Abstract. The billiard in the exterior of a finite disjoint union $K$ of strictly convex bodies in $\mathbb{R}^d$ with smooth boundaries is considered. The existence of global constants $0 < \delta < 1$ and $C > 0$ is established such that if two billiard trajectories have $n$ successive reflections from the same convex components of $K$, then the distance between their $j$th reflection points is less than $C(\delta^j + \delta^{n-j})$ for a sequence of integers $j$ with uniform density in $1, 2, \ldots, n$. Consequently, the billiard ball map (though not continuous in general) is expansive. As applications, an asymptotic of the number of prime closed billiard trajectories is proved which generalizes a result of T. Morita [Mor], and it is shown that the topological entropy of the billiard flow does not exceed $\log(s-1) a$, where $s$ is the number of convex components of $K$ and $a$ is the minimal distance between different convex components of $K$.

1. Introduction
Let $K$ be a compact subset of $\mathbb{R}^d$ ($d \geq 2$) of the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_s,$$

where $K_i$ are compact strictly convex disjoint bodies in $\mathbb{R}^d$ with smooth boundaries $\Gamma_i = \partial K_i$. We assume that $s \geq 3$. Let

$$Q = \mathbb{R}^d \setminus K.$$

Denote $D = \text{diam}(K)$ and $a = \min_{i \neq j} \text{dist}(K_i, K_j)$. Clearly $0 < a < D$. The outer unit normal to $\partial K$ at $q \in \partial K$ will be denoted by $n(q)$.

Let $M$ be the unit tangent bundle over $Q$. We will identify $M$ with $Q \times S^{d-1}$ and write the points of $M$ in the form $x = (q, v)$, where $q \in Q$ and $v \in S^{d-1}$. The natural projection $\pi : M \rightarrow Q$ is given by $\pi(q, v) = q$. Then $\partial M = \pi^{-1}(\partial Q)$ is the boundary of the manifold $M$. Let $S_t$ be the billiard flow on $M$ (see [CFS]). Given $x = (q, v) \in M$, the trajectory $\gamma(x) = \{S_t(x) : t \geq 0\}$, is the usual billiard curve described by the motion of a point mass issued from $q$ in direction $v$. Inside $Q$ the mass moves with constant velocity, while reaching the boundary it bounces off according to the usual law of geometrical optic “the angle of incidence equals the angle of reflection”. The billiard flow $S_t$ obtained in this way is a special case of a dispersing billiard. Similar billiards but in bounded domains in $\mathbb{R}^d$ or the torus $\mathbb{T}^d$ have been extensively studied in connection with some important problems in physics (see [Si1], [Si3], [CFS], [DS], [BSC], [Ch], [CvE], [Wo] and the references there). The motivation for the considerations in the present paper come from some problems in scattering theory (cf. for example [I], [Si], [Burq], [PS]).

The natural phase space of the billiard is

$$M_1 = \{x = (q, v) \in \partial M : \langle n(q), v \rangle \geq 0\}.$$
For \( x = (q, v) \in M_1 \) denote by \( \varphi(x) \) the angle between \( n(q) \) and \( v \); then \( \cos \varphi(x) = (n(q), v) \). Let \( x = (q, v) \in M_1 \) be such that the billiard trajectory \( \gamma(x) \) has a common point with \( \partial Q \) and let \( t > 0 \) be the minimal number for which \( S_t(x) \in \partial M \). Define \( B(x) = S_t(x) \). Clearly, if \( B(x) = (p, w) \), then \( p \in \partial Q \) is the point at which the ray issued from \( q \) in direction \( v \) hits \( \partial Q \) and \( w \) is the reflection of \( v \) with respect to the tangent plane \( T_p(\partial Q) \) to \( \partial Q \) at \( p \). Let \( M' \) be the set of all \( x \in M_1 \) for which \( B(x) \) is defined. The map \( B : M' \to M_1 \) is called the billiard ball map.

In order to get a continuous flow, one has to consider the billiard flow on the quotient space \( \tilde{M} = M/\sim \), where \( \sim \) is the following equivalence relation on \( M : (q, v) \sim (p, w) \) iff \( q = p \) and either \( v = w \) or \( q = p \in \Gamma \) and \( v \) and \( w \) are symmetric with respect to the tangent plane \( T_q(\Gamma) \) to \( \Gamma \) at \( q \). Clearly \( \tilde{M} = \tilde{\pi}(M_1) \), where \( \tilde{\pi} : M \to \tilde{M} \) is the natural projection. The projection of the billiard flow \( S_t \) on \( M \) is a continuous flow on \( \tilde{M} \) which is also called billiard flow. We will use the same notation \( S_t \) for it. Also, avoiding the cumbersome notation \( \tilde{\pi}(x), S_t(\tilde{\pi}(x)) \), etc., we will write \( x \) and \( S_t(x) \) instead. It will be clear from the context whether we consider points in \( M \) or in \( \tilde{M} \).

Throughout \( \tilde{\rho} \) will be a metric on \( \tilde{M} \) such that
\[
\tilde{\rho}(\tilde{\pi}(q, v), \tilde{\pi}(p, w)) \leq \max\{\|q - p\|, \|v - w\|\} \quad (q, v, p, w) \in M.
\]

Clearly such a metric exists. Given \( x = (q, v) \in M \), denote by \((q_j(x), v_j(x))\) the \( j \)th reflection point of the forward trajectory \( \{S_t(x) : t \geq 0\} \), provided there are at least \( j \) reflections, and by \( t_j(x) \) be the corresponding time of reflection, that is \( S_{t_j}(x) = (q_j(x), v_j(x)) \). If \( x \in M_1 \), then \( B^j(x) = (q_j(x), v_j(x)) \). In this case we will assume that \( q_0(x) = q \) and \( v_0(x) = v \).

Let \( \kappa_0 > 0 \) be the minimal sectional curvature of \( \Gamma = \partial K \) with respect to the outer unit normal field \( n(q) \) of \( \Gamma \). It follows from the strict convexity of \( K_j \) for each \( j \) that \( \kappa_0 > 0 \). Finally, introduce the notation
\[
\begin{align*}
l_0 &= \frac{D^2}{a^2} + 2, \quad \varphi_0 = \frac{\pi}{2}(1 - \frac{1}{l_0}), \quad \kappa = \min\{\kappa_0, 2\kappa_0 \cos \varphi_0, \frac{2}{a}\}, \\
\delta &= \frac{1}{(1 + a\kappa)^{\frac{1}{1 - \gamma}}}, \quad \delta_1 = \frac{1}{(1 + a\kappa)^{\frac{1}{2\gamma - 1}}}, \quad C = \frac{24(s - 1)D^2(1 + a\kappa)^{\frac{2}{1 - \gamma}}}{a \log(1 + a\kappa)}.
\end{align*}
\]

Our main result in this paper is the following.

**Theorem 1.1.** Let \( K \) be an obstacle of the form (1) and let \( n \in \mathbb{N} \) and \( x = (q_0, v_0), x' = (q'_0, v'_0) \in M_1 \) be such that for each integer \( j \) with \( 0 \leq j \leq n \) the points \( q_j(x) \) and \( q_j(x') \) are well-defined and both belong to \( \Gamma_i \), for some \( i_j \). Then
\[
\|v_j(x) - v_j(x')\| \leq \frac{12D}{a}(\delta^j + \delta^{n-j}) \quad (0 \leq j \leq n),
\]
and there exists \( \tau \) with \( |\tau| \leq \|q_0 - q'_0\| < D \) such that
\[
\tilde{\rho}(S_t(x), S_{t+\tau}(x')) \leq C(\delta_1^t + \delta_1^{T-\tau}) \quad (0 \leq t \leq T),
\]
where \( T = t_n(x) \). Moreover, there exists another global constant \( \hat{C} > 0 \), independent of \( x, x' \) and \( n \) such that for every \( i = 0, 1, \ldots, n - l_0 + 1 \) there exists \( j = i, i + 1, \ldots, i + l_0 - 1 \) with
\[
\|q_j(x) - q_j(x')\| < \hat{C}(\delta^j + \delta^{n-j}) \quad \text{and} \quad |t_j(x) - t_j(x') - \tau| \leq \hat{C}(\delta_2^j + \delta_2^{n-j}),
\]

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where \( \delta_2 = \delta_1^a \).

The obstacle \( K \) is said to satisfy the condition (H) of M.Ikawa if the convex hull of every two convex components of \( K \) does not have common points with any other convex component of \( K \). In the special case when the condition (H) is satisfied, a result similar to the above theorem can be easily derived from the papers of M. Ikawa [I] and J. Sjöstrand (see Sect. B in [Sj]).

In the proof of Theorem 1.1 (see Sect. 2 and 3) we essentially use some ideas from Ya. Sinai [Si1], [Si2], M. Ikawa [I] (see also [Burq]) and J. Sjöstrand [Sj].

The billiard ball map \( B \) defines a dynamical system on the set \( M_0 \) of those \( x \in M_1 \) so that for each \( m \in \mathbb{N} \) both \( B^m(x) \) and \( B^{-m}(x) \) are defined. Clearly \( M_0 \) is a compact subset of \( \partial M \) and \( B : M_0 \rightarrow M_0 \) is a bijection. It is also clear that in general \( B \) is not continuous.

A natural invariant measure for \( B \) on \( M_1 \) is the Liouville measure \( d\nu = \cos \varphi dqdv \). However in the situation under consideration, \( M_0 \) has Lebesgue measure zero in \( M_1 \), i.e. in \( Q \times S^{d-1} \), so the Liouville measure is of little help when one considers the restriction of \( B \) on \( M_0 \). In the special case when \( K \) is the union of three congruent disks in \( \mathbb{R}^2 \) centered at the vertices of an equilateral triangle, an invariant measure on \( M_0 \) was constructed by Lopes and Markarian [LM] (see also [ChM]).

In some degenerate cases \( M_0 \) is finite, and when \( M_0 \) is infinite it is probably the union of a finite set and finitely many Cantor sets of Lebesgue measure zero. Though the points of discontinuity of \( B \) on \( M_1 \) form a set of measure zero, it is not clear whether in general the points of discontinuity of \( B \) on \( M_0 \) form a relatively small subset of \( M_0 \) (say, a subset of first Baire category). So, it is not quite clear how to define the topological entropy of \( B : M_0 \rightarrow M_0 \).

On the other hand, the billiard flow \( S_t \) is always continuous, so its topological entropy is well-defined. It is reasonable to consider the billiard flow on its largest bounded invariant set

\[
\tilde{L} = \{ S_t(\tilde{x}) : x \in M_0, t \in \mathbb{R} \}
\]

(this is actually the non-wandering set of \( S_t \) in \( M_1 \), and define the topological entropy \( h(S_t) = h(S_t) \) of the flow \( S_t : \tilde{L} \rightarrow \tilde{L} \).

There is a natural way of coding the billiard trajectories using the boundary components \( \Gamma_i \). Let

\[
\Sigma = \prod_{j=-\infty}^{\infty} F_0,
\]

where \( F_0 = \{1,2,\ldots,s\} \) and let \( \sigma : \Sigma \rightarrow \Sigma \) be the Bernoulli shift on \( \Sigma \), i.e. \( \sigma(\{i_j\}) = \{i'_j\} \), where \( i'_j = i_{j+1} \) for each \( j \). Given \( x \in M_0 \), for every integer \( j \) there exists an (unique) \( i_j \in \{1,2,\ldots,s\} \) such that \( B^j(x) \in \Gamma_{i_j} \). Define \( f(x) = \{i_j\} \). In this way one gets a map \( f : M_0 \rightarrow \Sigma \) such that \( f \circ B = \sigma \circ f \). In particular, \( \Sigma_K = f(M_0) \) is an invariant subset for \( \sigma \). It follows from Theorem 1.1 that \( f \) is invertible, so it provides a coding for the billiard on \( M_0 \). Using it, in Sect. 4 we prove the following.

