\hat{U})$ and $m \geq 0$. It is also easy to check that $L_{f^{(0)}} \nu = \nu$, i.e. $\int L_{f^{(0)}} H \, d\nu = \int H \, d\nu$ for any $H \in \text{CLip}(\hat{U})$.

The hyperbolicity of the flow on $\Lambda$ implies the existence of constants $c_0 \in (0, 1]$ and $\gamma_1 > \gamma > 1$ such that

$$c_0 \gamma^m \, d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma^m}{c_0} \, d(u_1, u_2)$$

whenever $\sigma^j(u_1)$ and $\sigma^j(u_2)$ belong to the same $U_{ij}$ for all $j = 0, 1, \ldots, m$.

Set $\tilde{\tau} = \max\{ |\tau|_0, \text{Lip}^\circ(\tau|_\hat{U}) \}$. Assuming that the constant $a'_0 > 0$ is sufficiently small, there exists $T = T(a'_0)$ such that

$$T \geq \max\{ \|f^{(a)}\|_0, \text{Lip}^\circ(f^{(a)}|_\hat{U}), \tilde{\tau} \}$$

for all $|a| \leq a'_0$. Fix $a'_0 > 0$ and $T > 0$ and with these properties. Taking the constant $C_0 > 0$ sufficiently large, we have $|f^{(a)} - f^{(0)}| \leq C_0 |a|$ on $\hat{U}$ for $|a| \leq a'_0$. From now on we will assume that $a'_0$, $c_0$, $C_0$, $T$, $\gamma$ and $\gamma_1$ are fixed constants with the above properties.
3 Some properties of cylinders

Let again $\mathcal{R} = \{R_i\}_{i=1}^k$ be a fixed Markov family as in section 2. Define the matrix $A = (A_{ij})_{i,j=1}^k$ by $A_{ij} = 1$ if $\mathcal{P}(\text{Int}(R_i)) \cap \text{Int}(R_j) \neq \emptyset$ and $A_{ij} = 0$ otherwise. According to [BR] (see section 2 there), we may assume that $\mathcal{R}$ is chosen in such a way that $A^{M_0} > 0$ (all entries of the $M_0$-fold product of $A$ by itself are positive) for some integer $M_0 > 0$. In what follows we assume that the matrix $A$ has this property.

Given a finite string $\iota = (i_0, i_1, \ldots, i_m)$ of integers $i_j \in \{1, \ldots, k\}$, we will say that $\iota$ is admissible if for any $j = 0, 1, \ldots, m - 1$ we have $A_{i_ji_{j+1}} = 1$. Given an admissible string $\iota$, denote by $\hat{\mathcal{C}}[\iota]$ the set of those $x \in U$ so that $\sigma^j(x) \in \text{Int}^u(U_{i_j})$ for all $j = 0, 1, \ldots, m$. The set $C[\iota] = \hat{\mathcal{C}}[\iota] \subset \Lambda$ will be called a cylinder of length $m$ in $U$, while $\hat{\mathcal{C}}[\iota]$ will be called an open cylinder of length $m$. It follows from the properties of the Markov family that $\hat{\mathcal{C}}[\iota]$ is an open dense subset of $C[\iota]$. Any cylinder of the form $C[\iota_0, i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+q}]$ will be called a subcylinder of $C[\iota]$ of co-length $q$.

In what follows the cylinders considered are always defined by finite admissible strings.

The $\sigma$-invariant probability measure $\nu$ on $U$ defined in section 2 is a Gibbs measure related to $g = f - P\tau$ (cf. [Si], [R2] or [PP]). It follows from Pr$_\tau(g) = 0$, and the properties of Gibbs measures that there exist constants $c_2 > c_1 > 0$ such that

$$c_1 \leq \frac{\nu(C[\iota])}{e^{g_m(y)}} \leq c_2$$

for any cylinder $C[\iota]$ of length $m$ in $U$ and any $y \in C[\iota]$. It is well-known (see [B]) that the core $\hat{U}$ of $U$ (see section 2) is a residual subset of $U$ and $\nu(\hat{U}) = 1$. Notice that for any cylinder $C[\iota]$ the set $\hat{\mathcal{C}}[\iota] = C[\iota] \cap \hat{U}$ is dense in $C[\iota]$ and $\nu(\hat{\mathcal{C}}[\iota]) = \nu(C[\iota])$.

Given $x \in U_i$ for some $i$ and $r > 0$ we will denote by $B_U(x, r)$ the set of all $y \in U_i$ with $d(x, y) < r$.

The proof of the following proposition is straightforward.

**Proposition 3.1.** If for some integer $m \geq 1$ the map $\sigma^m : \mathcal{C} \rightarrow \mathcal{C}'$ defines a homeomorphism between two open cylinders $\mathcal{C}$ and $\mathcal{C}'$ and $w : \mathcal{C}' \rightarrow \mathcal{C}$ is its inverse map, then $w(\mathcal{C}'')$ is an open subcylinder of $\mathcal{C}$ of co-length $q \geq 1$ for every open subcylinder $\mathcal{C}''$ of $\mathcal{C}'$ of co-length $q$.

Recall the constants $c_0 \in (0, 1)$ and $\gamma_1 > \gamma > 1$ from section 2, and fix an integer $p_1 \geq 1$ with

$$\rho_1 = \frac{1}{c_0 \gamma_1^{p_1}} < \min \left\{ \frac{\text{diam}(U_i)}{\text{diam}(U_j)} : i, j = 1, \ldots, k \right\}.$$  

Then clearly $\rho_0 < 1$. Set $\rho_1 = \rho_0^{1/p_1}$ and fix a constant $r_0 > 0$ with $2r_0 < \min\{\text{diam}(U_i) : i = 1, \ldots, k\}$ and for each $i = 1, \ldots, k$ a point $\hat{z}_i \in \hat{U}_i$ such that $B_U(\hat{z}_i, r_0) \subset \text{Int}^u(U_i)$.

The following is an immediate consequence of (2.1).

**Lemma 3.2.** There exists a global constant $C_1 > 0$ such that for any cylinder $C[\iota]$ of length $m$ we have $\text{diam}(C[\iota]) \leq C_1 \rho_1^m$ and $\text{diam}(C[\iota]) \geq \frac{C_0 c_0}{\gamma_1^m}.$

From now on we will assume that the local stable holonomy maps through $\Lambda$ are uniformly Lipschitz. Then there exists a constant $K_1' > 0$ such that $d(\mathcal{H}_x^u(z), \mathcal{H}_x^u(z')) \leq K'_1 d(z, z')$ for all $x, y \in \Lambda$ with $d(x, y) < \epsilon_1$ and $z, z' \in \Lambda \cap W^s_{\epsilon_1}(x)$. 

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Given $i = 1, \ldots, k$, according to the choice of the Markov family $\{R_i\}$, the projection $\operatorname{pr}_{R_i'}: W_i = \phi_{[-\chi,\chi]}(R_i) \to R_i$ along the flow $\phi_t$ is well-defined and Lipschitz. Since the projection $\pi_i: R_i \to U_i$ along stable leaves is Lipschitz, the map $\psi_i = \pi_i \circ \operatorname{pr}_{R_i'}: W_i \to W_{R_i'}(z_i)$ is also Lipschitz. Thus, we may assume the constant $K' > 0$ is chosen sufficiently large so that $d(\psi_i(u), \psi_i(v)) \leq K'd(u,v)$ for all $u, v \in W_i$ and all $i = 1, \ldots, k$.

The following lemma describes the main consequences of the flow having a regular distortion along unstable manifolds that will be used in sections 4 and 5.

Proposition 3.3. Assume that $\phi_t$ has a regular distortion along unstable manifolds over the basic set $\Lambda$ and that the local stable holonomy maps through $\Lambda$ are uniformly Lipschitz. Then there exist global constants $0 < \rho < 1$ and $C_1 > 0$ and a positive integer $p_0 \geq 1$ such that:

(a) For any cylinder $C[i] = C[i_0, \ldots, i_m]$ and any subcylinder $C[i'] = C[i_0, i_1, \ldots, i_{m+1}]$ of $C[i]$ of co-length 1 we have $\rho \operatorname{diam}(C[i]) \leq \operatorname{diam}(C[i'])$.

(b) For any cylinder $C[i] = C[i_0, \ldots, i_m]$ and any subcylinder $C[i'] = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+p_0}]$ of $C[i]$ of co-length $p_0$ we have $\operatorname{diam}(C[i']) \leq \rho \operatorname{diam}(C[i])$.

Proof of Proposition 3.3. Notice that the properties of the Markov family (and the fact that $i$ is admissible) imply $\sigma^m(C[i_0, \ldots, i_m]) = \hat{U}_{i_m}$.

(a) Set $z = \hat{z}_{i_{m+1}}$ for brevity, and let $x \in \tilde{C}[i']$ be the point such that $\sigma^{m+1}(x) = z$. Set $r'_0 = c_0r_0/\sigma_1 > 0$. Let $R = R(r'_0/K', \epsilon_0) > 0$ be the constant from the definition of regular distortion along unstable manifolds with $\delta = r'_0/K'$ and $\epsilon = \epsilon_0$ in (1.1). Since $B_U(z, r_0) \subset \operatorname{Int}^u(U_{i_{m+1}})$, it follows from the properties of $\sigma$ that $B_U(\sigma^m(x), r_0') \subset \sigma^{-1}(U_{i_{m+1}})$. Thus, for $T = \tau_m(x)$ this implies $\Lambda \cap B^\sigma_T(x, r'_0/K') \subset C[i']$, so

$$\operatorname{diam}(C[i']) \geq \operatorname{diam}(\Lambda \cap B^\sigma_T(x, r'_0/K')) \geq \frac{1}{R} \operatorname{diam}(\Lambda \cap B^\sigma_T(x, \epsilon_0)).$$

On the other hand, $C[i] \subset \Lambda \cap B^\sigma_T(x, \epsilon_0)$. Indeed, if $y \in C[i]$, then $P^m(y) \in R_{i_m}$ and $P^m(y) = \phi_T(\phi_T(y))$ for some $|t| \leq \chi < \epsilon_0/2$, where $\phi_T(y) \in W^u_p(\phi_T(x))$. Since $\operatorname{diam}(R_{i_m}) \leq \chi < \epsilon_0/2$ and $\phi_T(x) = P^m(y)$, we get $d(\phi_T(y), \phi_T(y)) \leq d(P^m(x), P^m(y)) + d(P^m(y), \phi_T(y)) \leq 2\chi < \epsilon_0$. Thus, setting $\rho = 1/R$, we get $\rho \operatorname{diam}(C[i]) \leq \operatorname{diam}(C[i'])$.

(b) Choose an arbitrary $\rho \in (0, 1)$ (e.g. take the one from part (a) above), and set $\epsilon = r'_0/K'$. It follows from the condition (b) in the definition of regular distortion along unstable manifolds that there exists $\delta \in (0, r'_0/K')$ such that $\operatorname{diam}(\Lambda \cap B^\sigma_T(x, \delta)) \leq \rho \operatorname{diam}(\Lambda \cap B^\sigma_T(x, r'_0/K'))$ for $x \in \Lambda$ and $T \geq 0$. Choose the integer $p_0 \geq 1$ so that $C_1 K' \rho_1^{p_0} < \delta$.

Let $C[i] = C[i_0, i_1, \ldots, i_m]$ be an arbitrary cylinder and let $C[i'] = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+p_0}]$ be a subcylinder of $C[i]$ of co-length $p_0$. Let $x \in \tilde{C}[i']$ be the point such that $\sigma^{m+p_0}(x) = \hat{z}_{i_{m+p_0}}$ and let $T = \tau_m(x)$. For the cylinder $C' = C[i_m, i_{m+1}, \ldots, i_{m+p_0}] \subset U_{i_m}$, it follows from Lemma 3.2 that $\operatorname{diam}(C') \leq C_1 \rho_1^{p_0}$, so by the choice of $p_0$, $\operatorname{diam}(C') < \delta/K'$ and therefore $C' \subset B_U(\sigma(x), \delta/K')$. Next, we have $C[i] \subset \Lambda \cap B^\sigma_T(x, \delta)$. Indeed, if $y \in C[i]$, then $\sigma^m(y) \in C'$, so $d(\sigma^m(x), \sigma^m(y)) < \delta/K'$. For $T = \tau_m(x)$ we have $\phi_T(y) \in W^u_p(\phi_T(x))$, so $d(\phi_T(x), \phi_T(y)) = d(\mathcal{H}_{\sigma^m(x)}(\sigma^m(y)), \mathcal{H}_{\sigma^m(x)}(\sigma^m(y))) \leq K'd(\sigma^m(x), \sigma^m(y)) < \delta$. Thus, $C[i'] \subset \Lambda \cap B^\sigma_T(x, \delta)$ and therefore $\operatorname{diam}(C[i']) \leq \operatorname{diam}(\Lambda \cap B^\sigma_T(x, \delta)) \leq \rho \operatorname{diam}(\Lambda \cap B^\sigma_T(x, r'_0/K'))$. On the other hand, $B_U(\sigma^m(x), r_0) \subset \operatorname{Int}^u(U_{i_m})$ gives $\Lambda \cap B^\sigma_T(x, r_0/K') \subset C[i]$, so $\operatorname{diam}(C[i']) \leq \rho \operatorname{diam}(C[i])$.

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7Since $r'_0/K' < r_0$, for any $y \in \Lambda \cap B^\sigma_T(x, r'_0/K')$ one derives that $\sigma^j(y) \in U_{i_j}$ for all $j = 0, 1, \ldots, m + 1$. 
4 The temporal distance function

Throughout we assume that ϕt is a C^2 Axiom A flow on M and Λ is a basic set for ϕt satisfying the condition (LNIC) stated in section 2 and such that the local holonomy maps along stable laminations through Λ are uniformly Lipschitz.

Fix an arbitrary point z_0 ∈ Λ and constants ε_0 > 0 and θ_0 ∈ (0, 1) with the properties described in (LNIC). Following [B] and [BR], we can choose the Markov family \( R = \{ R_i \}_{i=1}^k \) from section 2 so that z_0 ∈ Int(\( R_1 \)) and \( A^{M_0} > 0 \) for some integer \( M_0 > 0 \), where A is the matrix defined by R (see the beginning of section 3). We then choose \( U_1 \) and \( S_1 \) so that \( U_1 \subset Λ \cap W_z^u(z_0) \) and \( S_1 \subset Λ \cap W_z^u(z_0) \). This implies \( z_0 \in \text{Int}(U_1) \). From now on we will assume that the Markov family \( \mathcal{R} = \{ R_i \}_{i=1}^k \) from section 2 is chosen in this way.

Fix an arbitrary constant \( θ_1 \) such that

\[ 0 < θ_0 < θ_1 < 1. \]

Next, fix an arbitrary orthonormal basis \( e_1, \ldots, e_n \) in \( E^u(z_0) \) and a C^1 parametrization \( r(s) = \exp_{z_0}(s) \), \( s \in V'_0 \), of a small neighbourhood \( W_0 \) of \( z_0 \) in \( W^u_{z_0}(z_0) \) such that \( V'_0 \) is a convex compact neighbourhood of 0 in \( \mathbb{R}^n \approx \text{span}(e_1, \ldots, e_n) \). Then \( r(0) = z_0 \) and \( \frac{∂}{∂t} r(s)|_{s=0} = e_i \) for all \( i = 1, \ldots, n \). Set \( U'_0 = W_0 \cap Λ \). Shrinking \( W_0 \) (and therefore \( V'_0 \) as well) if necessary, we may assume that \( U'_0 \subset \text{Int}(U_1) \) and \( \left| \left( \frac{∂r}{∂x_i}(s), \frac{∂r}{∂y_j}(s) \right) - δ_{ij} \right| \) is uniformly small for all \( i, j = 1, \ldots, n \) and \( s \in V'_0 \), so that

\begin{align*}
(4.1) & \quad \frac{1}{2} \langle ξ, η \rangle ≤ \langle dr(s) \cdot ξ, dr(s) \cdot η \rangle ≤ 2 \langle ξ, η \rangle , \quad ξ, η \in E^u(z_0) , \quad s \in V'_0 , \\
(4.2) & \quad \frac{1}{2} \| s - s' \| ≤ d(r(s), r(s')) ≤ 2 \| s - s' \| , \quad s, s' \in V'_0 .
\end{align*}

In what follows we will construct, amongst other things, a sequence of unit vectors \( η_1, η_2, \ldots, η_0 \in E^u(z_0) \). For each \( ℓ = 1, \ldots, ℓ_0 \) set \( B_ℓ = \{ η \in S^{n-1} : \langle η, η_ℓ \rangle ≥ θ_0 \} \). For \( t \in \mathbb{R} \) and \( s \in E^u(z_0) \) set \( (l_{η, t}) \cdot g(s) = \frac{g(s + t) - g(s)}{t} \), \( t ≠ 0 \) (increment of g in the direction of η).

Definitions 4.1. (a) For a cylinder \( C \subset U'_0 \) and a unit vector \( η \in E^u(z_0) \) we will say that a separation by an \( η \)-plane occurs in \( C \) if there exist \( u, v \in C \) with \( d(u, v) ≥ \frac{1}{2} \text{diam}(C) \) such that

\[ \left( \frac{r^{-1}(u) - r^{-1}(v)}{∥r^{-1}(u) - r^{-1}(v)∥}, η \right) ≥ θ_1 . \]

Let \( S_η \) be the family of all cylinders \( C \) contained in \( U'_0 \) such that a separation by an \( η \)-plane occurs in \( C \).

(b) Given an open subset \( V \) of \( U'_0 \) which is a finite union of open cylinders and \( δ > 0 \), let \( ℓ_1, \ldots, ℓ_p \) (\( p = p(δ, V) ≥ 1 \)) be the family of maximal closed cylinders in \( \overline{V} \) with \( \text{diam}(C_m) ≤ δ \). For any unit vector \( η \in E^u(z_0) \) set \( M_η(δ)(V) = \bigcup\{ C_m : C_m ∈ S_η, 1 ≤ m ≤ p \} \).

Our aim in this section is to prove the following:

Lemma 4.2. (Main Lemma) There exist integers \( 1 ≤ n_1 ≤ N_0 \) and \( ℓ_0 ≥ 1 \), a sequence of unit vectors \( η_1, η_2, \ldots, η_0 \in E^u(z_0) \) and a non-empty open subset \( U_0 \) of \( U'_0 \) which is a finite union of open cylinders of length \( n_1 \) such that \( U = a^{n_1}(U_0) \) is dense in \( U \) and we have:

(a) For any integer \( N ≥ N_0 \) there exist Lipschitz maps \( v_1^{(ℓ)}, v_2^{(ℓ)} : U → U \ (ℓ = 1, \ldots, ℓ_0) \) such that \( a^N(v_i^{(ℓ)}(x)) = x \) for all \( x ∈ U \) and \( v_i^{(ℓ)}(U) \) is a finite union of open cylinders of length \( N \) (\( i = 1, 2; ℓ = 1, 2, \ldots, ℓ_0 \)).
(b) There exists a constant $\delta > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \delta$ and $\eta \in B_{\ell}$ with $s + h \eta \in r^{-1}(U_0 \cap \Lambda)$ we have

$$\left[I_{\eta, h} \left(\tau_N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau_N(v_1^{(\ell)}(\tilde{r}(\cdot)))\right)\right](s) \geq \frac{\delta}{2},$$

where $\tilde{r}(s) = \sigma^{m_1}(r(s))$.

(c) We have $v_i^{(\ell)}(U) \cap v_i^{(\ell')}(U) = \emptyset$ whenever $(i, \ell) \neq (i', \ell')$.

(d) For any open cylinder $V$ in $U_0$ there exists a constant $\delta' = \delta'(V) > 0$ such that

$$V \subset M_{n_1}^{(\delta)}(V) \cup M_{n_2}^{(\delta)}(V) \cup \ldots \cup M_{n_0}^{(\delta)}(V)$$

for all $\delta \in (0, \delta']$.

Remark. Notice that if $y' \in W_{\epsilon}^s(\tilde{z}) \cap \Lambda$ is sufficiently close to $y$, then for any $z, z' \in \Lambda \cap W_{\epsilon}^s(\tilde{z})$ we have

$$|\Delta(z', \pi_{y'}(z)) - \Delta(z', \pi_y(z))| < \delta d(z, z').$$

We now proceed with the main step in the proof of the Main Lemma 4.2. This is where the non-integrability condition (LNIC) is used.

**Lemma 4.3.** For any $z \in \Lambda$ and any $\delta > 0$ there exists $\epsilon > 0$ such that for any $y \in W_{\epsilon}^s(z) \cap \Lambda$, if $y' \in W_{\epsilon}^s(z) \cap \Lambda$ is sufficiently close to $y$, then for any $z, z' \in \Lambda \cap W_{\epsilon}^s(z)$ we have

$$|\Delta(z', \pi_y(z)) - \Delta(z', \pi_y(z))| < \delta d(z, z').$$

Proof of Lemma 4.4. Clearly if $x, y \in S^{n-1}$ are such that $\langle x, y \rangle \leq \theta_1$, then $\|x - y\| \geq \sqrt{2(1 - \theta_1)}$. Thus there exists a positive integer $\ell_0$, depending on $n$ and $\theta_1$ only, such that for any finite set $\{x_1, \ldots, x_k\} \subset S^{n-1}$ with $\langle x_i, x_j \rangle \leq \theta_1$ for all $i \neq j$ we have $k \leq \ell_0$. Fix $\ell_0$ with this property.
As another preparatory step, fix $2\ell'_0$ distinct points $x^{(\ell)}_i \in \text{Int}^u(U_1) \setminus \overline{U}_0^\ell$ ($\ell = 1, \ldots, \ell'_0$; $i = 1, 2$) and for each $x^{(\ell)}_i$ **fix a small open neighbourhood** $\tilde{V}^{(\ell)}_i$ of $x^{(\ell)}_i$ in $\text{Int}^u(U_1)$ such that the sets $\tilde{V}^{(\ell)}_i$ ($\ell = 1, \ldots, \ell'_0$; $i = 1, 2$) are disjoint and contained in $\text{Int}^u(U_1)$.

We will construct the required objects by induction.

**Step 1.** Clearly there exists a unit vector $\eta_1 \in E^u(z_0)$ tangent to $\Lambda$ at $z_0$. It then follows from the condition (LNIC) and the choice of $z_0$ that there exist $\tilde{z} = r(\tilde{s}) \in U_0', \tilde{y}_1, \tilde{y}_2 \in W^s_{R_1}(\tilde{z})$ (so $\tilde{y}_1, \tilde{y}_2 \in \Lambda$) with $\tilde{y}_1 \neq \tilde{y}_2$, $\delta_1' > 0$ and $\epsilon_1' > 0$ such that

$$|\Delta(r(s + h \eta), \pi_{\tilde{y}_1}(r(s))) - \Delta(r(s + h \eta), \pi_{\tilde{y}_2}(r(s)))| \geq \delta_1' |h|$$

for all $r(s) \in U'_0$ with $\text{dist}(\tilde{z}, r(s)) < \epsilon_1'$, $\eta \in B_1$ and $h \in \mathbb{R}$ with $|h| < \epsilon_1'$ and $r(s + h \eta) \in U_0'$. We will assume that $\epsilon_1' > 0$ is so small that $B_U(\tilde{z}, \epsilon_1') \subset U_0'$.

Since $\tilde{V}^{(1)}_1$ and $\tilde{V}^{(1)}_2$ are open subsets of $U$ having common points with $\Lambda$, it follows that $\mathcal{P}^m(\tilde{V}^{(1)}_1)$ and $\mathcal{P}^m(\tilde{V}^{(1)}_2)$ fill in $R_1$ densely as $m \to \infty$. Using this, it follows that taking $m_1 \geq 1$ large enough we can find $y'_1 \in W^s_{R_1}(\tilde{z}) \cap \mathcal{P}^{m_1}(\tilde{V}^{(1)}_1)$ arbitrarily close to $\tilde{y}_1$ and $y'_2 \in W^s_{R_1}(\tilde{z}) \cap \mathcal{P}^{m_1}(\tilde{V}^{(1)}_2)$ arbitrarily close to $\tilde{y}_2$. By Lemma 4.3 we can make this choice so that for any $i = 1, 2$ and any $z = r(s), h$ and $\eta$ as above, we have

$$|\Delta(r(s + h \eta), \pi_{y'_1}(r(s))) - \Delta(r(s + h \eta), \pi_{y'_2}(r(s)))| \leq \frac{\delta_1' |h|}{4}.$$ 

Combining this with (4.3) one gets

$$|\Delta(r(s + h \eta), \pi_{y'_1}(r(s))) - \Delta(r(s + h \eta), \pi_{y'_2}(r(s)))| \geq \frac{\delta_1' |h|}{2}.$$ 

Thus, there exists an open cylinder $U_0^{(1)}$ contained in $B_U(\tilde{z}, \epsilon_1') \subset U_0'$ with $\tilde{z} \in U_0^{(1)}$ such that

$$|\Delta(r(s + h \eta), \pi_{y'_1}(r(s))) - \Delta(r(s + h \eta), \pi_{y'_2}(r(s)))| \geq \delta_1 |h|$$

whenever $r(s) \in U_0^{(1)}$, $\eta \in B_1$, $|h| \leq \delta_1$ and $r(s + h \eta) \in U_0^{(1)}$, where $\delta_1 = \min\{\delta_1'/2, \epsilon_1'\}$. Fix $m_1 \geq 1$, $y'_1$ and $y'_2$ with these properties.

Let $\mathcal{O}_1^{(1)}$ be a small open neighbourhood of $y'_1$ in $W^u_{R_1}(y'_1) \cap \mathcal{P}^{m_1}(\tilde{V}^{(1)}_1)$ and let $f_1^{(1)} : \mathcal{O}_1^{(1)} \to f_1^{(1)}(\mathcal{O}_1^{(1)}) \subset \tilde{V}^{(1)}_1$ be a Lipschitz homeomorphism (local inverse of $\mathcal{P}^{m_1}$) such that $\mathcal{P}^{m_1}(f_1^{(1)}(z)) = z$ for all $z \in \mathcal{O}_1^{(1)}$. Shrinking $U_0^{(1)}$ if necessary, we may assume that $\pi_{y'_1}(U_0^{(1)}) \subset \mathcal{O}_1^{(1)}$. Now define a Lipschitz homeomorphism $w_1^{(1)} : U_0^{(1)} \to w_1^{(1)}(U_0^{(1)}) \subset \tilde{V}^{(1)}_1$ by $w_1^{(1)}(x) = f_1^{(1)}(\pi_{y'_1}(x))$. We then have $\mathcal{P}^{m_1}(w_1^{(1)}(x)) = \pi_{y'_1}(x)$ and therefore $\sigma^{m_1}(w_1^{(1)}(x)) = x$ for all $x \in U_0^{(1)}$. Moreover, $\text{Lip}(w_1^{(1)}) \leq \frac{1}{\alpha_0 \gamma_{m_1}}$, so assuming $m_1$ is sufficiently large, $w_1^{(1)}$ is contracting and $w_1^{(1)}(U_0^{(1)})$ is a cylinder.

In the same way one constructs a Lipschitz homeomorphism $w_2^{(1)} : U_0^{(1)} \to w_2^{(1)}(U_0^{(1)}) \subset \tilde{V}^{(1)}_2$ (replacing $U_0^{(1)}$ by a smaller open cylinder if necessary; by Proposition 3.1, $w_1^{(1)}(U_0^{(1)})$ will continue to be a cylinder) such that $w_2^{(1)}(U_0^{(1)})$ is a cylinder and $\mathcal{P}^{m_1}(w_2^{(1)}(x)) = \pi_{y'_2}(x)$ for all $x \in U_0^{(1)}$. Then $\sigma^{m_1}(w_2^{(1)}(x)) = x$ for all $x \in U_0^{(1)}$. 