**Theorem 1.2.** For every obstacle \( K \) of the form (1) the topological entropy \( h \) of the billiard flow \( S_t \) does not exceed \( \frac{\log(s-1)}{a} \).

For bounded two-dimensional dispersing billiards such result was obtained by Chernov [Ch] using the estimate of the number of periodic points from [St]. In fact, Chernov [Ch] showed that \( \liminf \frac{P_n(B)}{n} \geq h(B) \), where \( P_n(B) \) is the number of periodic points of \( B \) with period \( n \) and \( h(B) \) is the topological entropy of \( B \) (in this case it is well-defined). It was also proved in [Ch] that for bounded two-dimensional dispersing billiards the periodic points form a dense subset.
of the phase space. An interesting (but probably difficult) question is whether the same holds in the situation considered above. Theorem 1.3 below shows that this is so when the condition (H) is satisfied.

Clearly in some cases we may have $h = 0$. A reasonable conjecture seems to be that $h > 0$ whenever the set $M_0$ is infinite. Let us mention that for dispersing billiards in bounded domains $Q$ the entropy $h$ is always positive (cf. Sinai [Si3] and [Ch]). For such billiards it is also known that a symbolic dynamics with an infinite alphabet always exists [BSC] (see also [GO]).

In our situation, in general the map $f$ is not continuous and $\Sigma_K$ is not compact, so $f$ does not provide a symbolic dynamics for the billiard ball map $B$. However, in the special case when the condition (H) is satisfied, the billiard flow $S_t$ is an Axiom A flow, $\Sigma_K$ is compact and $f$ is a Lipschitz homeomorphism, so it does provide symbolic dynamics for $B$. In fact, in this case it is easily seen that

$$\Sigma_K = \{\{i_j\} : i_j \neq i_{j+1} \text{ for all } j\},$$

so the restriction $\sigma_K$ of the Bernoulli shift $\sigma$ to $\Sigma_K$ is a topological Markov chain with entropy $\log(s - 1)$. Consequently, the billiard flow $S_t$ is naturally isomorphic to a special flow over $\sigma_K$ which gives an asymptotic for the distribution of the closed billiard trajectories. For $n = 2$ the latter was obtained by T. Morita [Mor] using a result of Parry and Pollicott [PP]. In fact, combining Theorem 1.1 above (or the earlier result of Ikawa and Sjöstrand) and a modification of the argument in Sect. 3 of [Mor] (see Sect. 5 below), one gets a generalization of Morita’s result to all dimensions $n \geq 2$.

**Theorem 1.3.** Let $K$ satisfy the condition (H) and let $h$ be the topological entropy of the billiard flow $S_t : \tilde{L} \rightarrow \tilde{L}$. Then the flow $S_t$ on $\tilde{L}$ is naturally isomorphic to a special flow over $\sigma_K$ such that the corresponding closed orbits have the same periods. Moreover

$$\#\{\gamma : e^{hT_\gamma} \leq \lambda\} \frac{\log \lambda}{\lambda} \rightarrow 1 \quad \text{as } \lambda \rightarrow \infty,$$

where $\gamma$ runs over the set of all prime closed trajectories of $S_t$ and $T_\gamma$ is the period of $\gamma$, and

$$\frac{\log(s - 1)}{D} \leq h \leq \frac{\log(s - 1)}{a}.$$


2. **Pseudo-billiard trajectories**

Let $K$ be an arbitrary obstacle in $\mathbb{R}^n$ of the form (1). Throughout we will assume that $0 \in K$ and will denote by $U_0$ the open ball with centre 0 and radius $2D$. Clearly $K \subset U_0$.

By a pseudo-billiard trajectory we mean a curve of the form $\gamma = \bigcup_{j=0}^{k} [q_j, q_{j+1}]$ such that:

(i) $q_0 \neq q_1$, $q_k \neq q_{k+1}$ and for each $j = 1, \ldots, k-1$, $q_j \in \Gamma_{i_j}$, where $i_j = 1, \ldots, s$ and $i_{j+1} \neq i_j$ for each $j$;

(ii) for each $j = 1, \ldots, k$ either $q_j$ lies on the segment $[q_{j-1}, q_{j+1}]$ (then $q_j$ will be called an intersection point or a tangent point for the trajectory) or $[q_{j-1}, q_j]$ and $[q_j, q_{j+1}]$ satisfy the law of reflection at $q_j$ with respect to $\Gamma_{i_j}$ (in which case $q_j$ will be called a reflection point for the trajectory).

It is clear from this definition that in general a pseudo-billiard trajectory may intersect the interior of $K$ (see Fig. 1). As we will see below, in some cases it is more convenient to consider certain pseudo-billiard trajectories instead of the corresponding proper billiard trajectories in the exterior $Q$ of $K$. 

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We begin with the following simple but rather important lemma.

**Lemma 2.1.** Let $l_0$ and $\varphi_0$ be defined by (2). Then for every pseudo-billiard trajectory $q_0, q_1, \ldots, q_{l_0+1}$ there exists $j = 1, 2, \ldots, l_0$ such that $q_j$ is a reflection point for the trajectory and $\varphi_j < \varphi_0$, where $\varphi_j$ is the angle between the segment $[q_j, q_{j+1}]$ and $n(q_j)$.

**Proof.** Given an arbitrary pseudo-billiard trajectory $q_0, q_1, \ldots, q_{l_0+1}$, assume that $\varphi_j \geq \varphi_0$ for all $j = 1, \ldots, l_0$. Then, according to (2), $\psi_j = \frac{\pi}{2} - \varphi_j \leq \frac{\pi}{2}$. Denote $e_j = q_{j+1} - q_j$; then $\|e_j\| \leq a$ for each $j = 1, \ldots, l_0$. Since for $1 \leq i < j \leq l_0$ the angle between $e_i$ and $e_j$ does not exceed

$$\psi_i + \psi_{i+1} + \ldots + \psi_{j-1} < (j-i) \frac{\pi}{2[D^2/a^2]} \leq \frac{\pi}{2},$$

it follows that $\langle e_i, e_j \rangle > 0$. Therefore

$$D^2 \geq \|q_0 - q_1\|^2 = \|e_1\|^2 \geq \sum_{j=1}^{l_0-1} \|e_j\|^2 \geq (l_0 - 1)a^2.$$  

This yields $l_0 \leq \frac{D^2}{2a^2} + 1 < \left[\frac{D^2}{2a^2}\right] + 2$ which is a contradiction with (2). □

Let $n \geq s$. An $n$-tuple

$$\alpha = (i_1, i_2, \ldots, i_n)$$

will be called a configuration of length $n$ if $i_j = 1, \ldots, s$ for each $j$ and $i_j \neq i_{j+1}$ for $j < n$.

Let (7) be a fixed configuration. Consider the open sets

$$U_\alpha = \{q \in U_0 : a < \text{dist}(q, K_{i_1})\}, \quad V_\alpha = \{p \in \mathbb{R}^d : a < \text{dist}(p, K_{i_n})\}.$$  

For $(q, p) \in U_\alpha \times V_\alpha$ define

$$F_\alpha(q, \hat{q}; p) = \|q - q_1\| + \sum_{i=1}^{n-1} \|q_i - q_{i+1}\| + \|q_n - p\|$$

for $\hat{q} = (q_1, \ldots, q_n) \in \hat{K} = K_{i_1} \times \ldots \times K_{i_n}$. Thus, one gets a map

$$F_\alpha : U_\alpha \times K_{i_1} \times \ldots \times K_{i_n} \times V_\alpha \rightarrow \mathbb{R}.$$  

For a fixed $(q, p) \in U_\alpha \times V_\alpha$ there exists $\hat{q} \in \hat{K}$ such that $F_\alpha(q, \hat{q}; p) = \min_{\hat{q} \in \hat{K}} F_\alpha(q, \hat{q}, p)$. In general $\hat{q}$ is not unique but the pseudo-billiard trajectory

$$\gamma^{(\alpha)}(q, p) = [q, q_1] \cup \left( \bigcup_{j=1}^{n-1} [q_i, q_{i+1}] \right) \cup [q_n, p]$$

is uniquely determined. This follows for example from the argument in Sect. 3 of [St]. Set

$$q_0(q, p) = q, \quad q_{n+1}(q, p) = p,$$

and proceeding by induction on $j$, denote by $q_j(q, p)$ the point on the segment $\gamma^{(\alpha)}(q, p) \cap K_{i_j}$ which is closest to $q_{j-1}(q, p)$ (see Fig. 1).

Define

$$v_j(q, p) = \frac{q_{j+1}(q, p) - q_j(q, p)}{\|q_{j+1}(q, p) - q_j(q, p)\|} \in \mathbb{S}^{d-1}.$$
It is easily seen that for each $1 \leq j \leq n$, either one of the segments $[q_{j-1}(q, p), q_j(q, p)]$ and $[q_j(q, p), q_{j+1}(q, p)]$ intersects the interior of $K_{I_j}$, in which case $q_j(q, p)$ will be called an intersection point of $\gamma^{(\alpha)}(q, p)$, or these two segments satisfy the law of reflection at $q_j(q, p)$ with respect to $\Gamma_{I_j} = \partial K_{I_j}$. In the latter case $q_j(q, p)$ will be called a reflection point of $\gamma^{(\alpha)}(q, p)$. Clearly a reflection point may be a point of transversal reflection or a tangent point.

Figure 1

It is easy to derive from this definition that each $q_j(q, p)$ is a continuous function of $(q, p) \in U_\alpha \times V_\alpha$, and if $\gamma(q, p)$ is not tangent to $K_{I_j}$ at $q_j(q, p)$, then $q_j(q, p)$ is differentiable with respect to $q$ and $p$ in a neighbourhood of $(q, p)$.

Let $\xi_t^{(\alpha)}(q, p)$ be the shift of $q$ along the trajectory $\gamma^{(\alpha)}(q, p)$ after time $t$. Denote by $\omega_t^{(\alpha)}(q, p) \in S^{d-1}$ the direction of $\gamma^{(\alpha)}(q, p)$ at $\xi_t^{(\alpha)}(q, p)$. At reflection points $\xi_t^{(\alpha)}(q, p)$, one has to identify pairs of directions symmetric with respect to the corresponding tangent plane. In others words, as for the billiard flow $S_t$, one has to consider $S_t^{(\alpha)}(q, p) = (\xi_t^{(\alpha)}(q, p), \omega_t^{(\alpha)}(q, p))$ as a point in $\bar{M}$. Clearly $S_t^{(\alpha)}(q, p)$ is well-defined for $0 \leq t \leq T^{(\alpha)}(q, p)$, where $T^{(\alpha)}(q, p)$ is the length of the trajectory $\gamma^{(\alpha)}(q, p)$. For $t > T^{(\alpha)}(q, p)$ we define $\omega_t^{(\alpha)}(q, p) = v_m(q, p)$ and $\xi_t^{(\alpha)}(q, p) = p + (t - T^{(\alpha)}(q, p))v_m(q, p)$, while for $t < 0$ we set $\omega_t^{(\alpha)}(q, p) = -v + 2\langle v, n(q)\rangle n(q)$ and $\xi_t^{(\alpha)}(q, p) = q + tv$. Given $p \in V_\alpha$ and $q, q' \in U_\alpha$, set

$$t_p(q, q') = T^{(\alpha)}(q', p) - T^{(\alpha)}(q, p).$$

The following estimates and some of the arguments in their proofs are the main ingredients for the proof of Theorem 1.1.