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Now for $z = r(s) \in U_0^{(1)}$, $\eta \in B_1$ and $h \in \mathbb{R}$ with $r(s + h \eta) \in U_0^{(1)}$ we get

$$\left[ \tau_m(w_1^{(1)}(r(s + h \eta))) - \tau_m(w_1^{(1)}(r(s))) \right] - \left[ \tau_m(w_1^{(1)}(r(s + h \eta))) - \tau_m(w_1^{(1)}(r(s))) \right]$$

$$\begin{align*}
&= \left[ \tau_m(w_1^{(1)}(r(s))) - \tau_m(w_1^{(1)}(r(s + h \eta))) \right] - \left[ \tau_m(w_1^{(1)}(r(s))) - \tau_m(w_1^{(1)}(r(s + h \eta))) \right] \\
&= \Delta(P_m^{(1)}(w_1^{(1)}(r(s + h \eta))), P_m^{(1)}(w_1^{(1)}(r(s)))) - \Delta(P_m^{(1)}(w_1^{(1)}(r(s + h \eta))), P_m^{(1)}(w_1^{(1)}(r(s)))) \\
&= \Delta(\pi^{(1)}_y(r(s + h \eta)), \pi^{(1)}_y(r(s))) - \Delta(\pi^{(1)}_y(r(s + h \eta)), \pi^{(1)}_y(r(s))) \\
&= \Delta(r(s + h \eta), \pi^{(1)}_y(r(s))) - \Delta(r(s + h \eta), \pi^{(1)}_y(r(s))).
\end{align*}$$

This and (4.4) give $||I_{\eta,h}(\tau_m(w_1^{(1)}(r(\cdot))) - \tau_m(w_1^{(1)}(r(\cdot))))(s)|| \geq \delta_1$ whenever $r(s) \in U_0^{(1)}$, $\eta \in B_1$, $0 < |h| \leq \epsilon_1$ and $r(s + h \eta) \in U_0^{(1)}$.

In this way we have completed the first step in our recursive construction. Whether we need to make more steps or not depends on which of the following two alternatives takes place.

**Alternative 1.A.** There exist an open cylinder $V$ contained in $U_0^{(1)}$ and a constant $\delta' \in (0, \delta_0)$ such that $M^{(1)}_{\eta_1}(V) \supset V$ for all $\delta \in (0, \delta']$.

**Alternative 1.B.** Alternative 1.A does not hold.

In the case of Alternative 1.A we simply terminate the recursive construction at this stage replacing $U_0^{(1)}$ by $V$.

If Alternative 1.B takes place, we need to make at least one more step.

**Inductive Step.** Suppose that for some $j \geq 1$ we have constructed open cylinders $U_0^{(j)} \subset \ldots \subset U_0^{(1)}$ contained in $U_0'$, and for each $\ell = 1, \ldots, j$, an integer $m_\ell \geq 1$ and a vector $\eta_\ell \in \mathbb{S}^{n-1}$ such that the conditions (i) and (ii) in the lemma are fulfilled with $\ell_0$ replaced by $j$.

There are two alternatives again.

**Alternative j.A.** There exist an open cylinder $V$ contained in $U_0^{(j)}$ and $\delta' \in (0, \delta_0)$ such that $M^{(j)}_{\eta_j}(V) \supset V$, $\delta \in (0, \delta']$.

**Alternative j.B.** Alternative j.A does not hold.

In the case of Alternative j.A we terminate the recursive construction at this stage replacing $U_0^{(j)}$ by $V$.

Next, assume that Alternative j.B takes place. One then needs to complete

**Step j+1.** Construct an open cylinder $U_0^{(j+1)}$ contained in $U_0^{(j)}$, an integer $m_{j+1} \geq 1$ and a unit vector $\eta_{j+1} \in E^n(z_0)$ such that the conditions (i) and (ii) in the lemma are fulfilled with $\ell_0$ replaced by $j + 1$.

Given an open cylinder $V$ in $U_0^{(j)}$ and $\delta > 0$, set $A_{\delta}(V) = M^{(j)}_{\eta_j}(V) \cup M^{(j+1)}_{\eta_{j+1}}(V) \cup \ldots \cup M^{(j)}_{\eta_j}(V)$. It follows from Alternative j.B that for any open cylinder $V$ contained in $U_0^{(j)}$ we have $V \setminus A_{\delta}(V) \neq \emptyset$ for arbitrarily small $\delta$.

Notice that, since there are only countably many cylinders in $U_0^{(j)}$, there exists a decreasing sequence $\delta_j = \mu_0 > \mu_1 > \mu_2 > \ldots > \mu_k > \ldots$ converging to zero such that for any cylinder $C$ in $U_0^{(j)}$ we have $\text{diam}(C) = \mu_k$ for some $k$. Then for any open cylinder $V \subset U_0^{(j)}$ and any $m \geq 1$ we have $A_{\delta}(V) = A_{\mu_m}(V)$ whenever $\mu_m \leq \delta < \mu_{m-1}$.

Let $V = U_0^{(j)}$ and for any $m \geq 1$ consider the compact subset $F_m = \bigcap_{k \geq m} A_{\mu_k}(V)$ of $\overline{V}$. Clearly $F_m' \subset F_m$ whenever $m' < m$. We claim that $\text{Int}^u(F_m) = \emptyset$ for all $m \geq 1$. Indeed, assume
that $\text{Int}^u(F_{m'}) \neq \emptyset$ for some $m' \geq 1$; then there exists a non-empty open cylinder $W$ contained in $F_{m'}$. There exists $k' \geq 1$ with $\mu_{k'} = \text{diam}(W)$. Setting $m = \max\{m', k'\} + 1$, we have $W \subset F_m$. Moreover, for $0 < \delta \leq \mu_m$ we have

\[ M_{\delta}^{(j)}(V) \cap W \subset M_{\delta}^{(j)}(W) , \quad i = 1, \ldots, j . \]

Consequently, for all $0 < \delta \leq \mu_m$ we have $A_i(V) \cap W \subset A_i(W)$. Now $W \subset F_m$ implies $W = F_m \cap W = \bigcap_{k \geq m} A_k(V) \cap W \subset \bigcap_{k \geq m} A_k(W)$ . Thus, $W \subset A_k(W)$ for all $k \geq m$, which means that $W \subset A_\delta(W)$ for all $0 < \delta \leq \mu_m$. This is a contradiction with Alternative j.B.

Hence $\text{Int}^u(F_m) = \emptyset$ for all $m \geq 1$. Thus, $U_0^{(j)} \setminus F_m$ are open and dense subsets of $U_0^{(j)}$, so $G = \cap_{m=1}^\infty (U_0^{(j)} \setminus F_m)$ is a residual (even a $G_\delta$) subset of $U_0^{(j)}$ . The properties of $\hat{U}$ now imply $G \cap \hat{U} \neq \emptyset$.

Choose an arbitrary $\hat{z} \in G \cap \hat{U}$ and fix it. Given any $m \geq 1$, we have $\hat{z} \notin F_m$, so there exists $\mu'_m \in (0, \mu_m]$ with $\hat{z} \notin A_{\mu'_m}(V)$, i.e. $\hat{z} \notin \bigcup_{\ell=1}^j M_{\mu'_m}(V)$. So, if $C^{(m)}$ is the maximal cylinder in $V$ with $\text{diam}(C^{(m)}) \leq \mu'_m$ such that $\hat{z} \in C^{(m)}$, then $C^{(m)} \notin S_m$ for any $\ell = 1, \ldots, j$.

Fix $m \geq 1$ for a moment, and let $u_m, v_m \in C^{(m)}$ be such that $d(u_m, v_m) = \text{diam}(C^{(m)})$. Since $\hat{z} \in C^{(m)}$, we may assume $d(u_m, \hat{z}) \geq \frac{1}{2} \text{diam}(C^{(m)})$. Then $\langle \frac{r^{-1}(u_m) - r^{-1}(\hat{z})}{\|r^{-1}(u_m) - r^{-1}(\hat{z})\|}, \eta_\ell \rangle < \theta_1$ for all $\ell = 1, \ldots, j$, since $C^{(m)} \notin S_m$. Clearly, $u_m \to \hat{z}$ as $m \to \infty$. Choose a subsequence $\{u_{n_p}\}$ so that $\eta_{j+1} = \lim_{p \to \infty} \frac{r^{-1}(u_{n_p}) - r^{-1}(\hat{z})}{\|r^{-1}(u_{n_p}) - r^{-1}(\hat{z})\|} \in S^{n-1}$ exists. Let $\hat{z} = r(\hat{\hat{z}})$; then $\eta_{j+1}' = dr(\hat{\hat{z}}) \cdot \eta_{j+1}$ is tangent to $\Lambda$ at $\hat{z}$, and according to the above, $\langle \eta_{j+1}, \eta_{j+1} \rangle \leq \theta_1$ for all $\ell = 1, \ldots, j$.

Repeating the argument from the proof of (4.4) in Step 1, one derives that there exist $\hat{z} = r(\hat{\hat{z}}) \in V = U_0^{(j)}, \hat{y}_1, \hat{y}_2 \in W_{R_1}(\hat{z}) \subset U_1$ with $\hat{y}_1 \neq \hat{y}_2$, $\delta_{j+1}' \in (0, \delta_j)$ and $\ell_{j+1}' > 0$ such that

$$\Delta(r(s + h \eta), \pi_{\hat{y}_1}(r(s))) - \Delta(r(s + h \eta), \pi_{\hat{y}_2}(r(s))) \geq \delta_{j+1}' |h|$$

for all $r(s) \in V$ with $d(\hat{z}, r(s)) < \delta_{j+1}'$, $\eta \in B_{j+1}$ and $h \in \mathbb{R}$ with $|h| < \delta_{j+1}'$ and $r(s + h\eta) \in U_0^{(j)}$. We will assume that $\ell_{j+1}' > 0$ is so small that $B_U(\hat{z}, \delta_{j+1}') \subset V = U_0^{(j)}$.

Then, again as in Step 1, one constructs an open cylinder $U_0^{(j+1)} \subset U_0^{(j)}$ and contracting homeomorphisms $w_i^{(j+1)} : U_0^{(j+1)} \to w_i^{(j+1)}(U_0^{(j+1)}) \subset \hat{V}_i^{(j+1)} (i = 1, 2)$ with $\sigma^{m_{j+1}}(w_i^{(j+1)}(x)) = x$ for all $x \in U_0^{(j+1)}$ and such that

$$\left| [I_{n, \eta}(\tau_{m_{j+1}}(w_2^{(j+1)}(r(\cdot)))) - \tau_{m_{j+1}}(w_1^{(j+1)}(r(\cdot))))] ](s) \right| \geq \delta_{j+1}' \frac{\delta_{j+1}'}{2}$$

for all $r(s) \in U_0^{(j+1)}$, $\eta \in B_{j+1}$, $0 < |h| \leq \delta_{j+1}'$ and $r(s + h\eta) \in U_0^{(j+1)}$. This completes Step $j+1$.

By the definition of $\ell_0'$, it is clear that this inductive procedure terminates after not more than $\ell_0'$ steps. That is, for some $\ell_0 \leq \ell_0'$ the Alternative $\ell_0.A$ holds, and then we terminate the construction at that step. }

In what follows we use the objects constructed in Lemma 4.4. Set $\delta = \min_{1 \leq \ell \leq \ell_0'} \delta_{j}$, $n_0 = \max_{1 \leq \ell \leq \ell_0'} m_{\ell}$, and fix an arbitrary point $\hat{z}_0 \in U_0^{(\ell_0)} \cap \hat{U}$.

**Lemma 4.5.** There exist an integer $n_1 \geq 1$ and an open neighbourhood $U_0$ of $\hat{z}_0$ in $U_0^{(\ell_0)}$ such that $\text{Int}^u(U) = \sigma^{n_1}(U_0)$, $\sigma^{n_1} : U_0 \to \sigma^{n_1}(U_0)$ is a homeomorphism and $U_0$ is a finite union of open cylinders of length $n_1$.

**Proof of Lemma 4.5.** Let $U_0^{(\ell_0)} = C[0, \ell_0] = C[i_0, \ldots, i_m]$. By construction, $\hat{z}_0 \in \hat{C}[i] \subset U_0^{(\ell_0)}$. For the matrix $A$ we have $A^{M_0} > 0$ for some integer $M_0 \geq 1$ (see the beginning of section 3), so for
each \( j = 1, \ldots, k \) there exists an admissible string \( s^{(j)} = (i_{m+1}^{(j)}, \ldots, i_{m+M_0}^{(j)} \) such that \( i_{m+M_0}^{(j)} = j \) and \( A_{s_{m+M_0}^{(j)}} \). Fix an arbitrary string \( s^{(j)} \) with this property. For the particular \( j' \) such that 

\[
\sigma_{m+M_0}(\tilde{z}_0) \in U_{j'}, \ 	ext{choose} \ s^{(j')} \ 	ext{in} \ 	ext{such} \ \text{a} \ \text{way} \ \text{that} \ \tilde{z}_0 \in C[i; s^{(j')}].
\]

Now set \( n_1 = m + M_0 \) and \( U_0 = \bigcup_{j=1}^k C[i; s^{(j)}] \). Clearly, \( U_0 \) is an open subset of \( U_{j}(\tilde{z}_0) \) containing \( \tilde{z}_0 \) and one shows easily that \( \sigma^{n_1} : U_0 \to \sigma^{n_1}(U_0) \) is a homeomorphism and \( \Int^U(U) = \bigcup_{j=1}^k \sigma^{n_1}(C[i; s^{(j)}]) = \sigma^{n_1}(U_0) \).