**Lemma 2.2.** Let $m \in N$, $\alpha$ be a configuration of the form (7) and let $(q_0, p_0) \in U_\alpha \times V_\alpha$ be such that for each $j = 1, \ldots, m$, $q_j(q_0, p_0)$ is not a tangent point of $\gamma^{(\alpha)}(q_0, p_0)$ to $\Gamma_{I_j}$. Then

$$\|\partial_p \xi_{t_p(q, q')}^{(\alpha)}(q, p_0)\|_{q=q_0} < C_1 \delta^t, \quad \|\partial_p \xi_{t_p(q, q')}^{(\alpha)}(q, p_0)\|_{p=p_0} < C_1 \delta^{T-t}$$

(9)

for $0 \leq t \leq T$, where $T = T^{(\alpha)}(q_0, p_0)$ and $C_1$ is given by

$$C_1 = \frac{8D(s - 1)(1 + a\kappa)^{\frac{1}{s-1}}}{a \log(1 + a\kappa)}.$$  

(10)

To prove Lemma 2.2, we first obtain a local estimate about $\|v_j(q_0, p_0) - v_j(q_0, p)\|$. Clearly this will provide an estimate for $\|\partial_p v_j(q_0, p_0)\|$. In the same way one obtains an estimate for $\|\partial_q v_j(q_0, p_0)\|$. 

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Let

\[ 0 = \tau_0(q,p) < \tau_1(q,p) < \ldots < \tau_n(q,p) < \tau_{n+1}(q,p) = T^{(a)}(q,p) \]

be such that \( \xi_j^{(a)}(q,p) = q_j(q,p) \) for each \( j = 0, 1, \ldots, n, n + 1 \).

Fix for a moment \((q_0, p_0) \in U_\alpha \times V_\alpha\) such that the trajectory \( \gamma^{(a)}(q_0, p_0) \) does not contain points tangent to \( \Gamma \). For \( b > 0 \), denote by \( U_b \) the ball with center \( q_0 \) and radius \( b \) and by \( V_b \) the ball with center \( p_0 \) and radius \( b \). Take \( b < \frac{a}{7} \) so small that for each \((q, p) \in U_b \times V_b\) the trajectory \( \gamma(q, p) \) has no tangencies to \( \Gamma \). We may as well assume that \( b \) is so small that for each \((q, p) \in U_\alpha \times V_\alpha\) we have

\[ \tau_{j-1}(q_0, p_0) < \tau_j(q, p) < \tau_{j+1}(q_0, p_0) \quad (j = 1, \ldots, n). \]  

(11)

Given \( j = 1, \ldots, n \), denote by \( \varphi_j(q, p) \) the (smallest positive) angle between the vector \( q_{j+1}(q, p) - q_j(q, p) \) and the unit normal \( n(q_j(q, p)) \) to \( \Gamma \) at \( q_j(q, p) \). Let \( j_1 < j_2 < \ldots < j_k \) be the sequence of all \( j = 1, \ldots, n \) with \( \varphi_j(q_0, p_0) < \varphi_0 \). It follows from Lemma 2.1 that \( j_{i+1} - j_i \leq l_0 \) \((1 \leq l \leq k)\), therefore

\[ k \geq \frac{n}{l_0}. \]  

(12)

In what follows we also assume that \( b > 0 \) is chosen so small that

\[ \varphi_{j_l}(q, p) < \varphi_0 \quad \text{for all } (q, p) \in U_b \times V_b, \quad l = 1, \ldots, k. \]  

(13)

Denote by \( X \) a sphere with center \( q_0 \) and radius \( r \leq \frac{a}{7} \); then the sectional curvature of \( X \) is \( \frac{1}{r} \geq \frac{2}{7} \). Denote by \( X' \) the set of those \( q \in X \) so that there exists \( p \in V_b \) for which the first segment \([q_0, q_1(q, p)]\) of \( \gamma^{(a)}(q_0, p) \) intersects \( X \) at \( q \). Then \( X' \) is an open subset of \( X \) and \( n_X(q) = \frac{q - q_0}{\|q - q_0\|} \) is the outer normal field to \( X' \). Set \( X'_T = \{ \xi_l^{(a)}(q_0, p) : p \in V_b \} \), and for \( t \in [r, T] \), denote by \( \kappa_{t}(p) \) the minimal sectional curvature of \( X'_T \) at \( \xi_l^{(a)}(q_0, p) \). When \( \xi_l^{(a)}(q_0, p) \) is a reflection point of \( \gamma^{(a)}(q_0, p) \), denote

\[ \kappa_{lT}^+(p) = \lim_{t' \uparrow t} \kappa_{t'}(p), \quad \kappa_{lT}^-(p) = \lim_{t' \downarrow t} \kappa_{t'}(p). \]

Lemma 2.3. We have

\[ \kappa_{t}(p) \geq \frac{\kappa}{1 + l_0 D\kappa} \]  

(14)

for all \( p \in V_b \) and \( t \in [r, T] \).

Proof. Fix \( p \in V_b \) and denote \( t_l = \tau_{j_l}(q_0, p) \). Since \( j_{l+1} - j_l \leq l_0 \), we have \( t_{l+1} \leq t_l + l_0 D \) for all \( l = 1, \ldots, k - 1 \).

Let \( r \leq t < t_1(q_0, p) \). Then \( \kappa_t(p) = \frac{\kappa'}{1 + \kappa' t}, \) where \( \kappa' = \frac{1}{r} \) is the sectional curvature of \( X' \). Since for every \( z > 0 \), the function \( x \mapsto \frac{z}{1 + z x} \) is increasing on \([0, \infty)\), \( \kappa' \geq \frac{2}{7} \geq \kappa \) and \( t \leq l_0 D \)

imply \( \kappa_t(p) \geq \frac{\kappa}{1 + l_0 D\kappa} \).

Assume \( 1 < j_1 \). Then \( t_2 \leq l_0 D \) and \( \kappa_t^+ \geq \kappa_t^- \geq \frac{\kappa}{1 + t_1 D\kappa} \) (cf. Proposition A.1 in the Appendix) imply that for \( t \in [t_1, t_2] \) we have

\[ \kappa_t(p) \geq \frac{\kappa_t^+ \kappa_t^-}{1 + (t - t_1) \kappa_t^-} \geq \frac{\kappa}{1 + (t - t_1) \kappa_{t_1}^+} = \frac{\kappa}{1 + t\kappa} \geq \frac{\kappa}{1 + l_0 D\kappa}. \]

Proceeding in this way, one shows that (14) holds for all \( t \in [r, t_1] \). Next, Proposition A.1, (13) and (2) yield \( \kappa_t^+ \geq 2\kappa_0 \cos \varphi_{j_1}(q_0, p) \geq 2\kappa_0 \cos \varphi_0 \geq \kappa \). Now, repeating the above
argument, one gets that (14) holds for all \( t \in [t_1, t_2] \), etc. Using a simple induction, it follows that (14) holds for all \( t \in [r, T] \).

**Lemma 2.4.** (Ikawa [I], Lemma 3.7) Let \( X \) and \( Y \) be the boundaries of two disjoint compact strictly convex bodies in \( \mathbb{R}^n \). Suppose \( X \) and \( Y \) are smooth and denote by \( n_X \) the unit outer normal field to \( X \). Given \( \eta \in X \), let \( \eta(t) (t \geq 0) \) be the shift of \( \eta \) along the billiard trajectory in the exterior of \( Y \) issued from \( (\eta, n_X(\eta)) \). Suppose \( \eta, \zeta \in X \) are such that the trajectories \( \{ \eta(t) : t \geq 0 \} \) and \( \{ \zeta(t) : t \geq 0 \} \) hit transversally \( Y \) at \( \eta(t_1) \) and \( \zeta(t_2) \), respectively, and \( t_1 \leq t_2 \). Then \( \| \eta^{(i)}(t_2) - \zeta(t_2) \| \leq \| \eta(t_2) - \zeta(t_2) \| \), where \( \eta^{(i)}(t_2) = \eta + t_2 n_X(\eta) \).

**Figure 2**

We also need the following lemma the first part of which is a local version of Lemma 3.6 of Ikawa [I]. The proof is simple and we omit it.

**Lemma 2.5.** Let \( Y \) be a smooth convex surface in \( \mathbb{R}^d \) with an unit normal field \( n_Y(p) \), let \( p_0 \in Y \) and let \( \kappa' > 0 \) be such that the minimal sectional curvature of \( Y \) at \( p \) is greater than \( \kappa' \) for each \( p \in Y \).

(a) There exists an open neighbourhood \( W \) of \( p_0 \) in \( Y \) such that whenever \( q, p \in W \) we have

\[
\| (q + t n_Y(q)) - (p + t n_Y(p)) \| \geq (1 + t \kappa') \| q - p \|.
\]

(b) If the neighbourhood \( W \) of \( p_0 \) is sufficiently small, then for all \( q, p \in W \) and \( t \geq 0 \) we have \( \| q - p \| \leq \| (q + t n_Y(q)) - p \| \).

The following is the main part in the proof of the central Lemma 2.2.

**Lemma 2.6.** (a) For \( \delta \) given by (3) we have

\[
\| v_i(q_0, p_0) - v_i(q_0, p) \| < \frac{12}{a} \delta^{n-i} \| p_0 - p \|, \quad 0 \leq i \leq n.
\]  

(15) for each \( p \in V_b \). Moreover, if \( i = j \) for some \( l = 1, \ldots, k \), then we have

\[
\| q_i(q_0, p_0) - q_i(q_0, p) \| < \frac{3}{\cos \varphi_0} \delta^{n-i} \| p_0 - p \|.
\]  

(16)

(b) For every \( p \in V_b \) we have

\[
| T(q_0, p_0) - T(q_0, p) | \leq \| p_0 - p \|,
\]  

(17)

and

\[
| \tau_j(q_0, p_0) - \tau_j(q_0, p) | \leq \| q_j(q_0, p_0) - q_j(q_0, p) \| \leq D
\]  

(18)
for each $j = 1, \ldots, n$.

(c) Let $\delta_1$ be defined by (3). Then

$$\tilde{\rho} \left( S_t^{(\alpha)} (q_0, p_0), S_t^{(\alpha)} (q_0, p) \right) \leq C_1 \delta_1^{t-t} \| p_0 - p \|, \quad 0 \leq t, p \in V_0,$$

where $T = T^{(\alpha)} (q_0, p_0)$ and $C_1$ is given by (10).

**Proof.** Denote $\delta_0 = \frac{1}{1+\alpha}$, and notice that for $\delta$ given by (3) we have $\delta = \delta_0^{1-1}$.

Let $l_1 \leq l_2 \leq \ldots \leq l_m$ be the sequence of all indices $l = 0, 1, \ldots, n$ such that $q_l (q_0, p_0)$ is a reflection point of $\gamma^{(\alpha)} (q_0, p_0)$. For convenience set $l_0 = 0$ and $l_{m+1} = n$. Clearly, if $n \geq 1$, then $l_{i+1} - l_i \leq s - 1$ for each $i$, so $i \leq l_i \leq (s-1)i$ for each $i \geq 1$. In particular $n \geq (m+1)(s-1)$.