Using the above lemma, fix \( n_1 > 0 \) and an open neighbourhood \( U_0 \) of \( \tilde{z}_0 \) in \( U_{j}(\tilde{z}_0) \) such that \( U_0 \) is a finite union of open cylinders of length \( n_1 \), \( U = \sigma^{n_1}(U_0) = \Int^u(U) \) and \( \sigma^{n_1} : U_0 \to U \) is a homeomorphism. The inverse homeomorphism \( \psi : U \to U_0 \) is Lipschitz, so it has a Lipschitz extension

\[
\psi : U \to U_0 \ 	ext{such that} \ \sigma^{n_1}(\psi(x)) = x, \ x \in U .
\]

Then \( r(s) = \sigma^{n_1}(r(s)), s \in V_0 \), where \( V_0 = r^{-1}(U_0) \subset V_0 \), gives a Lipschitz parametrization of \( U \) with \( \psi(r(s)) = r(s) \) for all \( s \in V_0 \). Finally, set

\[
V_i^{(\ell)} = w_i^{(\ell)}(U_0) \subset V_i^{(\ell)}, \ i = 1, 2 ; \ \ell = 1, \ldots, \ell_0 .
\]

It follows from the choice of \( U_0 \), the properties of \( w_i^{(\ell)} \) (see (i) in Lemma 4.4) and Proposition 3.1 that \( V_i^{(\ell)} \) is a finite union of open cylinders of lengths \( n_1 + m_\ell \).

The following two lemmas are proved essentially by using arguments from [D2] and Lemma 4.4 above. We omit most of the details.

**Lemma 4.6.** For every \( \delta'' > 0 \) there exists an integer \( n_2 > 0 \) such that for any \( m \geq n_0 + n_2 \), any \( \ell = 1, \ldots, \ell_0 \) and \( i = 1, 2 \) there exist contracting maps \( \tilde{v}_i^{(\ell)} : V_i^{(\ell)} \to U \) with \( \sigma^{m-m_\ell}(\tilde{v}_i^{(\ell)}(w)) = w \) for all \( w \in V_i^{(\ell)} \) such that

\[
\Lip(\tau_{m-m_\ell} \circ \tilde{v}_i^{(\ell)}) \leq \delta'' \ 	ext{on} \ V_i^{(\ell)} ,
\]

\( \tilde{v}_i^{(\ell)}(V_i^{(\ell)}) \) is a finite union of open cylinders of length \( n_1 + m \) and \( \tilde{v}_i^{(\ell)}(V_i^{(\ell)}) \cap \tilde{v}_i^{(\ell)'}(V_i^{(\ell)'}) = \emptyset \) whenever \( (i, \ell) \neq (i', \ell') \). \( \blacksquare \)

Set \( \delta'' = \frac{c_6}{c_8} \), fix \( n_2 = n_2(\delta'') > 0 \) with the properties listed in Lemma 4.6, and denote \( N_0 = n_0 + n_1 + n_2 . \)

**Proof of Lemma 4.2.** Let \( N \geq N_0 \). Then \( m = N - n_1 \geq n_0 + n_2 \), so by Lemma 4.6 for any \( \ell = 1, \ldots, \ell_0 \) and any \( i = 1, 2 \) there exists a contracting homeomorphism

\[
v_i^{(\ell)} : V_i^{(\ell)} \to V_i^{(\ell)}(V_i^{(\ell)}) \subset U \ 	ext{with} \ \sigma^{N-m_\ell-n_1}(v_i^{(\ell)}(w)) = w , \ w \in V_i^{(\ell)} ,
\]

such that (4.8) holds with \( m = N - n_1 \) and \( \delta'' \) as above. Moreover, we can choose the maps \( \tilde{v}_i^{(\ell)} \) so that \( \tilde{v}_i^{(\ell)}(V_i^{(\ell)}) \cap \tilde{v}_i^{(\ell)'}(V_i^{(\ell)'}) = \emptyset \) whenever \( (i, \ell) \neq (i', \ell') \). Now define Lipschitz maps

\[
v_i^{(\ell)} : U \to U \ 	ext{such that} \ v_i^{(\ell)}(x) = \tilde{v}_i^{(\ell)}(w_i^{(\ell)}(\psi(x))), \ x \in U .
\]

It follows immediately from the above that \( v_i^{(\ell)}(U) \cap v_i^{(\ell)'}(U) = \emptyset \) whenever \( (i, \ell) \neq (i', \ell') \), while Proposition 3.1 shows that each \( v_i^{(\ell)}(U) \) is a finite union of open cylinders of length \( N \).
Moreover, for any \( x \in \mathcal{U} \), according to (i) in Lemma 4.4, (4.10) and (4.9), we have \( \sigma^{N-m_1}(v_i^{(\ell)}(x)) = \sigma^{m_1}(\sigma^{N-m_1-m_2}(v_i^{(\ell)}(x)) = \sigma^{m_1}(w_i^{(\ell)}(\psi(x))) = \psi(x) \), which is the same for all \( \ell \) and \( i \). Consequently, \( \sigma^p(v_1^{(\ell)}(x)) = \sigma^p(v_2^{(\ell)}(x)) \) for all \( p \geq N-n_1 \) and \( x \in \mathcal{U} \). Thus, \( \tau_N(v_2^{(\ell)}(x)) - \tau_N(v_1^{(\ell)}(x)) = \tau_{N-n_1}(v_2^{(\ell)}(x)) - \tau_{N-n_1}(v_1^{(\ell)}(x)) \) for \( x \in \mathcal{U} \), and given \( \eta \in \mathcal{B}_\ell \) and \( h > 0 \), we have

\[
[I_{\eta, h}(\tau_N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau_N(v_1^{(\ell)}(\tilde{r}(\cdot))))](s) = \langle I_{\eta, h}(\tau_{N-n_1}(\tilde{v}_2^{(\ell)}(w_1^{(\ell)}(r(\cdot)))) - \tau_{N-n_1}(\tilde{v}_1^{(\ell)}(w_1^{(\ell)}(r(\cdot)))))](s) .
\]

Now Lemma 4.4(ii), (4.2) and (4.8) imply \( |I_{\eta, h}(\tau_N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau_N(v_1^{(\ell)}(\tilde{r}(\cdot))))](s) \geq \frac{\delta}{7} \).

## 5 Dolgopyat operators

In this section we prove Theorem 1.1.

Throughout we assume that \( \Lambda \) is a basic set for a \( C^2 \) Axiom A flow \( \phi_t : M \rightarrow M \) satisfying the condition (LNIC) which has regular distortion along unstable manifolds over \( \Lambda \) and uniformly Lipschitz local stable holonomy maps. We use the notation from section 2, in particular the fixed real-valued function \( f \in C^{\text{Lip}}(\hat{U}) \), the function \( g = f - P \nu \), where \( P \in \mathbb{R} \) is such that \( P\nu(g) = 0 \), and the \( \sigma \)-invariant probability measure \( \nu \) on \( \Lambda \) such that \( L_{f(0)}^{*}\nu = \nu \). As in the beginning of section 4, fix an arbitrary point \( z_0 \in \text{Int}^u(U_1) \) and constants \( \epsilon_0 > 0 \) and \( 0 < \theta_0 < \theta_1 < 1 \), a \( C^1 \) parametrization \( r(s) \) of a small neighbourhood \( W_0 \) of \( z_0 \) in \( W^{u}_{\nu}(z_0) \) with (4.1) and (4.2). In what follows we assume that the Markov family \( \mathcal{R} = \{ R_i \}_{i=1}^k \) from section 2 satisfies the assumptions made in the beginning of section 4.

The central point here is to prove the \( L^1 \)-contraction property of the normalized operator \( L_{ab} \) with respect to the Gibbs measure \( \nu \) and the norm \( \| h \|_{\text{Lip}, b} \).

**Theorem 5.1.** There exist a positive integer \( N \) and constants \( \hat{\rho} \in (0, 1) \) and \( a_0 > 0 \) such that for any \( a, b \in \mathbb{R} \) with \( |a| \leq a_0 \) and \( |b| \geq 1/a_0 \) and any \( h \in C^{\text{Lip}}(\hat{U}) \) with \( \| h \|_{\text{Lip}, b} \leq 1 \) we have

\[
\int_{\hat{U}} |L_{ab}^{N\nu} h|^2 \, d\nu \leq \hat{\rho}^m \quad \text{for every positive integer } m.
\]

Theorem 1.1 is derived from the above in the same way as in [D2] (see also the proof of Corollary 3.3(a) in [St1]). Indeed, the assumptions of Theorem 1.1 are exactly the ones we have in this section.

Define a new metric \( D \) on \( \hat{U} \) by

\[
D(x, y) = \min\{ \text{diam}(\mathcal{C}) : x, y \in \mathcal{C} , \mathcal{C} \text{ a cylinder contained in } U_i \}
\]

if \( x, y \in U_i \) for some \( i = 1, \ldots, k \), and \( D(x, y) = 1 \) otherwise. Recall that \( \text{diam}(U_i) < 1 \) for all \( i \) by the choice of the Markov family.

The proof of the following lemma is straightforward.

**Lemma 5.2.** (a) \( D \) is a metric on \( \hat{U} \), and if \( x, y \in \hat{U}_i \) for some \( i \), then \( d(x, y) \leq D(x, y) \).

(b) For any cylinder \( \mathcal{C} \) in \( U \) the characteristic function \( \chi_{\mathcal{C}} \) of \( \hat{C} \) on \( \hat{U} \) is Lipschitz with respect to \( D \) and \( \text{Lip}_D(\chi_{\mathcal{C}}) \leq 1/\text{diam}(\mathcal{C}) \). ■

We will denote by \( C^{\text{Lip}}_{D}(\hat{U}) \) the space of all Lipschitz functions \( h : \hat{U} \rightarrow \mathcal{C} \) with respect to the metric \( D \) on \( \hat{U} \) and by \( \text{Lip}_D(h) \) the Lipschitz constant of \( h \) with respect to \( D \).
Given $A > 0$, denote by $K_A(\hat{U})$ the set of all functions $h \in C^\text{Lip}_D(\hat{U})$ such that $h > 0$ and $\frac{|h(u) - h(u')|}{h(u)} \leq A D(u, u')$ for all $u, u' \in \hat{U}$ that belong to the same $\hat{U}_i$ for some $i = 1, \ldots, k$. Notice that $h \in K_A(\hat{U})$ implies $|\ln h(u) - \ln h(v)| \leq A D(u, v)$ and therefore $e^{-A D(u, v)} \leq \frac{h(u)}{h(v)} \leq e^{A D(u, v)}$ for all $u, v \in \hat{U}_i$, $i = 1, \ldots, k$.

Theorem 5.1 is derived from the following lemma which is the analogue of Lemma 10$''$ in [D2]. It should be stressed that replacing the standard metric$^8$ $d$ by the metric $D$ is significant here.

**Lemma 5.3.** There exist a positive integer $N$ and constants $\hat{\rho} = \hat{\rho}(N) \in (0, 1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b \in \mathbb{R}$ with $|a| \leq a_0$, $|b| \geq b_0$, there exists a finite family $\{N_j\}_{j \in J}$ of operators $N_j = N_j(a, b) : C^\text{Lip}_D(\hat{U}) \to C^\text{Lip}_D(\hat{U})$, where $J = J(a, b)$ is a finite set depending on $a$ and $b$, with the following properties:

(a) The operators $N_j$ preserve the cone $K_{E|b|}(\hat{U})$.

(b) For all $H \in K_{E|b|}(\hat{U})$ and $J \in J$ we have $\int_{\hat{U}} (N_j H)^2 \, d\nu \leq \hat{\rho} \int_{\hat{U}} H^2 \, d\nu$.

(c) If $h, H \in C^\text{Lip}_D(\hat{U})$ are such that $H \in K_{E|b|}(\hat{U})$, $|h(u)| \leq H(u)$ for all $u \in \hat{U}$ and $|h(u) - h(u')| \leq E|b| H(u') D(u, u')$ whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in J$ such that $|L_{ab}^N h(u)| \leq (N_j H)(u)$ for all $u \in \hat{U}$ and

$$|(L_{ab}^N h)(u) - (L_{ab}^N h)(u')| \leq E|b|(N_j H)(u') D(u, u')$$

whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$.

The remainder of this section if devoted to the proof of Lemma 5.3. We begin with a technical lemma containing two specific versions of Lasota-Yorke type of inequalities. Its proof is given in the Appendix. Here we use the constants $a_0' > 0$, $T > 0$ and $\gamma_1 > \gamma > 1$ from section 2.

**Lemma 5.4.** There exists a constant $A_0 > 0$ such that for all $a \in \mathbb{R}$ with $|a| \leq a_0'$ the following hold:

(a) If $H \in K_B(\hat{U})$ for some $B > 0$, then $\frac{|(M^m_a H)(u) - (M^m_a H)(u')|}{(M^m_a H)(u')}$ $\leq A_0 \left[ \frac{B}{\gamma^m} + \frac{T}{\gamma - 1} \right] D(u, u')$ for all $m \geq 1$ and all $u, u' \in \hat{U}_i$, $i = 1, \ldots, k$.

(b) If the functions $h$ and $H$ on $\hat{U}$ and $B > 0$ are such that $H > 0$ on $\hat{U}$ and $|h(u) - h(u')| \leq B H(v') D(v, v')$ for any $v, v' \in \hat{U}_i$, $i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b \in \mathbb{R}$ with $|b| \geq 1$ we have $|L_{ab}^m h(u) - L_{ab}^m h(u')| \leq A_0 \left[ \frac{B}{\gamma^m} (M^m_a H)(u') + |b| (M^m_a h)(u') \right] D(u, u')$ whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$.

Fix integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^n(z_0)$ and a non-empty open subset $U_0$ of $W_0$ which is a finite union of open cylinders of length $n_1$ with the properties described in Lemma 4.2. We will also use the set $U = \sigma^{n_1}(U_0) = \text{Int}^a(U)$ and the constants $\rho \in (0, 1)$ and $p_0 \geq 1$ from Lemma 3.3. The choice of $U_0$ shows that $\sigma^{n_1} : U_0 \to U$ is one-to-one and has an inverse map $\psi : U \to U_0$, which is Lipschitz.

We will now impose certain conditions on the numbers $N$, $\epsilon_1$, $b$ and $\mu$ that will be used throughout. Where these conditions come from will become clear later on.

---

$^8$In fact, it is not clear at all whether a similar lemma will be true for general basic sets on manifolds of arbitrary dimension with the metric $d$ in the place of $D$. 
Set \( E = \max \left\{ 4A_0, \frac{2A_0 T}{\gamma^2 - 1} \right\} \), where \( A_0 \geq 1 \) is the constant from Lemma 5.4, and fix an integer \( N \geq N_0 \) such that

\[
\gamma^N \geq \max \left\{ 6A_0, \frac{200 \gamma_{11}^n A_0 \gamma_{11}^n}{c_0^2}, \frac{512 \gamma_{11}^n E}{\gamma_{11}^n} \right\}.
\]

Now fix maps \( v_i^{(\ell)} : U \rightarrow U \) (\( \ell = 1, \ldots, \ell_0, i = 1, 2 \)) with the properties (a), (b), (c) and (d) in Lemma 4.2. In particular, (c) gives

\[
\overline{v_i^{(\ell)}}(U) \cap \overline{v_{i'}^{(\ell')}}(U) = \emptyset, \quad (i, \ell) \neq (i', \ell').
\]

Since \( U_0 \) is a finite union of open cylinders, it follows from Lemma 4.2(d) that there exist a constant \( \delta' = \delta'(U_0) > 0 \) such that

\[
M^{(\delta)}(U_0) \cup \ldots \cup M^{(\delta)}(U_0) \supset U_0, \quad \delta \in (0, \delta'].
\]

Fix \( \delta' \) with this property. Set

\[
\epsilon_1 = \min \left\{ \frac{1}{32C_0}, c_1, \frac{1}{4E}, \frac{1}{\delta \rho^{p_0+2}}, \frac{c_0 r_0}{\gamma_{11}^n}, \frac{c_0^2 (\gamma - 1)}{16T \gamma_{11}^n} \right\},
\]

and let \( b \in \mathbb{R} \) be such that \( |b| \geq 1 \) and

\[
\epsilon_1 \leq \delta'.
\]

Let \( C_m = C_{m(c_1/|b|)} \) (\( 1 \leq m \leq p \)) be the family of maximal closed cylinders contained in \( \overline{U_0} \) with \( \text{diam}(C_m) \leq \frac{c_1}{|b|} \) such that \( U_0 \subset \bigcup_{j=m}^{p} C_m \) and \( \overline{U_0} = \bigcup_{m=1}^{p} C_m \) (see Definitions 4.1). It follows from (5.5), (5.4) and Lemma 3.2, that the length of each \( C_m \) is not less than \( n_1 \), so \( \sigma^{n_1} \) is expanding on \( C_m \). Moreover, Proposition 3.3(a) implies that \( \text{diam}(C_m) \geq \rho^{\delta \rho_0} \) for all \( m \), so

\[
\rho^{\epsilon_1/|b|} \leq \text{diam}(C_m) \leq \rho^{\epsilon_1/|b|}, \quad 1 \leq m \leq p.
\]

Fix an integer \( q_0 \geq 1 \) such that

\[
\theta_0 < \theta_1 - 32 \rho^{q_0-1}.
\]

Next, let \( D_1, \ldots, D_q \) be the list of all closed cylinders contained in \( \overline{U_0} \) that are subcylinders of co-length \( p_0 q_0 \) of some \( C_m \) (\( 1 \leq m \leq p \)). That is, if \( k_m \) is the length of \( C_m \), we consider the subcylinders of length \( k_m + p_0 q_0 \) of \( C_m \), and we do this for any \( m = 1, \ldots, p \). Then \( \overline{U_0} = C_1 \cup \ldots \cup C_p = D_1 \cup \ldots \cup D_q \). Moreover, it follows from the properties of \( C_m \) and Proposition 3.3 that

\[
\rho^{p_0 q_0+1} \frac{\epsilon_1}{|b|} \leq \text{diam}(D_j) \leq \rho^{p_0} \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq q.
\]

Given \( j = 1, \ldots, q, \ell = 1, \ldots, \ell_0 \) and \( i = 1, 2 \), set \( \hat{D}_j = D_j \cap \hat{U} \), \( Z_j = \sigma^{n_1}(\hat{D}_j) \), \( \hat{Z}_j = Z_j \cap \hat{U} \), \( X_{i,j}^{(\ell)} = v_i^{(\ell)}(\hat{Z}_j) \), and \( \hat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \hat{U} \). It then follows that \( D_j = \psi(Z_j) \), and \( U = \bigcup_{j=1}^{q} Z_j \). Moreover, \( \sigma^{N-n_1}(v_i^{(\ell)}(x)) = \psi(x) \) for all \( x \in \mathcal{U} \), and by Proposition 3.1, all \( X_{i,j}^{(\ell)} \) are cylinders.
Remark 5.5. It follows from (5.2) that \( X_{i,j}^{(\ell)} \cap X_{i',j'}^{(\ell')} = \emptyset \) whenever \((i,j,\ell) \neq (i',j',\ell')\).

By Lemma 5.2(b), the characteristic function \( \omega_{i,j}^{(\ell)} = \chi_{\tilde{X}_{i,j}^{(\ell)}} : \hat{U} \to [0,1] \) of \( \tilde{X}_{i,j}^{(\ell)} \) belongs to \( C^\text{Lip}_D(\hat{U}) \) and \( \text{Lip}_D(\omega_{i,j}^{(\ell)}) \leq 1/\text{diam}(X_{i,j}^{(\ell)}) \). Since \( \sigma^N(\tilde{X}_{i,j}^{(\ell)}) = \hat{Z}_j = \sigma_n^1(\bar{D}_j) \) and \( \sigma^N \) is expanding and one-to-one on \( \tilde{X}_{i,j}^{(\ell)} \), it follows that \( \sigma^{N-n_1}(\tilde{X}_{i,j}^{(\ell)}) = \bar{D}_j \) and (2.1) gives \( \text{diam}(D_j) \leq \frac{N-n_1}{c_0} \) \text{diam}(X_{i,j}^{(\ell)}) \leq \frac{N-n_1}{c_0} \text{diam}(X_{i,j}^{(\ell)}) \). Combining this with (5.8) gives

\[
\text{diam}(X_{i,j}^{(\ell)}) \geq \frac{c_0 \rho^{\rho_0 q_0 + 1}}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|}.
\]

for all \( i, j, \ell \in \{ 1, 2, 1, \ldots, q \} \) and \( \ell = 1, \ldots, \epsilon_0 \).

Let \( J \) be a subset of the set \( \Xi = \Xi(a,b) = \{ (i,j,\ell) : 1 \leq i \leq 2, 1 \leq j \leq q, 1 \leq \ell \leq \epsilon_0 \} \). Set

\[
\mu = \mu(N) = \min \left\{ \frac{1}{4}, \frac{c_0 \rho^{\rho_0 q_0 + 2}}{4 \gamma_1^N}, \frac{1}{4 e^{2TN}} \sin^2 \left( \frac{\delta \rho \epsilon_1}{256} \right) \right\},
\]

and define the function \( \beta = \beta_J : \hat{U} \to [0,1] \) by \( \beta = 1 - \mu \sum_{(i,j,\ell) \in J} \omega_{i,j}^{(\ell)} \). Clearly \( \beta \in C^\text{Lip}_D(\hat{U}) \) and

\[
1 - \mu \leq \beta(u) \leq 1 \text{ for any } u \in \hat{U}.
\]

Using Remark 5.5, Lemma 5.2(b) and (5.9) one derives that

\[
\text{Lip}_D(\beta) \leq \Gamma = \frac{2 \mu \gamma_1^N}{c_0 \rho^{\rho_0 q_0 + 2}} \cdot \frac{|b|}{\epsilon_1}.
\]

Next, define the operator \( \mathcal{N} = \mathcal{N}_J(a,b) : C^\text{Lip}_D(\hat{U}) \to C^\text{Lip}_D(\hat{U}) \) by \( \mathcal{N}h = \mathcal{M}^N(\beta \cdot h) \).

The following lemma contains statements similar to Proposition 6 and Lemma 11 in [D2] and by means of Lemma 5.4 their proofs are also very similar, so we omit them.

Lemma 5.6. Under the above conditions for \( N \) and \( \mu \) the following hold:

(a) \( \mathcal{N}h \in K_{E|b}(\hat{U}) \) for any \( h \in K_{E|b}(\hat{U}) \);

(b) If \( h \in C^\text{Lip}_D(\hat{U}) \) and \( h \in K_{E|b}(\hat{U}) \) are such that \( |h| \leq H \) in \( \hat{U} \) and \( |h(v) - h(v')| \leq E|b|H(v')D(v,v') \) for any \( v, v' \in U_j, j = 1, \ldots, k \), then for any \( i = 1, \ldots, k \) and any \( u, u' \in \hat{U} \) we have \( |(L_{ab}^N h)(u) - (L_{ab}^N h)(u')| \leq E|b|(N|H|)(u')D(u,u') \).

Definition. Given \( t > 0 \) and \( S > 0 \), a subset \( W \) of \( \hat{U} \) will be called \((t,S)\)-dense in \( \hat{U} \) if for every \( u \in \hat{U} \) there exist a cylinder \( C \) containing \( u \) with \( \text{diam}(|C|) \leq S \) and a cylinder \( C' \) with \( \text{diam}(C') \geq t \) such that \( C \subseteq W \cap C' \).

Below we use the constants \( c_1, c_2 \) from (3.1) and \( \|g\|_0 = \sup_{u \in \hat{U}} |g(u)| \).

Lemma 5.7. Let \( A > 0, S \geq 1 \) and let \( \epsilon = \epsilon(S,A) = \frac{d_S}{e^{S^2 A^2}} \), where \( d_S = c_1 e^{-\rho_0 \|g\|_0 (\frac{\ln S}{|\ln \rho|} + 1)} \).

Then for any \( t > 0 \), any \((t,S)\)-dense subset \( W \) of \( \hat{U} \) and any \( H \in K_{A/t}(\hat{U}) \), we have

\[
\int_W H^2 \, d\nu \geq \epsilon \int_{\hat{U}} H^2 \, d\nu.
\]

Proof of Lemma 5.7. Let \( t > 0 \), let \( W \) be a \((t,S)\)-dense subset of \( \hat{U} \) and let \( H \in K_{A/t}(\hat{U}) \). Let \( B_1, \ldots, B_n \) be the maximal cylinders in \( U \) with \( \text{diam}(B_j) \leq S \) for any \( j = 1, \ldots, n \). Then \( \bigcup_{j=1}^n \hat{B}_j = \hat{U} \), and so \( \sum_{j=1}^n \nu(B_j) = 1 \). Setting \( m_j = \inf_{u \in B_j} H(u) \) and \( M_j = \sup_{u \in B_j} H(u) \), it follows from \( H \in K_{A/t}(\hat{U}) \) that \( \frac{M_j}{m_j} \leq e^{\frac{2\rho_0}{\epsilon}} = e^{AS} \). Thus, \( H(u) \geq m_j \geq M_j e^{-AS} \) for all \( u \in B_j \).
For each \( j = 1, \ldots, n \) choose an arbitrary point \( u_j \in \tilde{B}_j \). Since \( W \) is \((t,S)\)-dense in \( \tilde{U} \), there exists a subcylinder \( B_j' \) of \( B_j \) such that \( \text{diam}(B_j') \geq t \) and \( \tilde{B}_j' \subset W \cap B_j \). If \( q_j \) is the co-length of \( B_j' \) in \( B_j \), and \( q_j = r_j p_0 + s_j \) for some integers \( r_j \geq 0, 0 \leq s_j < p_0 \), by Proposition 3.3(b) we have \( t \leq \text{diam}(B_j') \leq \rho^{r_j} \text{diam}(B_j) \leq \rho^{r_j} St \), so \( \rho^{r_j} \geq 1/S \), i.e. \( r_j \leq \ln S/|\ln \rho| \). Thus, \( q_j < p_0 (r_j + 1) \leq p_0 (\ln S/|\ln \rho| + 1) \). If \( p_j \) is the length of the cylinder \( B_j \) and \( p_j' = p_j + q_j \) that of \( B_j' \), (3.1) gives \( \nu(B_j') \geq c_2 e^{-q_j \|h\|} \geq d_S \). Hence \( \nu(B_j') \geq d_S \nu(B_j) \) for all \( j = 1, \ldots, n \).

It now follows that \( \nu(W \cap B_j) \geq \nu(B_j') \geq d_S \nu(B_j) \) for any \( j \), so

\[
\int_W H^2(u) \, d\nu(u) = \sum_{j=1}^n \int_{W \cap B_j} H^2(u) \, d\nu(u) \geq \sum_{j=1}^n M_j^2 e^{-\lambda_2 S^2} \nu(W \cap B_j) \\
\geq e^{-\lambda_2 S^2} \sum_{j=1}^n M_j^2 d_S \nu(B_j) \geq \frac{d_S}{e^{\lambda_2 S^2}} \int_{\tilde{U}} H^2(u) \, d\nu(u) = \epsilon \int_{\tilde{U}} H^2(u) \, d\nu(u).
\]

This proves the assertion. \( \blacksquare \)

**Definitions.** A subset \( J \) of \( \Xi \) will be called dense if for any \( m = 1, \ldots, p \) there exists \((i, j, \ell) \in J \) such that \( D_j \subset C_m \). Denote by \( J = J(a,b) \) the set of all dense subsets \( J \) of \( \Xi \).

**Lemma 5.8.** Given the number \( N \), there exist \( \epsilon_2 = \epsilon_2(N) \in (0,1) \) and \( a_0 = a_0(N) > 0 \) such that \( \int_{\tilde{U}} (N_j H)^2 \, d\nu \leq (1 - \epsilon_2) \int_{\tilde{U}} H^2 \, d\nu \) whenever \( |a| \leq a_0 \), \( J \) is dense and \( H \in K_{E|b|}(\tilde{U}) \).

More precisely, we can take \( \epsilon_2 = c_0 \sqrt{\mu} e^{-NT} \) and \( a_0 = \min \left\{ a'_0, \frac{1}{C_0 \gamma^4} \ln \left( 1 + \epsilon' \mu e^{-NT} \right) \right\} \), where

\[
\epsilon' = \frac{c_0}{\sqrt{\mu} e^{-NT}} \text{ and } S = \frac{c_0 \gamma^4}{\epsilon' \mu e^{-NT}}.
\]

**Proof of Lemma 5.8.** The definition of \( \mathcal{N} = N_j \) and the Cauchy-Schwartz inequality imply

\[
(NH)^2(u) = (N^2(u) BH)^2(u) \leq (M^2(u) \beta^2(u)) \cdot (M^2(u) H^2)(u) \leq (M^2(u) H^2)(u)
\]

for all \( u \in \tilde{U} \).