Given $1 \leq j \leq m$, denote $\eta_j = q_j (q_0, p_0)$, $\eta_i^{(r)} = \xi_{\gamma_j (q_0, p)} (q_0, p_0)$ and $\xi_j = q_j (q_0, p)$ if $\tau_j (q_0, p_0) \leq \tau_j (q_0, p)$ and $\eta_j = q_j (q_0, p)$, $\eta_i^{(r)} = \xi_{\gamma_j (q_0, p)} (q_0, p)$ and $\xi_j = q_j (q_0, p)$ otherwise.

Let $\eta_j^{(i)}$ be the point symmetric to $\eta_j^{(r)}$ with respect to the tangent plane $T_{\eta_j} (\partial K)$.

Let us prove (b) first. Let $1 \leq j \leq m$. We may assume $\tau_j (q_0, p_0) \leq \tau_j (q_0, p)$: the other case is similar. Let $t = \tau_j (q_0, p_0)$, $t' = \tau_j (q_0, p) - t$. Then the points $q_j (q_0, p_0) = \xi_{\gamma_j (q_0, p)}$ and $q' = \xi_{\gamma_j (q_0, p)} (q_0, p)$ lie on the (strictly) convex surface $Y = \{ \xi_{\gamma_j (q_0, p)} : p' \in V_0 \}$ and $q_j (q_0, p) = q' + t' n_Y (q', q)$, where $n_Y (q', q)$ is the outer unit normal to $Y$ at $q'$. This clearly implies $t' \leq \| q_j (q_0, p_0) - q_j (q_0, p) \|$, which proves (18).

To prove (17), we may assume that $T = T^{(\alpha)} (q_0, p_0) \leq T^{(\alpha)} (q_0, p)$; the other case is similar. Then $t = T^{(\alpha)} (q_0, p) - T \geq 0$. Consider the strictly convex surface $Y = \{ \xi_{\gamma_j (q_0, p)} : p' \in V_0 \}$. Clearly $p_0 \in Y$ and $p'' = \xi_{\gamma_j (q_0, p)} (q_0, p) \in Y$, while $p' = p'' + t n_Y (p'')$. The convexity of $Y$ now implies $\| p_0 - p \| \geq \| p' - p'' \| = t$. Thus $| T - T^{(\alpha)} (q_0, p) | = t \geq \| p_0 - p \|$, which proves (17).

Next, we are going to prove (a). First notice that Lemma 2.4 yields $\| \eta_j^{(r)} - \xi_1 \| \geq \| \eta_j^{(i)} - \xi_1 \|$, clearly, if $n \geq 1$. Applying Lemmas 2.4 and 2.5 (see Fig. 3 for one possible configuration), one gets $\| \eta_j^{(r)} - \xi_1 \| \geq \| \eta_j^{(i)} - \xi_1 \| \geq 1 + \alpha \| \eta_j^{(r)} - \xi_1 \|$. Finally, for $l = n$, using Lemma 2.5 (b) and (a), we have

$$\| p_0 - p \| \geq (1 + \alpha \kappa) \| \eta_j^{(r)} - \xi_1 \| .$$

A simple induction shows now that $\| p_0 - p \| \geq (1 + \alpha \kappa)^{m-j} \| \eta_j^{(r)} - \xi_1 \|$ for $1 \leq j < m$. This yields

$$\| \eta_j^{(r)} - \xi_1 \| \leq \delta_0^{m-j} \| p_0 - p \| .$$

Consequently, for $j < m$, $\| \eta_j^{(i)} - \xi_1 \| \leq \| \eta_j^{(r)} - \xi_1 \| \leq \delta_0^{m-j} \| p_0 - p \| .

Without loss of generality we may assume that $\xi_j$ and $\eta_{j+1}$ are on the same trajectory (the case on Fig. 3: the other case is similar). Denote $L = \| \eta_{j+1}^{(i)} - \xi_1 \|$. Then $L = \| \xi_j + \eta_j^{(r)} \|$ by construction and so one of the vectors $v_{i_j} (q_0, p_0)$ and $v_{i_j} (q_0, p)$ coincides with $\frac{1}{L} (\eta_{j+1}^{(i)} - \xi_1)$ and the other one with $\frac{1}{L} (\eta_{j+1}^{(r)} - \xi_1)$.

Now combining (21) and (22) gives

$$\| v_{i_j} (q_0, p_0) - v_{i_j} (q_0, p) \| \leq \frac{1}{L} \| (\eta_{j+1}^{(i)} - \xi_1) - (\eta_{j+1}^{(r)} - \xi_1) \| \leq \frac{1}{L} \left( \| \eta_j^{(r)} - \xi_1 \| + \| \eta_j^{(i)} - \xi_1 \| \right) \leq \frac{1}{L} \left( \| \eta_j^{(r)} - \xi_1 \| + \| \eta_j^{(i)} - \xi_1 \| \right) .$$

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\begin{align*}
&\leq \frac{1}{a} \delta_0^{m-j} \|p_0 - p\| + \frac{1}{a} \delta_0^{m-j-1} \|p_0 - p\| < \frac{4}{a} \delta_0^{m-j} \|p_0 - p\|.
\end{align*}

Hence the inequality
\begin{equation}
\|v_l(q_0, p_0) - v_l(q_0, p)\| < \frac{4}{a} \delta_0^{m-j} \|p_0 - p\| \tag{23}
\end{equation}
holds for any $j < m$. It also holds for $j = m$; this follows trivially from (20).

\textbf{Figure 3}

To prove (15), consider an arbitrary $i = 0, 1, \ldots, n$. The case $m = 0$ is trivial, so we will assume that $m \geq 1$. There exists $j = 0, 1, \ldots, m$ with $l_j \leq i < l_j + 1$. Then $n - i \leq l_{m+1} - l_j \leq (m + 1 - j)(s - 1)$. From this, (23) and (3) one gets
\begin{align*}
\|v_l(q_0, p_0) - v_l(q_0, p)\| &= \|v_l(q_0, p_0) - v_l(q_0, p)\| < \frac{4}{a} \delta_0^{m-j} \|p_0 - p\|
&< \frac{4}{a} \delta_0^{m-j} \|p_0 - p\| = \frac{12}{a} \delta_0^{m-j} \|p_0 - p\|,
\end{align*}
which proves (15).

\textbf{Figure 4}

Finally, let $l = r_i$ for some $i = 1, \ldots, k$. Then $\varphi_l < \varphi_0$ and also $l = l_j$ for some $j = 1, \ldots, m$. Consider the triangle $\eta_j \zeta_j \eta_j^{(r)}$ (see Fig. 4). We have $\psi \geq \frac{\pi}{2} - \varphi_0$. Clearly if $\psi > \frac{\pi}{2}$, then by (21)
\begin{equation}
\|\eta_j - \zeta_j\| < \|\eta_j^{(r)} - \zeta_j\| < \delta_0^{m-j} \|p_0 - p\|. \tag{21}
\end{equation}
Let $\psi \leq \frac{\pi}{2}$. Then, using (21) again, we get
\begin{equation}
\|q_l(q_0, p_0) - q_l(q_0, p)\| = \|\eta_j - \zeta_j\| = \frac{\sin \psi'}{\sin \psi} \|\eta_j^{(r)} - \zeta_j\| \leq \frac{1}{\cos \varphi_0} \delta_0^{m-j} \|p_0 - p\|.
\end{equation}
As above, $n - l = n - l_j \leq (m + 1 - j)(s - 1)$, therefore
\begin{equation}
\|q_l(q_0, p_0) - q_l(q_0, p)\| \leq \frac{1}{\delta_0 \cos \varphi_0} \delta_0^{\frac{n-i}{2}} \|p_0 - p\| \leq \frac{3}{\cos \varphi_0} \delta_0^{n-i} \|p_0 - p\|,
\end{equation}
which completes the proof.
which proves (16).

It remains to prove (c). Given \( t \in (0, T] \), take \( r \) with \( 0 < r < \min\{t, \frac{T}{2}\} \) and, as before, denote by \( X \) the sphere with centre \( q_0 \) and radius \( r \). Then the inequalities in (a) hold. There exists \( j = 0, 1, \ldots, m \) such that

\[
t' = \tau_j \leq t < \tau'' = \tau_{j+1}.
\]  

(24)

Set \( l = l_j \). Without loss of generality we may assume that \( \eta_j = q_l(q_0, p_0) \) and \( \zeta_j = q_l(q_0, p) \); the other case is considered similarly. With this assumption, (11) implies \( t' \leq \tau(q_0, p) < t'' \). Then there exists \( p' \in [p_0, p] \) such that \( \tau_j(q_0, p') = t \).

Figure 5

First, suppose that \( j < n \). Set \( \tilde{t} = t - t' \), \( v' = v_{j-1}(q_0, p_0) \), \( \eta' = \eta_j + \tilde{t}v' \), \( w = v_j(q_0, p) \), \( w'' = v_j(q_0, p_0) \), \( \eta'' = \eta_j + \tilde{t}w'' \), \( \zeta' = q_j(q_0, p') \), \( w' = v_j(q_0, p') \) (see Fig. 5). Using (22) and (24) for the pairs \( p_0, p' \) and \( p', p \), we get

\[
\|\xi_t^{(a)}(q_0, p_0) - \xi_t^{(a)}(q_0, p)\| \leq \|\xi_t^{(a)}(q_0, p_0) - \xi_t^{(a)}(q_0, p')\| + \|\xi_t^{(a)}(q_0, p') - \xi_t^{(a)}(q_0, p)\|
\]

\[
= \|\eta' - \zeta'\| + \|\zeta' - \eta''\| \leq \delta_0^{m-j}\|p_0 - p'\| + \delta_0^{m-j}\|p' - p\| = \delta_0^{m-j}\|p_0 - p\|.
\]

Therefore

\[
\tilde{\rho}(\xi_t^{(a)}(q_0, p_0), S_t^{(a)}(q_0, p)) \leq \tilde{\rho}(\xi_t^{(a)}(q_0, p_0), S_t^{(a)}(q_0, p')) + \tilde{\rho}(S_t^{(a)}(q_0, p'), S_t^{(a)}(q_0, p))
\]

\[
= \tilde{\rho}(\eta', v'), (\zeta', v'') + \tilde{\rho}(\zeta', w'), (\eta'', w''))
\]

\[
\leq \frac{4}{a}\delta_0^{m-j}\|p_0 - p'\| + \frac{4}{a}\delta_0^{m-j}\|p' - p\| = \frac{4}{a}\delta_0^{m-j}\|p_0 - p\|.
\]

Notice that \( a \leq t_{l+1} - t_l \leq D \). As in the proof of (a) above we have \( n - l = t_{m+1} - l_j \leq (m + 1 - j)(s - 1) \). Moreover \( t_l \leq t < t_{l+1} \) implies \( t + (n - l)D > T \), so \( n - l > \frac{T - T}{D} \). Therefore

\[
m + 1 - j > \frac{a^{-1}}{a^{-1}} > \frac{T - T}{(s-1)D},
\]

which implies \( \tilde{\rho}(S_t^{(a)}(q_0, p_0), S_t^{(a)}(q_0, p)) < \frac{4}{a}\delta_0^{m-j}\|p_0 - p\| < \frac{12}{a}\delta_1^{T-t}\|p_0 - p\| \). This proves (19) in the case when \( j < m \).