Denote \( W = \cup_{(i, j, \ell) \in J} \tilde{B}_j \). Then \( u \in W \) means that there exists \((i, j, \ell) \in J \) with \( u_{(i, j, \ell)} (u) \in X_{(i, j, \ell)} \), and so \( \beta(u_{(i, j, \ell)} (u)) = 1 - \mu \).

We will now show that \( W \) is \((t, S)\)-dense in \( \tilde{U} \), where \( t = c_0 \gamma^{n_1} \rho^{p_0 \gamma_0 + 1} \cdot \|h\| \) and \( S \) is as in the statement of the lemma. Let \( u \in \tilde{U} \). Since \( \tilde{U} \subset U \subset \cup_{m=1}^p \sigma^{n_1}(C_m) \), we have \( u \in C = \sigma^{n_1}(C_m) \) for some \( m \). Since \( J \) is dense, there exists \((i, j, \ell) \in J \) so that \( D_j \subset C_m \). Then \( Z_j = \sigma^{n_1}(D_j) \subset C \), so \( Z_j \subset W \cap C \). Now (2.1), (5.6) and (5.8) yield \( \text{diam}(Z_j) \geq t \) and \( \text{diam}(C) \leq St \), so \( W \) is \((t, S)\)-dense in \( \tilde{U} \).

Let \( H \in K_{E|b|}(\tilde{U}) \). Setting \( A = E c_0 \gamma^{n_1} \rho^{p_0 \gamma_0 + 1} \epsilon_1 \), we have \( H \in K_{A/4}(\tilde{U}) \). Since \( \epsilon' = \frac{c_0}{\epsilon' e^{S/4T}}, \) it follows that \( \epsilon' = \epsilon(S, A) \), the number defined in Lemma 5.7. By Lemma 5.6 (a), \( NH \in K_{A/4}(\tilde{U}) \), so Lemma 5.7 implies

\[
\int_W (NH)^2 \, d\nu \geq \epsilon' \int_{\tilde{U}} (NH)^2 \, d\nu.
\]

It follows from the definition of \( \beta \) that \( M^2(u) \beta^2(u) \leq 1 - \mu e^{-NT} \) on \( W \). Using this, (5.12) and (5.13), as in [D2] (see also the proof of Lemma 7.4 in [St1]) we get

\[
\int_{\tilde{U}} (NH)^2 \, d\nu \leq \int_{\tilde{U}} (M^2(u) H^2) \, d\nu - \mu e^{-NT} \epsilon' \int_{\tilde{U}} (NH)^2 \, d\nu.
\]
and therefore \( \int_{\tilde{U}} (\mathcal{N}H^2) \, d\nu \leq \frac{1}{1 + \mu e^{e^{-NT}}} \int_{\tilde{U}} (\mathcal{M} N H^2) \, d\nu \). Since \( \mathcal{M} N H^2 = L_{j(0)}^N \left( e^{(f^{(a)} - f^{(0)})N} H^2 \right) \leq e^{N|a|C_0} L_{j(0)}^N H^2 \) on \( \tilde{U} \), it now follows from \(|a| \leq a_0 \) and \( L_{j(0)}^N \nu = \nu \) (see section 2) that

\[
\int_{\tilde{U}} (\mathcal{N}H^2) \, d\nu \leq \frac{e^{N|a|C_0}}{1 + \mu e^{e^{-NT}}} \int_{U} (L_{j(0)}^N H^2) \, d\nu \leq \frac{1 + \epsilon_2}{1 + 4\epsilon_2} \int_{\tilde{U}} H^2 \, d\nu \leq (1 - \epsilon_2) \int_{\tilde{U}} H^2 \, d\nu .
\]

This completes the proof of the lemma. \( \blacksquare \)

In what follows we assume that \( h, H \in C_D^{Lip}(\tilde{U}) \) are such that

\[
H \in K_E|b(\tilde{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \tilde{U} ,
\]

and

\[
|h(u) - h(u')| \leq E|b|H(u') D(u,u') \quad \text{whenever} \quad u,u' \in \tilde{U}_i , \quad i = 1, \ldots , k .
\]

Define the functions \( \chi_{\ell}^{(i)} : \tilde{U} \rightarrow \mathbb{C} \) (\( \ell = 1, \ldots , j_0, i = 1, 2 \)) by

\[
\chi_{\ell}^{(1)}(u) = \frac{|e^{(f^{(a)} + ib \tau_\ell \xi^{(i)}(u))} h(v_1^{(i)}(u)) + e^{(f^{(a)} + ib \tau_\ell \xi^{(i)}(u))} h(v_2^{(i)}(u))|}{(1 - \mu) e^{f^{(a)}(v_i^{(i)}(u))} H(v_1^{(i)}(u)) + e^{f^{(a)}(v_i^{(i)}(u))} H(v_2^{(i)}(u))} ,
\]

\[
\chi_{\ell}^{(2)}(u) = \frac{|e^{(f^{(a)} + ib \tau_\ell \xi^{(i)}(u))} h(v_1^{(i)}(u)) + e^{(f^{(a)} + ib \tau_\ell \xi^{(i)}(u))} h(v_2^{(i)}(u))|}{e^{f^{(a)}(v_i^{(i)}(u))} H(v_1^{(i)}(u)) + (1 - \mu) e^{f^{(a)}(v_i^{(i)}(u))} H(v_2^{(i)}(u))} ,
\]

and set \( \gamma_\ell(u) = b [\tau_N(v_2^{(i)}(u)) - \tau_N(v_1^{(i)}(u))] , \quad u \in \tilde{U} . \)

**Remark.** It is easy to see that for any \( m \) and any \( \ell \) the set \( \{ \gamma_\ell(u) : u \in \sigma^{n_1}(C_m) \} \) is contained in an interval of length \( < 1/8 \). Indeed, given \( m, \ell, i = 1, 2 \) and \( u, u' \in \sigma^{n_1}(C_m) \), set \( x = v_i^{(i)}(u) , x' = v_i^{(i)}(u') \). Since \( d(\sigma^N(x), \sigma^N(x')) = d(u, u') \leq \text{diam}(\sigma^{n_1}(C_m)) \leq \frac{\tau_{n_1}^{n_1}}{c_0 \gamma^{n_1} \gamma^{-1}} \), it follows from (2.1) that

\[
d(\sigma^j(x), \sigma^j(x')) \leq \frac{\tau_{n_1}^{n_1}}{c_0 \gamma^{n_1} \gamma^{-1}} \frac{1}{c_0 \gamma^{n_1} \gamma^{-1} \gamma} .
\]

This, (2.3) and (5.4) give \( |\tau_N(x) - \tau_N(x')| \leq \sum_{j=0}^{N-1} |\tau(\sigma^j(x)) - \tau(\sigma^j(x'))| \leq \frac{T_{\tau_{n_1}^{n_1}}}{c_0 \gamma^{n_1} \gamma^{-1}} \frac{1}{c_0 \gamma^{n_1} \gamma^{-1}} \frac{1}{c_0 \gamma^{n_1} \gamma^{-1}} < \frac{1}{16 \gamma^2} \). Thus, the set \( \{ \gamma_\ell(v_1^{(i)}(u)) : u \in \sigma^{n_1}(C_m) \} \) is contained in an interval of length \( < 1/8 \), and therefore \( \{ \gamma_\ell(u) : u \in \sigma^{n_1}(C_m) \} \) is contained in an interval of length \( < 1/8 \).

**Definitions.** We will say that the cylinders \( D_j \) and \( D_j' \) are adjacent if they are subcylinders of the same \( C_m \) for some \( m \). If \( D_j \) and \( D_j' \) are contained in \( C_m \) for some \( m \) and for some \( \ell = 1, \ldots , \ell_0 \) there exist \( u \in D_j \) and \( v \in D_j' \) such that \( d(u, v) \geq \frac{1}{2} \text{diam}(C_m) \) and \( \| r_{-1}(v) - r_{-1}(u) \| , \eta_\ell \geq \theta_1 \), we will say that \( D_j \) and \( D_j' \) are \( \eta_\ell \)-separable in \( C_m \).

**Lemma 5.9.** Let \( j, j' \in \{ 1, 2, \ldots , q \} \) be such that \( D_j \) and \( D_j' \) are contained in \( C_m \) and are \( \eta_\ell \)-separable in \( C_m \) for some \( m = 1, \ldots , p \) and \( \ell = 1, \ldots , \ell_0 \). Then \( |\gamma_\ell(u) - \gamma_\ell(u')| \geq c_2 \epsilon_1 \) for all \( u \in \tilde{Z}_j \) and \( u' \in \tilde{Z}_{j'} \), where \( c_2 = \frac{\epsilon}{16} \).

**Proof of Lemma 5.9.** Let \( u \in \tilde{Z}_j \) and \( u' \in \tilde{Z}_{j'} \); then \( x = \psi(u) \in \tilde{D}_j \) and \( x' = \psi(u') \in \tilde{D}_{j'} \). Also \( x = r(s) \) and \( x' = r(s') \) for some \( s, s' \in V_0 \). Set \( \eta = \frac{s-s'}{\| s-s' \|} \in S_0^{n-1} \).
Since $D_j$ and $D_{j'}$ are $\eta_{\ell}$-separable in $C_m$, there exist $x_0 = r(s_0) \in D_j$ and $x'_0 = r(s'_0) \in D_{j'}$ such that $d(x_0, x'_0) \geq \frac{1}{2} \text{diam}(C_m)$ and $\langle \eta_0, \eta_{\ell} \rangle \geq \theta_1$, where $\eta_0 = \frac{s_0 - s'_0}{\|s_0 - s'_0\|} \in S^{n-1}$. By (4.2), (5.8) and (5.6), $\|s - s_0\| \leq 2 d(r(s), r(s_0)) \leq 2 \text{diam}(D_j) \leq 2 \rho_0^{-1} \text{diam}(C_m)$, and similarly $\|s' - s'_0\| \leq 2 \rho_0^{-1} \text{diam}(C_m)$. This implies

$$\|s - s'\| - \|s - s_0\| \leq \|s - s_0\| + \|s' - s'_0\| \leq 4 \rho_0^{-1} \text{diam}(C_m).$$

Hence $\|\eta_0 - \eta\| = \frac{|s - s'_0|}{\|s_0 - s'_0\|} \leq \frac{8 \rho_0^{-1} \text{diam}(C_m)}{\|s_0 - s'_0\|} \leq \frac{16 \rho_0^{-1} \text{diam}(C_m)}{d(x_0, x'_0)} \leq 32 \rho_0^{-1}$. Combining this with (5.7) gives $\langle \eta, \eta_{\ell} \rangle = \langle \eta_0, \eta \rangle + \langle \eta - \eta_0, \eta_{\ell} \rangle \geq \theta_1 - 32 \rho_0^{-1} > \theta_0$. Thus, $\eta \in B_{\epsilon}$, and Lemma 4.2 implies $\left| I_{\eta, h} \left( \tau_N(v_2^{(l)}(\tilde{r}(\cdot))) - \tau_N(v_1^{(l)}(\tilde{r}(\cdot))) \right) \right| (\tilde{s}) \geq \frac{\delta}{2}$ for all $\tilde{s} \in r^{-1}(U_0)$ and $h \neq 0$ such that $\tilde{s} + h\eta \in r^{-1}(U_0 \cap \Lambda)$.

Since $u = \sigma^m x = \tilde{r}(s)$ and $u' = \tilde{r}(s')$, we have $s, s' \in r^{-1}(U_0 \cap \Lambda)$ and $s = s' + h\eta$ with $h = \|s - s'\|$. It then follows from the above, (5.16) and (5.6) that

$$\frac{1}{|b|} |\gamma_{\ell}(u) - \gamma_{\ell}(u')| = \frac{1}{4} \left( \frac{1}{\text{diam}(C_m)} - 4 \rho_0^{-1} \text{diam}(C_m) \right) \geq \frac{\delta \rho \epsilon_1}{16|b|}.$$ 

The following lemma is central for this section.

**Lemma 5.10.** Assume $b$ is chosen in such a way that (5.5) holds. Then for any $j = 1, \ldots, q$ there exist $i \in \{1, 2\}$, $j' \in \{1, \ldots, q\}$ and $\ell \in \{1, \ldots, \ell_0\}$ such that $D_j$ and $D_{j'}$ are adjacent and $\chi_{\ell}^{(i)}(u) \leq 1$ for all $u \in \hat{Z}_{j'}$.

To prove this we need the following lemma which coincides with Lemma 14 in [D2] and its proof is almost the same, so we omit it.

**Lemma 5.11.** If $h$ and $H$ satisfy (5.14)-(5.15), then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have:

(a) $\frac{1}{2} \leq \frac{H(v_1^{(i)}(u'))}{H(v_1^{(i)}(u''))} \leq 2$ for all $u', u'' \in \hat{Z}_j$;

(b) Either for all $u \in \hat{Z}_j$ we have $|h(v_1^{(i)}(u))| \leq \frac{3}{4} H(v_1^{(i)}(u))$, or $|h(v_1^{(i)}(u))| \geq \frac{1}{4} H(v_1^{(i)}(u))$ for all $u \in \hat{Z}_j$.

**Proof of Lemma 5.10.** Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $D_j \subset C_m$. By (5.5), $\delta = \epsilon_1/|b| \in (0, \delta']$, so it follows from (5.3) that $C_m \subset M_{\eta_{\ell}}(U_0)$ for some $\ell = 1, \ldots, \ell_0$. This means that there exist $u, v \in C_m$ such that $d(u, v) \geq \frac{1}{2} \text{diam}(C_m)$ and $\langle \eta_{\ell}, \eta \rangle \geq \theta_1$. Let $j', j'' = 1, \ldots, q$ be such that $u \in D_{j'}$ and $v \in D_{j''}$. (Notice that we may have $j' = j$ or $j'' = j$.) Then $D_{j'}$ and $D_{j''}$ are $\eta_{\ell}$-separable in $C_m$.

Fix $\ell, j'$ and $j''$ with the above properties, and set $\hat{Z} = \hat{Z}_j \cup \hat{Z}_{j'} \cup \hat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 5.11(b) holds for $\hat{Z}_t$, $\ell$ and $i$, then $\mu \leq 1/4$ implies $\chi_{\ell}^{(i)}(u) \leq 1$ for any $u \in \hat{Z}_t$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 5.11(b) holds for $\hat{Z}_t$, $\ell$ and $i$, i.e. $|h(v_1^{(i)}(u))| \geq \frac{1}{4} H(v_1^{(i)}(u))$, $u \in \hat{Z}_t$.  

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Thus, it follows from Lemma 5.9 and the above that and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2\sin \theta_i(u) < \frac{\epsilon_1}{8}$.

The difference between the arguments of the complex numbers $e^{ib\tau_N(v_i(u))}h(v_i(u))$ and $e^{ib\tau_N(v_i(u'))}h(v_i(u'))$ is given by the function

$$\Gamma_\ell(u) = [b\tau_N(v_i(u)) + \theta_2(u) + \lambda_2] - [b\tau_N(v_i(u')) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1 + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u)).$$

Given $u' \in \hat{Z}_j$ and $u'' \in \hat{Z}_{j''}$, since $\hat{D}_j$ and $\hat{D}_{j''}$ are contained in $C_m$ and are $\eta_\ell$-separable in $C_m$, it follows from Lemma 5.9 and the above that

$$|\Gamma_\ell(u') - \Gamma_\ell(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{\epsilon_1}{2}.$$ 

Thus, $|\Gamma_\ell(u') - \Gamma_\ell(u'')| \geq \frac{\epsilon_1}{2}$ for all $u' \in \hat{Z}_{j''}$ and $u'' \in \hat{Z}_{j''}$. Hence either $|\Gamma_\ell(u')| \geq \frac{\epsilon_1}{2}$ for all $u' \in \hat{Z}_{j'}$ or $|\Gamma_\ell(u'')| \geq \frac{\epsilon_1}{2}$ for all $u'' \in \hat{Z}_{j''}$.

Assume for example that $|\Gamma_\ell(u)| \geq \frac{\epsilon_1}{4}$ for all $u \in \hat{Z}_{j'}$. Since $\hat{Z} \subset \sigma^{n_1}(C_m)$, using again the Remark before Lemma 5.9, for any $u \in \hat{Z}$ we get

$$|\Gamma_\ell(u)| \leq |\lambda_2 - \lambda_1 + \lambda| + |\gamma_\ell(u) - \gamma_\ell(u_0)| + |\theta_2(u) - \theta_1(u)| \leq \pi + \frac{1}{8} + \frac{\pi}{3} = \frac{3\pi}{2}.$$ 

Thus, $\frac{\epsilon_1}{8} \leq |\Gamma_\ell(u)| < \frac{3\pi}{2}$ for all $u \in \hat{Z}_{j'}$. Now as in [D2] (see also [St1]) one shows that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \hat{Z}_{j'}$. $\blacksquare$

**Proof of Lemma 5.3.** As before, we assume that $N$ and $\mu > 0$ satisfy (5.1) and (5.10). Define $\epsilon_1$ by (5.4), take $\epsilon_2 = \epsilon_2(N) > 0$ and $a_0 = a_0(N) > 0$ as in Lemma 5.8 and set $\hat{\rho} = 1 - \epsilon_2$.

Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ be such that $|a| \leq a_0$ and $|b| \geq b_0 = \epsilon_1/\delta'$. Then for any $J \in J(a,b)$, Lemma 5.6 (a) implies property (a) in Lemma 5.3 for the operator $N_J$, while Lemma 5.8 gives property (b) in Lemma 5.3.

To check (c) in Lemma 5.3, assume that $h, H \in C^\text{Lip}(\hat{U})$ satisfy (5.14) and (5.15). Now define the subset $J$ of $J(a,b)$ in the following way. First, include in $J$ all $(1, j, \ell) \in \Xi$ such that $\chi^{(1)}_\ell(u) \leq 1$ for all $u \in \hat{Z}_j$. Then for any $j = 1, \ldots, q$ and $\ell = 1, \ldots, j_0$ include $(2, j, \ell)$ in $J$ if and only if $(1, j, \ell)$ has not been included in $J$ (that is, $\chi^{(1)}_\ell(u) > 1$ for some $u \in \hat{Z}_j$) and $\chi^{(2)}_\ell(u) \leq 1$ for all $u \in \hat{Z}_j$. It follows from Lemma 5.10 that $J$ is dense. (Clearly, $J$ depends not only on $N$, $a$ and $b$, but on $h$ and $H$ as well.)
Consider the operator $N = N^j(a,b) : C^{\text{Lip}}_D(\tilde{U}) \to C^{\text{Lip}}_D(\tilde{U})$. Then Lemma 6.6 (b) implies $|(L_{ab}^N)(u) - (L_{ab}^N)(u')| \leq E|b|(NH)(u') D(u, u')$ whenever $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k$. So, it remains to show that
\[
|(L_{ab}^N)(u)| \leq (NH)(u), \ u \in \tilde{U}.
\]

Let $u \in \tilde{U}$. If $u \notin \tilde{Z}_j$ for any $(i, j, \ell) \in J$, then $\beta(v) = 1$ whenever $\sigma^N v = u$ and therefore $|(L_{ab}^N)(u)| \leq (M_0^N(\beta H))(u) = (NH)(u)$.

Assume that $u \in \tilde{Z}_j$ e.g. for $(1, j, \ell) \in J$; then $(2, j, \ell) \notin J$. Since $\lambda^{(1)}(u) \leq 1$, $\beta(v_1^\ell)(u)) \geq 1 - \mu$ and $\beta(v_2^\ell)(u)) = 1$, using (5.14) one derives
\[
|(L_{ab}^N)(u)| \leq \sum_{\sigma^N v = u, v \neq v_1(u), v_2(u)} e^{f^{(a)}_N(v)}|h(v)| + \left[e^{f^{(a)}_N(v_1(u))}\beta(v_1(u))H(v_1(u)) + e^{f^{(a)}_N(v_2(u))}\beta(v_2(u))H(v_2(u))\right] \leq (NH)(u),
\]
which proves (5.17). This completes the proof of Lemma 5.3. ■

6 Non-integrability conditions

Throughout we assume that $\phi_t$ is a $C^2$ contact flow on $M$ with a $C^2$ invariant contact form $\omega$. The following condition says that $d\omega$ is in some sense non-degenerate on the ‘tangent space’ of $\Lambda$ near some of its points:

(ND): There exist $z_0 \in \Lambda$ and $\mu_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, any $\tilde{z} \in \Lambda \cap W^u(z_0)$ and any unit vector $\eta \in E^u(\tilde{z})$ tangent to $\Lambda$ at $\tilde{z}$ there exist $\tilde{z} \in \Lambda \cap W^u(\tilde{z})$, $\tilde{y} \in W^s(\tilde{z})$ and a unit vector $\xi \in E^s(\tilde{y})$ tangent to $\Lambda$ at $\tilde{y}$ with $|d\omega_{\tilde{z}}(\xi, \eta_{\tilde{z}})| \geq \mu_0$, where $\eta_{\tilde{z}}$ is the parallel translate of $\eta$ along the geodesic in $W^u(\tilde{z})$ from $\tilde{z}$ to $\tilde{z}$, while $\xi_{\tilde{z}}$ is the parallel translate of $\xi$ along the geodesic in $W^s(\tilde{z})$ from $\tilde{y}$ to $\tilde{z}$.

Remark. It appears the above condition would become significantly more restrictive if one requires the existence of a unit vector $\xi \in E^s(\tilde{z})$ tangent to $\Lambda$ at $\tilde{z}$ with $|d\omega_{\tilde{z}}(\xi, \eta_{\tilde{z}})| \geq \mu_0$. The reason for this is that in general the set of unit tangent vectors to $\Lambda$ does not have to be closed in the bundle $E^s_{\Lambda}$. That is, there may exist a point $\tilde{z} \in \Lambda$, a sequence $\{z_m\} \subset W^s(\tilde{z}) \cap \Lambda$ and for each $m$ a unit vector $\xi_m$ tangent to $\Lambda$ at $z_m$ such that $z_m \to z$ and $\xi_m \to \xi$ as $m \to \infty$, however $\xi$ is not tangent to $\Lambda$ at $\tilde{z}$.

Proposition 6.1. For contact flows $\phi_t$ with Lipschitz local stable holonomy maps, the condition (ND) implies (LNIC).

Since (ND) is always satisfied when $\dim(M) = 3$ or $\Lambda = M$, we get the following.

Corollary 6.2. For contact flows $\phi_t$ with either $\dim(M) = 3$ (and an arbitrary basic set $\Lambda$) or $\Lambda = M$ the condition (LNIC) is always satisfied on $\Lambda$. ■

To prove Proposition 6.1 we will make use of the following lemma which is a consequence of Lemma B.7 in [L2].

Lemma 6.3. ([L2]) Let $\phi_t$ be a contact flow on $M$ with a contact form $\omega$ and let $\Lambda$ be a basic set for $\phi_t$ with Lipschitz local (un)stable holonomy maps. Then for every $\tilde{z} \in \Lambda$ and every $\delta > 0$ there exists $\tilde{e} \in (0, \epsilon_0)$ such that for any $z \in \Lambda \cap W^u_{\tilde{e}}(\tilde{z})$, any $x \in W^u_{\tilde{e}}(z) \cap \Lambda$ and any $y \in W^s_{\tilde{e}}(z) \cap \Lambda$ we have $|\Delta(x, y) - d\omega_{\tilde{z}}(u, v)| \leq \delta \|u\| \|v\|$, where $u \in E^u(z)$ and $v \in E^s(z)$ are such that $\exp_{\tilde{z}}^u(u) = x$ and $\exp_{\tilde{z}}^s(v) = y$. 

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Proof of Proposition 6.1. Assume that $\Lambda$ is a basic set for a contact flow $\phi_t$ such that (ND) holds on $\Lambda$. Let $z_0 \in \Lambda$, $1 > \epsilon_0 > 0$ and $\mu_0 > 0$ be as in the statement of (ND) above. Fix a constant $C > 0$ with $|d\omega_z(u,v)| \leq C \|u\|\|v\|$ for all $z \in \Lambda$ and $u,v \in T_zM$, and then fix an arbitrary constant $\theta_0$ such that
\begin{equation}
1 - \frac{\mu_0^2}{128C^2} \leq \theta_0 < 1.
\end{equation}

To check (LNIC), consider arbitrary $\epsilon \in (0,\epsilon_0]$ and $\tilde{z} \in \Lambda \cap W^u_\epsilon(z_0)$, and let $\eta \in E^u(\tilde{z})$ be an arbitrary tangent vector to $\Lambda$ at $\tilde{z}$ with $\|\eta\| = 1$. It follows from the condition (ND) that there exist $\tilde{z} \in \Lambda \cap W^u_\epsilon(\tilde{z})$, $\tilde{y} \in W^s_\epsilon(\tilde{z})$ and a unit vector $\xi \in E^s(\tilde{y})$ tangent to $\Lambda$ at $\tilde{y}$ such that
\begin{equation}
|d\omega_\epsilon(\xi,\eta)| \geq \mu_0.
\end{equation}

Set $\tilde{y}_1 = \tilde{y}$. Let $0 < \delta < \min\{\epsilon/2, \sqrt{\mu_0}/2\}$ (some additional condition on $\delta$ will be imposed later). Since $\xi$ is tangent to $\Lambda$ at $\tilde{y}$, there exists $w \in E^s(\tilde{y};\delta)$ such that $\tilde{y}_2 = \exp_{\tilde{y}}(w) \in \Lambda$ and
\begin{equation}
\|w/\|w\| - \xi\| < \delta.
\end{equation}
Assuming $\epsilon_0$ (and therefore $\epsilon$) is sufficiently small, there exist $w_1, w_2 \in E^s(\tilde{z};\epsilon)$ such that $\tilde{y}_i = \exp_{\tilde{y}}(w_i)$ for $i = 1,2$. Moreover we will assume$^9$ that $C > 0$ is taken sufficiently large and $\epsilon > 0$ and then $\delta > 0$ sufficiently small so that $\|w_2 - (w_1 + \|w\|\xi)\| \leq \frac{\mu_0}{8\epsilon} \|w\|$. Fix $w$ with this property and (6.3), and set $\delta = \frac{\mu_0\|w\|}{10\epsilon}$.

Given any $z \in \Lambda \cap W^u_\epsilon(\tilde{z})$, for $i = 1,2$ there exists a unique $w_i(z) \in E^s(z)$ such that $\pi_{\tilde{y}}(z) = \exp_{\tilde{y}}(w_i(z))$. Clearly $w_i(z)$ is a continuous function of $z$ with $w_i(\tilde{z}) = w_i$. Take $0 < \epsilon \leq \min\{\epsilon, \sqrt{\mu_0}/\|w\|\}$ so small that the conclusion of Lemma 6.3 holds and moreover $\|w_i(z)\| < \epsilon$ and $\|w_i\|/\|w\| \leq \|w_i(z)\| \leq 2\|w_i\|$ for all $z \in \Lambda \cap W^u_\epsilon(\tilde{z})$ and $i = 1,2$.