Finally, consider the case when \( j = m \), i.e. \( t \geq \tau_m(q_0, p_0) \). Let \( l = t_m \). If \( t \leq \tau_l(q_0, p) \), then (19) follows trivially from the above argument. So, we will assume that \( t > \max\{\tau_l(q_0, p), \tau_l(q_0, p)\} \). Notice that in this case (23) implies \( \omega_t^{(a)}(q_0, p_0) - \omega_t^{(a)}(q_0, p) = \|v_m(q_0, p_0) - v_m(q_0, p)\| < \frac{\|p_0 - p\|}{a} \). So, in order to prove (19), it remains to show that

\[
\|\xi_t^{(a)}(q_0, p_0) - \xi_t^{(a)}(q_0, p)\| < C_1\delta_1^{T-t}\|p_0 - p\|.
\]  

(25)

Again without loss of generality one may assume that \( \eta_m = q_l(q_0, p_0) \) and \( \zeta_m = q_l(q_0, p) \). Then for \( t' = t - \tau_l(q_0, p) \) we have \( \xi_t^{(a)}(q_0, p_0) = \eta_m + t'v_l(q_0, p_0) \) and \( \xi_t^{(a)}(q_0, p) = \zeta_m + t'v_l(q_0, p) \), and
clearly $\|\xi_{t}^{(a)}(q_0, p_0) - \xi_{t}^{(a)}(q_0, p)\| \leq \|p_0 - p\|$ holds for $t \leq T' = \min\{T, T^{(a)}(q_0, p)\}$. In this case we have $T - t \leq 2D$, so $3\delta_{1}^{t-D} \geq 3\delta_{1}^{t} = \frac{3}{(1+\alpha\tau)^{t}} \geq \frac{3}{1+\alpha\tau} \geq 1$. Therefore (cf. (10))

$$
\|\xi_{t}^{(a)}(q_0, p_0) - \xi_{t}^{(a)}(q_0, p)\| \leq \|p_0 - p\| \leq 3\delta_{1}^{T-t}\|p_0 - p\| < C_{1}\delta_{1}^{T-t}\|p_0 - p\|.
$$

Let $t > T'$. Consider the points $p' = \xi_{T'}^{(a)}(q_0, p_0)$ and $p'' = \xi_{T'}^{(a)}(q_0, p)$. For $v' = v_{t}(q_0, p_0)$ and $v'' = v_{t}(q_0, p)$, according to (23) we have $\|v' - v''\| < \frac{1}{2}\|p_0 - p\|$. Then

$$
\|\xi_{t}^{(a)}(q_0, p_0) - \xi_{t}^{(a)}(q_0, p)\| = \|(p' + (t - T')v') - (p'' + (t - T')v'')\|
\leq \|p' - p''\| + (t - T')\|v' - v''\| \leq \|p_0 - p\| + (t - T')\frac{4}{a}\|p_0 - p\|
\leq \left(1 + \frac{4(t - T')}{a}\right)\|p_0 - p\|.
$$

Using the fact that $\delta_{1}^{-1} = (1+\alpha\tau)^{(s-1)/s}$, one easily checks that $\lambda(u) = \frac{4D(s-1)}{a\log(1+\alpha\tau)}\delta_{1}^{-u} - \frac{4u}{a} - 1 \geq 0$ for all $u \geq 0$. Thus, $1 + \frac{4u}{a} \leq C_{2}\delta_{1}^{-u}$ for all $u \geq 0$, where

$$
C_{2} = \frac{4D(s-1)}{a\log(1+\alpha\tau)}.
$$

Consequently, $\|\xi_{t}^{(a)}(q_0, p_0) - \xi_{t}^{(a)}(q_0, p)\| < C_{2}\delta_{1}^{T-t}\|p_0 - p\|$ for all $t \geq T'$. Since $T' \leq T - D$, it follows that (cf. (10) again)

$$
\|\xi_{t}^{(a)}(q_0, p_0) - \xi_{t}^{(a)}(q_0, p)\| < C_{2}\delta_{1}^{T-t-D}\|p_0 - p\| = C_{2}(1 + \alpha\tau)^{-1}\delta_{1}^{-t}\|p_0 - p\|
\leq C_{1}\delta_{1}^{T-t}\|p_0 - p\|.
$$

This proves (25) which implies (19). \Box

Recall that for $q, q' \in U_{a}$ and $p \in V_{a}$, $t_{p}(q, q') = T^{(a)}(q, p) - T^{(a)}(q', p)$. This clearly implies $t_{p}(q, q') + t_{p}(q', q'') = t_{p}(q, q'')$.

**Corollary 2.7.** (a) For each $q \in U_{b}$ we have $\|v_{i}(q_0, p_0) - v_{i}(q, p_0)\| < \frac{12}{\cos\varphi_{0}}\delta_{i}\|q_0 - q\|$ for $1 \leq i \leq n$, and if $i = j_{i}$ for some $l = 1, \ldots, k$, then

$$
\|q_{i}(q_0, p_0) - q_{i}(q, p_0)\| < \frac{3}{\cos\varphi_{0}}\delta_{i}\|q_0 - q\|.
$$

(b) For every $q \in U_{b}$ we have $|T(q_0, p_0) - T(q, p_0)| \leq \|q_0 - q\|$, and

$$
|\tau_{j}(q_0, p_0) - \tau_{j}(q, p_0)| \leq \|q_{j}(q_0, p_0) - q_{j}(q, p_0)\| \leq D
$$

for $j = 1, \ldots, n$.

(c) Let $\delta_{1}$ be defined by (3). Then for every $q \in U_{b}$ we have

$$
\hat{\rho}(\xi_{t}^{(a)}(q_0, p_0), \xi_{t+\tau}^{(a)}(q_0, p_0)) \leq C_{1}\delta_{1}\|q_0 - q\| , \quad 0 \leq t \leq T,
$$

where $T = T^{(a)}(q_0, p_0)$, $\tau = t_{p}(q_0, q)$ and $C_{1}$ is given by (10).

**Proof.** Parts (a) and (b) follow immediately from the corresponding parts of Lemma 2.6, changing the roles of $q$ and $p$ and replacing the configuration $\alpha$ by $-\alpha = (i_{n}, i_{n-1}, \ldots, i_{2}, i_{1})$. 

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In this section Theorem 1.1 is proved. Exponential instability of the billiard (26).

Proof of Lemma 2.2. The second inequality in (9) follows from (19), while the first follows from (26). □

3. Exponential instability of the billiard
In this section Theorem 1.1 is proved.

Given \( x = (q,v) \in M' \), let \((q_j(x),v_j(x)) = B^j(x)\), provided \(B^j(x)\) is defined. Denote by \(t_j(x)\) the corresponding times of reflection, that is \(S_{t_j(x)}(x) = B^j(x)\).

Let \( x = (q_0,v_0) \in M_1, x' = (q_0',v_0') \in M_1 \) and \( n \in \mathbb{N} \) be as in Theorem 1.1. Then for each \( j = 0,1,\ldots,n \) there exists \( i_j \in \{1,2,\ldots,s\} \) with \( q_j(x),q_j(x') \in \Gamma_{i_j} \). Define \( \alpha \) by (7). Since \( q_0 = q_0(x), q_0' = q_0(x') \in \Gamma_{i_0} \), it is clear that both \( q_0 \) and \( q_0' \) belong to the set \( U_\alpha \) (cf. Sect. 2). In fact the whole segment \([q_0,q_0']\) is in \( U_\alpha \). Also, the points

\[
p_0 = q_0(x) \quad \text{and} \quad p_0' = q_0(x')
\]

are in \( \Gamma_{i_\alpha} \), so \([p_0,p_0'] \subset V_\alpha\).

It follows from the assumptions about \( x \) and \( x' \) that \( S^{(\alpha)}_{t_0}(q_0,p_0) = S_{t_0}(x) \) for \( 0 \leq t \leq T^{(\alpha)}(q_0,p_0) = t_n(x) \) and \( S^{(\alpha)}_{t_0}(p_0,q_0') = S_{t_0}(x') \) for \( 0 \leq t \leq T^{(\alpha)}(q_0',p_0') = t_n(x') \). Moreover

\[
\|q_0 - q_0'\| \leq \text{diam}(K_{i_0}) < D \quad \text{and} \quad \|p_0 - p_0'\| \leq \text{diam}(K_{i_\alpha}) < D.
\]

Given \( q \in U_\alpha, p \in V_\alpha \), denote by \( \Sigma_q \) (resp. \( \Sigma_p \)) the set of those \( p \in V_\alpha \) (resp. \( q \in U_\alpha \)) such that \( \gamma^{(\alpha)}(q,p) \) is tangent to \( \Gamma \) at \( q_j(q,p) \) for at least one \( j = 1,\ldots,m \). Clearly \( \Sigma_q \) is closed in \( V_\alpha \). Using the strict convexity of the components of \( \partial K \) and some arguments from ch.3 in [PS], it can be easily shown that \( \Sigma_q \) is always contained in a finite disjoint union of submanifolds of \( V_\alpha \) of dimension \( d-1 \). In particular \( \Sigma_q \) has Lebesgue measure zero in \( V_\alpha \). Moreover, for arbitrary points \( p,p' \in V_\alpha \), Thom’s transversality theorem (cf. for example ch. 2 in [GG]) shows that almost all (with respect to the \( C^\infty \) Whitney topology) smooth curves \( p(r), 0 \leq r \leq 1 \), with \( p(0) = p \) and \( p(1) = p' \) are transversal to \( \Sigma_q \) and therefore have only finitely many common points with \( \Sigma_q \). The sets \( \Sigma_p \) have similar properties.

Given \( p,p' \in V_\alpha \), denote by \( L(p,p') \) the infimum of the lengths of smooth curves \( p(r), 0 \leq r \leq 1 \), in \( V_\alpha \) such that \( p(0) = p \) and \( p(1) = p' \). In the same way one defines \( L(q,q') \) for \( q,q' \in U_\alpha \). It follows easily from the definitions of \( U_\alpha \) and \( V_\alpha \) that in general

\[
L(q,q') < 11D, \quad L(p,p') < 11D \quad (q,q' \in U_\alpha; p,p' \in V_\alpha).
\]

Since \([q_0,q_0'] \subset U_\alpha \) and \([p_0,p_0'] \subset V_\alpha \), we have

\[
L(q_0,q_0') = \|q_0 - q_0'\|, \quad L(p_0,p_0') = \|p_0 - p_0'\|.
\]

Lemma 3.1. For all \( q,q' \in U_\alpha \) and \( p,p' \in V_\alpha \) we have

\[
|T^{(\alpha)}(q,p) - T^{(\alpha)}(q,p')| \leq L(p,p') \quad \text{and} \quad |T^{(\alpha)}(q,p) - T^{(\alpha)}(q',p)| \leq L(q,q').
\]
Lemma 3.2. Let \( q, q' \in U_\alpha \) and \( p, p' \in V_\alpha \) be arbitrary points and let \( \tau = t_{\delta}(q, q') \). Then

\[
\|v_i(q, p) - v_i(q', p')\| \leq \frac{12}{a} (\delta^{n-i} L(p, p') + \delta^i L(q, q')) \quad (0 \leq i \leq n),
\]

and

\[
\tilde{\rho} \left( S_\alpha(p, q), S_\alpha'(q', p') \right) \leq C_1 \left( L(p, p') \delta_0 \frac{1}{\delta_1} + L(q, q') \delta_1 \frac{1}{\delta_0} \right)
\]

for all \( 0 \leq t \leq T - L(p, p') - L(q, q') \), where \( T = T_\alpha(q, p) \).

Proof. Consider a smooth curve \( p(r), 0 \leq r \leq 1 \), in \( V_\alpha \) with \( p(0) = p \) and \( p(1) = p' \) such that \( F = \{ r \in [0, 1] : p(r) \in \Sigma_q \} \) is finite. It then follows from Lemma 2.6 (b) that the function \( I = [0, 1] \setminus F \mapsto T_\alpha(q, p(r)) \) is differentiable and \( \| \frac{d}{dr} T_\alpha(q, p(r)) \| \leq \| p'(r) \| \) for each \( r \in I \). Since \( T_\alpha(q, p(r)) \) is continuous on \([0, 1]\), it follows that

\[
\| T_\alpha(q, p) - T_\alpha(q, p') \| \leq \int_0^1 \| \frac{d}{dr} T_\alpha(q, p(r)) \| dr \leq \int_0^1 \| p'(r) \| dr.
\]

This holds for every curve \( p(r) \) connecting \( p \) and \( p' \) and transversal to \( \Sigma_q \), therefore \( |T_\alpha(q, p) - T_\alpha(q, p')| \leq L(p, p') \) for all \( p, p' \in V_\alpha \).

In the same way one shows that \( |T_\alpha(q, p) - T_\alpha(q', p)| \leq L(q, q') \). Consequently

\[
|T_\alpha(q, p) - T_\alpha(q', p')| \leq |T_\alpha(q, p) - T_\alpha(q, p')| + |T_\alpha(q, p') - T_\alpha(q', p')| \leq L(p, p') + L(q, q'),
\]

which completes the proof of the assertion. \( \square \)

Lemma 3.2. \( q, q' \in U_\alpha \) and \( p, p' \in V_\alpha \) be arbitrary points and let \( \tau = t_{\delta}(q, q') \). Then

\[
\|v_i(q, p) - v_i(q', p')\| \leq \frac{12}{a} (\delta^{n-i} L(p, p') + \delta^i L(q, q')) \quad (0 \leq i \leq n),
\]

and

\[
\tilde{\rho} \left( S_\alpha(p, q), S_\alpha'(q', p') \right) \leq C_1 \left( L(p, p') \delta_0 \frac{1}{\delta_1} + L(q, q') \delta_1 \frac{1}{\delta_0} \right)
\]

for all \( 0 \leq t \leq T - L(p, p') - L(q, q') \), where \( T = T_\alpha(q, p) \).

Proof. Consider an arbitrary curve \( p(r), 0 \leq r \leq 1 \), in \( V_\alpha \) with \( p(0) = p \) and \( p(1) = p' \) which is transversal to \( \Sigma_q \). Then \( F = \{ r \in [0, 1] : p(r) \in \Sigma_q \} \) is finite. For \( r \in I = [0, 1] \setminus F \) it follows from Lemma 2.6 (a) that

\[
\| \frac{d}{dr} v_i(q, p(r)) \| \leq \frac{12}{a} \delta^{n-i} \| p'(r) \|.
\]

Therefore

\[
\|v_i(q, p) - v_i(q, p')\| \leq \int_0^1 \| \frac{d}{dr} v_i(q, p(r)) \| dr \leq \frac{12}{a} \delta^{n-i} \int_0^1 \| p'(r) \| dr,
\]

which yields

\[
\|v_i(q, p) - v_i(q', p')\| \leq \frac{12}{a} \delta^{n-i} L(p, p') \quad (0 \leq i \leq n).
\]

Let again \( r \in I = [0, 1] \setminus F \). Denote

\[
T' = \inf \{ T_\alpha(q, p(r)) : 0 \leq r \leq 1 \}.
\]

It then follows from Lemma 3.1 that \( 0 \leq T - T' \leq L(p, p') \).

Let \( 0 \leq t \leq T' \) and let \((r', r'')\) be an open interval contained in \( I \). Fix for a moment an arbitrarily small \( \epsilon > 0 \). For every \( r \in (r', r'') \) there exists \( b = b(r) > 0 \) so that (19) holds with \( q_0 = q \) and \( p_0 = p(r) \). Hence there exists a finite sequence \( r' < r_0 < r_1 < \ldots < r_k < r'' \) such that \( \| r' - r_0 \| < \epsilon, \| r'' - r_k \| < \epsilon, \)

\[
\left| \sum_{i=0}^{k-1} \| p(r_{i+1}) - p(r_i) \| - \int_{r'}^{r''} \| p'(r) \| dr \right| < \epsilon
\]

(30)
and \( \hat{\rho}\left(S_t^{(\alpha)}(q, p(r_i)), S_t^{(\alpha)}(q, p(r_{i+1}))\right) < C_1 \delta_1^{T'-t}||p(r_i) - p(r_{i+1})|| \) for each \( i = 0, 1, \ldots, k - 1 \). Set 
\[
l_i = \sum_{j=0}^{i-1} ||p(r_j) - p(r_{j+1})||.
\]
Then 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q, p(r_0)), S_t^{(\alpha)}(q, p(r_k))\right) \leq \sum_{i=0}^{k-1} \hat{\rho}\left(S_t^{(\alpha)}(q, p(r_i)), S_t^{(\alpha)}(q, p(r_{i+1}))\right) \leq C_1 \delta_1^{T'-t} \sum_{i=0}^{k-1} ||p(r_j) - p(r_{j+1})||.
\]
Combining this with (30), gives 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q, p(r_0)), S_t^{(\alpha)}(q, p(r_k))\right) \leq C_1 \delta_1^{T'-t}\left(\int_{r'}^{r''} ||p'(r)|| dr - \epsilon\right),
\]
and letting \( \epsilon \to 0 \), one gets 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q, p(r')), S_t^{(\alpha)}(q, p(r''))\right) \leq C_1 \delta_1^{T'-t}\int_{r'}^{r''} ||p'(r)|| dr.
\]
Summing up over all connected components \((r', r'')\) of \(I\), one obtains 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q, p), S_t^{(\alpha)}(q, p')\right) \leq C_1 \delta_1^{T'-t}\int_0^1 ||p'(r)|| dr
\]
for all \( t \geq 0 \). From this and \( \delta^{T'-t} \leq \delta^{-L(p,p')}\delta^{T-t} \) it follows that 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q, p), S_t^{(\alpha)}(q, p')\right) \leq C_1 \delta_1^{-L(p,p')}\delta_1^{T-t}L(p,p'), \ t \geq 0.
\]
(31)

Next, fix a curve \( q(r), 0 \leq r \leq 1, \) in \( U_\alpha \) which is transversal to \( \Sigma_p \) and such that \( q(0) = q, q(1) = q' \). Then \( E = \{r \in [0,1] : q(r) \in \Sigma_p\} \) is finite and \( J = [0,1] \setminus E \) is open in \([0,1]\).

As above, this time using Corollary 2.7 (a), one gets 
\[
||v_i(q, p') - v_i(q, p')|| \leq \frac{12}{a} \delta^i \int_0^1 ||q'(r)|| dr,
\]
for all \( i = 0, 1, \ldots, m \) which yields \( ||v_i(q, p') - v_i(q', p')|| \leq \frac{12}{a} \delta^i L(q, q') \) for \( 0 \leq i \leq m \). This and (29) clearly imply (27).

We are now going to get an estimate for \( \hat{\rho}\left(S_t^{(\alpha)}(q, p'), S_t^{(\alpha)}(q', p')\right) \) similar to (31). This is a bit more subtle than (31).

It follows from Lemma 3.1 that for 
\[
T'' = \inf\{T^{(\alpha)}(q(r), p') : 0 \leq r \leq 1\}
\]
we have \( T \geq T'' \geq T^{(\alpha)}(q, p') - L(q, q') \geq T - L(p, p') - L(q, q') \). Let \( 0 \leq t \leq T'' \) and let \((r', r'')\) be an open interval contained in \(I\). Fix for a moment an arbitrarily small \( \epsilon > 0 \). For every \( r \in (r', r'') \) there exists \( b = b(r) > 0 \) so that (26) holds with \( q_0 = q(r) \) and \( p_0 = p' \). Hence there exists a finite sequence \( r' < r_0 < r_1 < \ldots < r_k < r'' \) such that \( ||r' - r_0|| < \epsilon, ||r'' - r_k|| < \epsilon \), 
\[
\sum_{i=0}^{k-1} ||q(r_{i}+1) - q(r_i)|| - \int_{r'}^{r''} ||q'(r)|| dr \right) \leq \epsilon
\]
and 
\[
\hat{\rho}\left(S_t^{(\alpha)}(q(r_i), p'), S_t^{(\alpha)}(q(r_{i+1})(q(r_{i+1}), p'))\right) < C_1 \delta_1^i ||q(r_i) - q(r_{i+1})||
\]
(33)
for each \( i = 0, 1, \ldots, k - 1 \). Introduce the notation

\[
s_i = t_{p'}(q(r_0), q(r_i)) \quad \text{and} \quad s'_i = t_{p'}(q(r_i), q(r_{i+1})).
\]

Clearly \( |s_i| \leq \|q(r_0) - q(r_i)\| \leq l_i \) for all \( i \).

We will prove by induction on \( i \) that

\[
\hat{\rho} \left( S_{t}^{(a)}(q(r_0), p'), S_{t+s_i}^{(a)}(q(r_i), p') \right) < C_1 \delta_1^{-l_i} \delta_i^t l_i.
\]

(34)

It follows from (33) that (34) holds for \( i = 1 \). Assume that (34) holds for some \( i \) with \( 1 \leq i \leq k - 1 \).

Using (33) with \( t \) replaced by \( t + s_i \) and the fact that \( s_i + s'_i = s_{i+1} \), we get

\[
\hat{\rho} \left( S_{t+s_i}^{(a)}(q(r_0), p'), S_{t+s_i+1}^{(a)}(q(r_{i+1}), p') \right) \leq C_1 \delta_1^{s_{i+1}} \|q(r_i) - q(r_{i+1})\| \leq C_1 \delta_1^{s_{i+1}} \|q(r_i) - q(r_{i+1})\|.
\]

This and (34) for \( i \) imply

\[
\hat{\rho} \left( S_{t}^{(a)}(q(r_0), p'), S_{t+s_i}^{(a)}(q(r_i), p') \right) \leq \hat{\rho} \left( S_{t}^{(a)}(q(r_0), p'), S_{t+s_i}^{(a)}(q(r_i), p') \right)
\]

\[
+ \hat{\rho} \left( S_{t+s_i}^{(a)}(q(r_0), p'), S_{t+s_i+1}^{(a)}(q(r_{i+1}), p') \right)
\]

\[
\leq C_1 \delta_1^{-l_i} \delta_i^t \sum_{j=0}^{i-1} \|q(r_j) - q(r_{j+1})\| + C_1 \delta_1^{-l_i} \delta_i^t |q_i - q_{i+1}| \leq C_1 \delta_1^{-l_i} l_{i+1}.
\]

This proves (34) for \( i + 1 \). Thus, (34) holds for all \( i = 1, \ldots, k \). From (34) with \( i = n \) and (32) it follows that

\[
\hat{\rho} \left( S_{t}^{(a)}(q(r_0), p'), S_{t+p}(q(r_n), p') \right) < C_1 \delta_1^{-l_k} \delta_i^t l_k < C_1 \delta_1^{-\int_{r^*}^{r^*} \|q(r')\|dr} \delta_i^t \left( \int_{r^*}^{r^*} \|q(r')\|dr \right).
\]

Now letting \( \epsilon \to 0 \), we get

\[
\hat{\rho} \left( S_{t}^{(a)}(q(r'), p'), S_{t+p}^{(a)}(q(r'), p') \right) \leq C_1 \delta_1^{-\int_{r^*}^{r^*} \|q(r')\|dr} \delta_i^t \int_{r^*}^{r^*} \|q(r')\|dr.
\]

Finally, applying the latter to each of the finitely many connected components \( (r', r'') \) of \( I \) and summing up, one obtains

\[
\hat{\rho} \left( S_{t}^{(a)}(q, p'), S_{t+p}^{(a)}(q, p') \right) \leq C_1 \delta_1^{-\int_0^1 \|q(r')\|dr} \delta_i^t \int_0^1 \|q(r')\|dr.
\]

This clearly yields

\[
\hat{\rho} \left( S_{t}^{(a)}(q, p'), S_{t+p}^{(a)}(q', p') \right) \leq C_1 \delta_1^{-L(q,q')} L(q, q') \delta_i^t
\]

(35)

for all \( t \) with \( 0 \leq t \leq T'' \).