We will now use parallel translation on the Riemann manifold $W^u_\epsilon(z_0)$. For any $z \in W^u_\epsilon(\tilde{z})$ let $\Gamma_z : E^u(z) \rightarrow E^u(\tilde{z})$ be the parallel translation along the geodesic in $W^u_\epsilon(z_0)$ from $z$ to $\tilde{z}$. Then $\Gamma_z$ is an isometry which is Lipschitz in $z$ and $\Gamma_z = \text{id}$. Since the form $d\omega_\epsilon$ is $C^1$ in $z$ and $w(z) \rightarrow w$ as $z \rightarrow \tilde{z}$, taking $\epsilon > 0$ sufficiently small, for any $z \in W^u_\epsilon(\tilde{z}) \cap \Lambda$ with $d(z,\tilde{z}) < \epsilon$ we have
\begin{equation}
|d\omega_\epsilon(v, w_1(z)) - d\omega_\epsilon(\Gamma_z(v), w_1)| \leq \delta \|v\|, \quad v \in E^s(z;\epsilon), \quad i = 1,2.
\end{equation}
Moreover we can take $\epsilon > 0$ so small that $\|\Gamma_z(\eta_z) - \eta_z\| \leq \frac{\mu_0}{4\epsilon}$ for any $z \in W^u_\epsilon(\tilde{z}) \cap \Lambda$.

Let $z \in \Lambda \cap W^u_\epsilon(\tilde{z})$, $d(z,\tilde{z}) < \epsilon$, and let $v \in E^u(\tilde{z})$, be such that $\exp_\epsilon^s(v) \in \Lambda$, $\|v\| < \epsilon$, and $\langle v/\|v\|, \eta_z\rangle \geq \theta_0$. Setting $\tilde{v} = \Gamma_z(v)$ and $\tilde{\eta} = \Gamma_z(\eta_z)$, we have $\tilde{\eta} = \tilde{v} = \tilde{v}/\|\tilde{v}\|, \|\tilde{\eta}\| = 1, \langle \tilde{v}, \tilde{\eta}\rangle = \langle v, \eta_z\rangle$ and $\|\tilde{\eta} - \eta_z\| \leq \frac{\mu_0}{8\epsilon}$. Thus, $\langle \tilde{v}/\|\tilde{v}\|, \tilde{\eta}\rangle = \langle v/\|v\|, \eta_z\rangle \geq \theta_0$, which combined with (6.1) gives $\|\tilde{v}/\|\tilde{v}\| - \tilde{\eta}\|^2 = 2 - 2\langle \tilde{v}/\|\tilde{v}\|, \tilde{\eta}\rangle \leq 2(1 - \theta_0) \leq \frac{\mu_0}{48C^2}$, so $\|\tilde{v}/\|\tilde{v}\| - \tilde{\eta}\| < \frac{\mu_0}{8\epsilon}$. Using Lemma 6.3, (6.4), and (6.3) and $\delta < \frac{\mu_0}{10\epsilon}$ (which follows from the choice of $\delta$) we now get
\begin{align*}
|\Delta(\exp_\epsilon^s(v), \pi_{\tilde{y}}(z)) - \Delta(\exp_\epsilon^s(v), \pi_{\tilde{y}}(z))| &= |\Delta(\exp_\epsilon^s(v), \exp_\epsilon^s(w_1(z)) - \Delta(\exp_\epsilon^s(v), \exp_\epsilon^s(w_2(z)))| \\
&\geq |d\omega_\epsilon(v, w_1(z)) - d\omega_\epsilon(v, w_2(z))| - \delta \|v\| \|w_1(z) - w_2(z)\| \\
&\geq |d\omega_\epsilon(v, w_1(z)) - d\omega_\epsilon(v, w_2(z))| - \delta \|v\| ||w_1(z) - w_2(z)|| \\
&\geq |w_2 - w_1 - (v, w_2(z))| - 2\delta \|v\| - 2\delta \|\tilde{v}\| \geq |d\omega_\epsilon(v, w_1(z)) - w_2(z)| - 4\delta \|v\| \\
&\geq \|w_2 - w_1 - (v, w_2(z))| - \frac{\mu_0}{8} \|v\| - 4\delta \|v\| \geq \|w_2 - w_1 - (v, w_2(z))| - \frac{\mu_0}{4} \|v\| - 4\delta \|v\| \\
&\geq \|w_2 - w_1 - (v, w_2(z))| - \frac{\mu_0}{2} \|v\| - 4\delta \|v\| = \left(\frac{\mu_0}{2\epsilon} - 4\delta\right) \|v\| = \frac{\delta}{\epsilon} \|v\|.
\end{align*}

$^9$Using local coordinates on the Riemann manifold $X = W^u_\epsilon(\tilde{z})$ we can identify $E^s(\tilde{y}) = T_{\tilde{y}}X$ and $E^u(\tilde{z}) = T_{\tilde{z}}X$. Given $\omega > 0$, we can take $\epsilon > 0$ so small that $|d_{P_{\tilde{z}}y} - f_{\epsilon/2}| < \omega/2$ for any $y \in X$ with $d(y,\tilde{z}) < \epsilon$, where $P_{\tilde{z}}$ is the operator of parallel translation along the geodesic in $X$ from $y$ to $\tilde{z}$. Then, given $\tilde{y} = \exp_{\tilde{y}}(w_1)$ with $\|w_1\| < \epsilon$, a simple calculation shows that we can choose $\delta > 0$ so small that $\|\|\exp_{\tilde{y}}(\pi_{\tilde{y}}(\xi)) \| - (w_1 + \xi)\| \leq \omega\|\xi\|$ for every $\xi \in T_{\tilde{y}}X$ with $\|\xi\| < \delta$. 

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This proves the lemma. ■

7 Regular distortion along unstable manifolds

In this section we briefly describe the results in [St3] which give sufficient conditions for a flow over a basic set to have regular distortion along unstable manifolds. As we mentioned in section 1, there are good reasons to believe that this condition is satisfied for a very general class of flows on basic sets (perhaps even always).

As before, let $M$ be a $C^2$ complete Riemann manifold, $\phi_t$ be a $C^2$ flow on $M$, and let $\Lambda$ a basic set for $\phi_t$.

Assume that $\phi_t$ and $\Lambda$ satisfy the following lower unstable pinching condition:

(LUPC): There exist constants $C > 0$ and $0 < \alpha \leq \beta < \alpha_2 \leq \beta_2$, and for every $x \in \Lambda$ constants $\alpha_1(x) \leq \beta_1(x)$ with $\alpha \leq \alpha_1(x) \leq \beta_1(x) \leq \beta$ and $2\alpha_1(x) - \beta_1(x) \geq \alpha$ and a $d\phi_t$-invariant splitting $E_u(x) = E^1_u(x) \oplus E^2_u(x)$, continuous with respect to $x \in \Lambda$, such that

$$
\frac{1}{C} e^{\alpha_1(x) t} \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C e^{\beta_1(x) t} \|u\|, \quad u \in E^1_u(x), t > 0,
$$

and

$$
\frac{1}{C} e^{\alpha_2 t} \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C e^{\beta_2 t} \|u\|, \quad u \in E^2_u(x), t > 0.
$$

In (LUPC) the lower part of the spectrum of $d\phi_t$ over $E^u$ is (point-wisely) pinched, however there is no restriction on the rest of the spectrum, except that it should be uniformly separated from the lower part.

Under the above condition the distribution $E^u_\alpha(x) (x \in \Lambda)$ is integrable (see e.g. [Pes]), so (assuming $\epsilon_0 > 0$ is small enough) there exists a $\phi_t$-invariant family $W^{u,2}_\epsilon(x) (x \in \Lambda)$ of $C^2$ submanifolds of $W^u_\epsilon(x)$ such that $T_x(W^{u,2}_\epsilon(x)) = E^g_2(x)$ for all $x \in \Lambda$. Moreover (see Theorem 6.1 in [HPS] or the proof of Theorem B in [PSW]), for any $x \in \Lambda$, the map $\Lambda \cap W^u_\epsilon(x) \ni y \mapsto E^g_2(y)$ is $C^1$. However in general the distribution $E^1_\alpha(x) (x \in \Lambda)$ does not have to be integrable (see [Pes]).

We now make the additional assumption that $E^u_\alpha(x) (x \in \Lambda)$ is integrable:

(I): There exist $\epsilon_0 > 0$ and a continuous $\phi_t$-invariant family $W^{u,1}_\epsilon(x) (x \in \Lambda)$ of $C^2$ submanifolds of $W^u_\epsilon(x)$ such that $T_x(W^{u,1}_\epsilon(x)) = E^1_u(x)$ for all $x \in \Lambda$, and moreover for any $\epsilon > 0$ and any $x \in \Lambda$, $\Lambda \cap W^u_\epsilon(x)$ is not contained in $W^{u,2}_\epsilon(x)$.

Roughly speaking, the latter means that the distribution $E^1_\alpha(x)$ is significantly involved in the dynamics of the flow over $\Lambda$.

The main result in [St3] is the following.

**Theorem 7.1.** Let $\phi_t$ and $\Lambda$ satisfy the conditions (LUPC) and (I). Then $\phi_t$ has a regular distortion along unstable manifolds over $\Lambda$.

A simplified case is presented by the following pinching condition:

(P): There exist constants $C > 0$ and $\beta > 0$ such that for every $x \in \Lambda$ we have

$$
\frac{1}{C} e^{\alpha_x t} \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C e^{\beta_x t} \|u\|, \quad u \in E^u(x), t > 0,
$$

for some constants $\alpha_x, \beta_x > 0$ depending on $x$ but independent of $u$ and $t$ with $\alpha \leq \alpha_x \leq \beta_x \leq \beta$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in \Lambda$. 

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Clearly the condition (P) is \( \text{LUPC} \) in the special case when \( E_2^x(x) = 0 \) for all \( x \in \Lambda \). Notice that when the local unstable manifolds are one-dimensional the condition (P) is always satisfied.

In higher dimensions a well-known example when (P) holds is the geodesic flow on a manifold with strictly negative sectional curvature satisfying the so called \( \frac{1}{4} \)-pinching condition (see [HP]). For open billiard flows (in any dimension) it was shown in [St2] that if the distance between the scatterers is large compared with the maximal sectional curvature of the boundaries, then the condition (P) is satisfied over the non-wandering set.

As a special case of Theorem 7.1, it is shown in [St3] that if \( \phi_t \) satisfies the condition (P) on \( \Lambda \), then \( \phi_t \) has a regular distortion along unstable manifolds over \( \Lambda \).

8 Appendix: Proof of Lemma 5.4

(a) Let \( u, u' \in \hat{U}_i \) for some \( i = 1, \ldots, k \) and let \( m \geq 1 \) be an integer. Given \( v \in \hat{U} \) with \( \sigma^m(v) = u \), let \( C[i] = C[i_0, \ldots, i_m] \) be the cylinder of length \( m \) containing \( v \) (see the beginning of section 3). Since the sequence \( i = [i_0, \ldots, i_m] \) is admissible, the Markov property implies \( i_m = i \) and \( \sigma^m(C[i]) = \hat{U}_i \). Moreover, \( \sigma^m : C[i] \rightarrow \hat{U}_i \) is a homeomorphism, so there exists a unique \( v' = v'(v) \in C[i] \) such that \( \sigma^m(v') = u' \). By (2.1), \( d(\sigma^j(v), \sigma^j(v'(v))) \leq \frac{1}{c_0 \gamma^{m-j}} d(u, u') \) for all \( j = 0, 1, \ldots, m - 1 \). This and (2.2) imply

\[
|f^{(a)}_m(v) - f^{(a)}_m(v')| \leq \sum_{j=0}^{m-1} \frac{\text{Lip}(f^{(a)}_j)}{c_0 \gamma^{m-j}} d(u, u') \leq \frac{T}{c_0 (\gamma - 1)} d(u, u') \leq \frac{T}{c_0 (\gamma - 1)} D(u, u') .
\]

Also notice that if \( D(u, u') = \text{diam}(C') \) for some cylinder \( C' = C[i_m, \ldots, i_p] \), then \( v, v'(v) \in C'' = C[i_0, \ldots, i_p] \) for some cylinder \( C'' \) with \( \sigma^m(C'') = C' \), so \( D(v, v'(v)) \leq \text{diam}(C'') \leq \frac{1}{c_0 \gamma^m} \text{diam}(C') = \frac{D(u, u')}{c_0 \gamma^m} \).

Using the above, \( \text{diam}(U_i) \leq 1 \), the definition of \( \mathcal{M}_a \), and the fact that \( \mathcal{M}_a^m 1 = 1 \) (hence \( \mathcal{M}_a^m 1 = 1 \)), and assuming \( A_0 \geq e^{-\frac{T}{c_0 (\gamma - 1)}} / c_0 \), we get

\[
\left| (\mathcal{M}_a^m H)(u) - (\mathcal{M}_a^m H)(u') \right| \leq \sum_{\sigma^m v = u} e^{f^{(a)}_m(v)} H(v) - \sum_{\sigma^m v = u} e^{f^{(a)}_m(v'(v))} H(v'(v))
\]

\[
\leq \sum_{\sigma^m v = u} e^{f^{(a)}_m(v)} (H(v) - H(v'(v))) + \sum_{\sigma^m v = u} \left| e^{f^{(a)}_m(v)} - e^{f^{(a)}_m(v'(v))} \right| H(v'(v))
\]

\[
\leq \sum_{\sigma^m v = u} \frac{e^{f^{(a)}_m(v)} B H(v'(v)) D(v, v'(v))}{\mathcal{M}_a^m H(u')} + \sum_{\sigma^m v = u} \left| e^{f^{(a)}_m(v)} - e^{f^{(a)}_m(v'(v))} - 1 \right| e^{f^{(a)}_m(v'(v))} H(v'(v))
\]

\[
\leq e^{\frac{T}{c_0 (\gamma - 1)}} B D(u, u') + e^{\frac{T}{c_0 (\gamma - 1)}} \frac{T}{c_0 (\gamma - 1)} D(u, u') \leq A_0 \left[ \frac{B}{\gamma^m} + \frac{T}{\gamma - 1} \right] D(u, u') .
\]

(b) The proof of this part is very similar to the above and we omit it. ■

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Figure 1

Temporal distance function: $\Delta(x, y) = t$
(LNIC) with $z' = \exp^y_x(v)$, $y_i = \pi_{\tilde{y}_i}(z) = [z, \tilde{y}_i]$, $z'_i = [z', \pi_{\tilde{y}_i}(z)]$, $s_i = \Delta(z', \pi_{\tilde{y}_i}(z))$, $i = 1, 2$. 