Let \( 0 \leq t \leq T - L(p, p') - L(q, q') \). Then \( 0 \leq t \leq T'' \), and so (35) and (31) imply

\[
\hat{\rho} \left( S_{t}^{(a)}(q, p), S_{t+p}^{(a)}(q', p') \right) \leq \hat{\rho} \left( S_{t}^{(a)}(q, p), S_{t+p}^{(a)}(q, p') \right) + \hat{\rho} \left( S_{t}^{(a)}(q, p'), S_{t+p}^{(a)}(q', p') \right)
\]

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\[ \leq C_1 \delta_1^{-L(p,p')} \delta_1^{T-t} L(p,p') + C_1 \delta_1^{-L(q,q')} L(q,q') \delta_1^t. \]

This proves the lemma. \(\square\)

We go on with the proof of Theorem 1.1. Using (27) with \(q = q_0, q' = q_0', p = p_0, p' = p_0'\), and the fact that \(L(q_0, q_0') = \|q_0 - q_0'\| < D\) and \(L(p_0, p_0') = \|p_0 - p_0\| < D\), one immediately gets (4).

To prove (5), define \(\tau = t_{p'}(q_0, q_0')\). Then clearly \(|\tau| \leq \|q_0 - q_0'\| < D\). Notice that \(\pi(S_t(x)) = \xi_t^{(\alpha)}(x)\) for \(0 \leq t \leq T\) and \(\pi(S_{t+r}(x')) = \xi_t^{(\alpha)}(x')\) for \(0 \leq t + \tau \leq T^{(\alpha)}(q_0, p_0')\). Since \(T^{(\alpha)}(q_0, p_0') - \tau = T^{(\alpha)}(q_0, p_0') \leq T - D\), it follows that \(\pi(S_{t+r}(x')) = \xi_t^{(\alpha)}(x')\) holds for \(D \leq t \leq T - D\).

Let \(D \leq t \leq T - 2D\). Then \(t \leq T - L(q_0, q_0') - L(p_0, p_0')\) and so (28) holds. This and the previous remark imply
\[ \tilde{\rho}(S_t(x), S_{t+r}(x')) \leq \frac{C_1 D}{\delta_1^1} (\delta_1^{T-t} + \delta_1^t). \tag{36} \]

Next, assume that \(0 \leq t \leq D\). Then, using (36) with \(t = D\), we get
\[ \tilde{\rho}(S_t(x), S_{t+r}(x')) \leq \tilde{\rho}(S_t(x), S_{D}(x)) + \tilde{\rho}(S_D(x), S_{D+r}(x')) + \tilde{\rho}(S_{D+r}(x'), S_{t+r}(x')) \leq 2D + \frac{C_1 D}{\delta_1^1} (\delta_1^{T-D} + \delta_1^0) < 2D + 2C_1 D < 3DC_1 = \frac{3DC_1}{\delta_1^1} \delta_1^T < \frac{3DC_1}{\delta_1^1} (\delta_1^{T-t} + \delta_1^t). \]

In the same way, for \(T - 2D \leq t \leq T\), one obtains
\[ \tilde{\rho}(S_t(x), S_{t+r}(x')) \leq \tilde{\rho}(S_t(x), S_{D}(x)) + \tilde{\rho}(S_D(x), S_{D+r}(x')) + \tilde{\rho}(S_{D+r}(x'), S_{t+r}(x')) \leq 2D + \frac{C_1 D}{\delta_1^1} (\delta_1^{2D} + \delta_1^{T-2D}) < 2D + 2C_1 D \delta_1^1 < 3DC_1 \delta_1^D = \frac{3DC_1}{\delta_1^1} \delta_1^{2D} < \frac{3DC_1}{\delta_1^1} (\delta_1^{T-t} + \delta_1^t). \]

Hence for each \(t \in [0, T]\) we have \(\tilde{\rho}(S_t(x), S_{t+r}(x')) \leq \frac{3DC_1}{\delta_1^1} (\delta_1^{T-t} + \delta_1^t)\). Since the constant \(C\) determined by (3) satisfies \(C = \frac{3DC_1}{\delta_1^1}\), (5) follows.

It remains to prove (6). Given \(i\), it follows from Lemma 2.1 that there exists \(j = i, i + 1, \ldots, i + l_0 - 1\) with \(\varphi_j(x) < \varphi_0\), where \(\varphi_j(x)\) is the angle between the vector \(v_{j+1}(x) - v_j(x)\) and the unit normal \(n(q_j(x))\). Fix \(j\) with this property. The strict convexity of the components of \(K\) implies the existence of a global constant \(A > 0\) such that if \(\xi, \xi' \in \Gamma_i\) for some \(i = 1, \ldots, s\), then
\[ \|\xi - \xi'\| \leq A\|n(\xi) - n(\xi')\|. \tag{37} \]

Set
\[ \epsilon = \frac{12D}{a} \min\{\delta_1^{j-1} + \delta_1^{n-j+1}, \delta_1^j + \delta_1^{n-j}\}. \]

It is sufficient to consider the case \(\epsilon < \frac{1}{2} \cos \varphi_0\). Denote \(v = v_j(x) - v_{j-1}(x), w = v_{j+1}(x) - v_j(x), \xi = q_j(x), v' = v_j(x') - v_{j-1}(x'), w' = v_{j+1}(x') - v_j(x'), \xi' = q_j(x')\). Then \(\|v - v'\| < \epsilon\) and \(\|w - w'\| < \epsilon\). Notice that \(\|w - v\| = \langle w - v, n(\xi) \rangle = 2 \cos \varphi_j(x) \geq \cos \varphi_0\). Therefore
\[ \|n(\xi) - n(\xi')\| = \| \begin{pmatrix} w - v \\ w - v' \end{pmatrix} - \begin{pmatrix} w' - v' \\ w' - v'' \end{pmatrix} \| \leq \frac{\|w - v\| - \|w' - v'\|}{\|w - v''\|} \leq \frac{2\epsilon}{2 \cos \varphi_0} + \frac{2\epsilon}{2 \cos \varphi_0 (2 \cos \varphi_0 - \epsilon)} < \frac{3\epsilon}{\cos^2 \varphi_0}. \]
This and (37) imply
\[ \|q_j(x) - q_j(x')\| = \|\xi - \xi'\| \leq \frac{3Ac}{\cos^2 \varphi_0} \leq \frac{36AD}{a\cos^2 \varphi_0} (\delta_j^2 + \delta^{n-j}) , \]
which proves the first inequality in (6).

To prove the second inequality in (6), we may assume that \( t_j(x) \geq t_j(x') + \tau \); the other case is similar. Set \( \eta = \pi(S_{t_j(x)+\tau}(x')) \), and use (5) with \( t = t_j(x) \). This gives
\[ \|\eta - \eta\| = \|\pi(S_{t_j(x)}(x)) - \pi(S_{t_j(x)+\tau}(x'))\| \leq C \left( \delta^{s_j}(x) + \delta^{T_j-t_j(x)} \right) \leq C(\delta_1^{s_j} + \delta_1^{(n-j)a}) = C(\delta_2 + \delta_2^{n-j}), \]
where \( \delta_2 = \delta_1^a = \delta^{s_2} \). Since
\[ \cos \varphi_j(x') = \frac{1}{2} \|w' - v'\| > \cos \varphi_j(x) - \epsilon > \cos \varphi_0 - \frac{1}{2} \cos \varphi_0 = \frac{1}{2} \cos \varphi_0, \]
we have
\[ \|\eta - \eta'\| \leq \frac{\|\xi - \eta\|}{\sin(\pi - \varphi_j(x'))} = \frac{\|\xi - \eta\|}{\cos \varphi_j(x')} \leq \frac{2C}{\cos \varphi_0} (\delta_2 + \delta_2^{n-j}). \]
On the other hand, in this case clearly \( |t_j(x) - t_j(x') - \tau| = ||\eta - \xi'|| \), so this proves the second inequality in (6). \( \Box \)

4. Estimate of the entropy
In this section Theorem 1.2 is proved. In what follows we use the Bowen-Dinaburg definition of topological entropy (see for example Sect. 7.2 in [W]).

Consider the Bernoulli shift \( \sigma : \Sigma \rightarrow \Sigma \) and the map \( f : M_0 \rightarrow \Sigma_K \subset \Sigma \) defined in Sect. 1. Here \( \Sigma_K = f(M_0) \) is an invariant subset for the Bernoulli shift \( \sigma \) but \( \Sigma_K \) is not necessarily closed in \( \Sigma \). Define the metric \( \omega_0 \) on \( F_0 = \{1, 2, \ldots , s\} \) by \( \omega_0(i, i') = 0 \) if \( i = i' \) and 1 otherwise. Then
\[ \omega(\{i_j\}, \{i'_j\}) = \sum_{j=-\infty}^{\infty} \delta_{|j|+1} \omega_0(i_j, i'_j) \]
defines a metric on \( \Sigma \) which generates the Tichonov (product) topology on \( \Sigma \). Here \( \delta_1 \) is the constant defined by (3). It follows easily from Theorem 1.1 that \( g = f^{-1} : \Sigma_K \rightarrow M_0 \) is a continuous map. However in general it is not Lipshitz.

Consider the topological Markov chain \( \sigma_A : \Sigma_A \rightarrow \Sigma_A \) determined by the matrix \( A = \{a_{ij}\}_{i,j=1}^{s}, a_{ij} = 0 \) if \( i = j \) and 1 otherwise. That is,
\[ \Sigma_A = \{\{b_i\} : a_{b_i, b_{i+1}} = 1 \text{ for all } i\}, \]
and \( \sigma_A \) is the restriction of \( \sigma \) to \( \Sigma_A \). It is known (cf. Theorem 7.13 in [W] for example) that \( h(\sigma_A) = \log \lambda \), \( \lambda \) being the maximal eigenvalue of the matrix \( A \). In our case the maximal eigenvalue of \( A \) is \( s - 1 \), so \( h(\sigma_A) = \log(s - 1) \).

It follows from the definition of \( \Sigma_K \) that it is contained in \( \Sigma_A \), so \( \Sigma_K \subset \Sigma_A \). Consequently, \( h(\sigma, \Sigma_K) = h(\sigma, \Sigma_K) \leq h(\sigma_A) = \log(s - 1) \).

Let \( \epsilon = \delta_1^k \) for some \( k \in \mathbb{N} \). We assume that \( k \) is so big that \( 2C\epsilon < a \). Here \( \delta_1 \) and \( C \) are the constants defined by (3).

Set \( \epsilon' = 2C\epsilon + \epsilon \). Consider an arbitrary \( n \in \mathbb{N} \) and choose a subset \( E \) of \( \Sigma_K \) of minimal cardinality which is \( (n + 1, \epsilon) \) spanning for \( \sigma \) on \( \Sigma_K \), i.e. \( |E| = r_{n+1}(\epsilon, \Sigma_K, \sigma) \). Define \( m = \)
Since \( T \) is defined by \( \delta^t \), which implies \( i_l = i_l' \) for all \( l = -k, -k+1, \ldots, -1, 0, 1, \ldots, k+n \), where \( f(x) = (i_l) \) and \( f(y) = (i_l') \). Applying Theorem 1.1 to \( \hat{x} = B^{-k}(x) \) and \( \hat{y} = B^{-k}(y) \), we get that there exists \( \tau \) with \( \left| \tau \right| \leq D \) such that
\[
\tilde{p}(S_t(\hat{x}), S_{t+\tau}(\hat{y})) \leq C(\delta^t_1 + \delta^{T-t}_1) \quad (0 \leq t \leq T),
\]
where \( T = t_{n+2k+1}(\hat{x}) \). For \( t = t_k(\hat{x}) \) we have \( S_{t}(\hat{x}) = x \) and therefore
\[
\tilde{p}(x, S_{t_k(\hat{x})+\tau}(\hat{y})) < C(\delta^{T-t_k(\hat{x})}_1 + \delta^{t_k(\hat{x})}_1).
\]
Take \( j_0 \) such that \( \frac{k}{m} \leq t_k(\hat{x}) + \tau < \frac{k}{m} + 1 \). Then \( z = S_{j_0}(\hat{y}) \in E' \). For \( 0 \leq t \leq na \) we have
\[
\tilde{p}(S_t(x), S_t(z)) = \tilde{p}(S_{t+t_k(\hat{x})}(\hat{x}), S_{t+t_k(\hat{x})}(\hat{y})) \leq \tilde{p}(S_{t+t_k(\hat{x})}(\hat{x}), S_{t+t_k(\hat{x})+\tau}(\hat{y})) + \tilde{p}(S_{t+t_k(\hat{x})+\tau}(\hat{y}), S_{t+t_k(\hat{x})+\tau}(\hat{y})) < C(\delta^{t_k(\hat{x})}_1 + 1) \frac{1}{m}.
\]
Here we have taken into account that \( T - t_k(\hat{x}) = t_{n+2k+1}(\hat{x}) - t_k(\hat{x}) = t_{n+k+1}(x) \). Since \( t_k(\hat{x}) \geq ka \) and \( t_{n+k+1}(x) - t \geq (n+k+1)a - [na] > ka \) for \( 0 \leq t \leq na \), it follows from the above estimates that \( \tilde{p}(S_t(x), S_t(z)) < 2C\delta^{ka}_1 + \frac{1}{m} < \epsilon' \) for \( 0 \leq t \leq na \). Thus, for all integers \( j = 0, 1, \ldots, \lceil na \rceil \) we have \( \tilde{p}(S_j(x), S_j(z)) < \epsilon' \). This means that \( E' \) is \((m_n, \epsilon')\)-spanning for the map \( S_k \) on \( \tilde{L} \), and according to (39),
\[
r_{m_n}(\epsilon', \tilde{L}, S_1) \leq |E'| \leq r_{n+1}(\epsilon, \Sigma_K, \sigma) + 1 + \frac{1}{m}.
\]
Since \( \frac{m_n}{n} \to a \) as \( n \to \infty \), it now follows that \( ar(\epsilon', \tilde{L}, S_1) \leq r(\epsilon, \Sigma_K, \sigma) \), which implies \( ah(S_1) \leq h(\sigma, \Sigma_K) \leq \log(s-1) \). Therefore \( h(S_1) \leq \frac{1}{a} \log(s-1) \) which proves the theorem.

\[\Box\]

5. Billiards without singularities

Throughout we assume that \( K \) is of the form (1) and satisfies the condition (H). Let \( \rho \) be the metric on \( M \subset \mathbb{R}^n \times \mathbb{R}^n \) defined by
\[
\rho((q, v), (p, w)) = \max\{||q - p||, ||v - w||\}.
\]
Under the condition (H), one can add the following to the statement of Theorem 1.1.

**Theorem 5.1.** Let $K$ have the form (1) and satisfy the condition (H). Then there exists a global constant $C_2 > 0$ with the following property. If $n \in \mathbb{N}$ and $x = (q_0, v_0), x' = (q'_0, v'_0) \in M_1$ are such that for each integer $j$ with $1 \leq j \leq n$ the points $q_j(x)$ and $q_j(x')$ are well-defined and both belong to $\Gamma_{ij}$ for some $i_j$, then

$$r(B^j(x), B^j(x')) < C_2(\delta_1^j + \delta_2^{n-j}) \quad (0 \leq j \leq n-1)$$

(40)

and there exists $\tau$ with $|\tau| \leq \|q_0 - q'_0\| < D$ such that

$$|t_j(x) - t_j(x') - \tau| \leq C_2(\delta_1^j + \delta_2^{n-j}),$$

(41)

where $\delta_2 = \delta_1^a$ and $\delta_1$ is defined by (3).

The inequality (40) in the above theorem follows from [I] (see also [Burq]) and [Sj]. Here we derive both (40) and (41) from Theorem 1.1.

**Proof.** It follows from the condition (H) that there exists $\psi_0 \in (0, \frac{\pi}{2})$ such that for any three successive reflection points $q_1, q_2, q_3$ of a billiard trajectory in $Q$ the angle between the vectors $q_3 - q_2$ and $n(q_2)$ is less than or equal to $\psi_0$. Using this and (4), it follows from the last argument in Sect. 3 that there exists a constant $C_2 > 0$ such that both (40) and (41) hold. $\square$

In what follows we use the notation from the beginning of Sect. 4 and assume that $K$ satisfies the condition (H). Then the map $f : M_0 \rightarrow \Sigma_K$ is clearly continuous. Thus, $\Sigma_K$ is compact and $f$ is a homeomorphism between $M_0$ and $\Sigma_K$. In fact, it follows from Theorem 5.1 that $g = f^{-1} : \Sigma_K \rightarrow M_0$ is a Lipshitz map. We claim that

$$\Sigma_K = \Sigma_A,$$

(42)

$\Sigma_A$ being given by (38). Clearly $\Sigma_K \subset \Sigma_A$. Let $\xi = \{i_j\}$ be a periodic element of $\Sigma_A$. It then follows from Sect. 2 of [St] that there exists a periodic point $x \in M_0$ of the billiard ball map $B$ such that $f(x) = \xi$. Hence $\xi \in \Sigma_K$. Since the periodic elements of $\Sigma_A$ are dense in $\Sigma_A$ and both $\Sigma_K$ and $\Sigma_A$ are compact, (42) follows.

**Proof of Theorem 1.3.** The above remark shows that $f : M_0 \rightarrow \Sigma_K = \Sigma_A$ is a homeomorphism which conjugates the billiard ball map $B : M_0 \rightarrow M_0$ and the topological Markov chain $\sigma_K : \Sigma_K \rightarrow \Sigma_K$. Consider the map $\lambda : \Sigma_K \rightarrow \mathbb{R}$, defined by

$$\lambda(\xi) = t_1(g(\xi)) = \|q_1(g(\xi)) - q_0(g(\xi))\|,$$

and let

$$\Sigma^\lambda_K = \{ (\xi, t) \in \Sigma_K \times \mathbb{R} : 0 \leq t \leq \lambda(\xi) \}.$$

Then the map $\Lambda : \Sigma^\lambda_K \rightarrow L$, $\Lambda(\xi, t) = S_t(g(\xi))$, is a homeomorphism which defines a conjugacy between the special flow over $\sigma_K$ on $\Sigma^\lambda_K$ and the billiard flow $S_t : L \rightarrow L$.

Let $h$ be the topological entropy of $S_t$, i.e. $h = h(S_t)$. It follows from Theorem 1.2 that $h \leq \frac{1}{\delta} \log(s - 1)$. Since $B : M_0 \rightarrow M_0$ and $\sigma_K : \Sigma_K \rightarrow \Sigma_K$ are conjugate, we have $h(B) = h(\sigma_K) = \log(s - 1)$. Now applying a straightforward modification of the argument in Sect. 4, one gets $\frac{1}{\delta} \log(s - 1) \leq h$. We omit the details.
It remains to establish the asymptotic $\#\{\gamma : e^{\beta T_\gamma} \leq \lambda\} \frac{\log \lambda}{\lambda} \to 1$ as $\lambda \to \infty$. As mentioned in [Mor] (see Remark 3.1 there), this follows from Parry and Pollicott [PP] and the following lemma.

**Lemma 5.2.** There do not exist $c > 0$ and functions $\mu : \Sigma_K \to \mathbb{R}$ and $m : \Sigma_K \to \mathbb{Z}$ such that

$$\lambda = \mu \circ \sigma_K - \mu + cm. \quad (43)$$

**Proof of Lemma 5.2.** Suppose there exist $c > 0$ and functions $\mu$ and $m$ which satisfy (43). This implies that every periodic billiard trajectory in $Q$ has a period (length) of the form $ck$, $k$ being an integer. Using an appropriate dilation in $\mathbb{R}^d$, we may assume that $c = 1$. This means that every periodic billiard trajectory in $Q$ has integer length. We are now going to show that this is impossible when $s \geq 3$.

In what follows we consider configurations

$$\alpha = (i_0, i_1, \ldots, i_{n-1}, i_n = i_0) \quad (44)$$

such that $i_j \in \{1, 2, 3\}$ for all $j$. Clearly for every $\alpha$ of the form (44) there exists a unique $x = x(\alpha) \in M_0$ with $\pi(x) \in \Gamma_0$ such that $B^t(x) = x$ and $B^j(x) \in \Gamma_i$ for all $j = 0, 1, \ldots, n$. Let $q_1 \in \Gamma_1$, $q_2 \in \Gamma_2$ be such that $[q_1, q_2]$ is orthogonal to $\Gamma_1$ at $q_1$ and to $\Gamma_2$ at $q_2$, i.e. $d_0 = \|q_1 - q_2\| = \text{dist}(K_1, K_2)$. By assumption, we have $d_0 \in \mathbb{N}$. Given $k \in \mathbb{N}$, define

$$\alpha_k = (1, 2, 1, 2, \ldots, 1, 2, 3, 1), \quad 2k \text{ terms}$$

and let

$$q_1^{(k)} \in \Gamma_1, q_2^{(k)} \in \Gamma_2, \ldots, q_{4k-1}^{(k)} \in \Gamma_1, q_{4k}^{(k)} \in \Gamma_2, q_{4k+1}^{(k)} \in \Gamma_3, q_{4k+2}^{(k)} \in \Gamma_1$$

be the successive reflection points of the corresponding billiard trajectory. Denote by $T_k$ the length of this trajectory. Cutting the last two terms of $\alpha_k$, one gets the configuration

$$\beta_k = (1, 2, 1, 2, \ldots, 1, 2). \quad 2k \text{ terms}$$

Clearly the billiard trajectories

$$q_1^{(k)}, q_2^{(k)}, \ldots, q_{4k-1}^{(k)}, q_{4k}^{(k)} \quad \text{and} \quad q_1, q_2, \ldots, q_1, q_2$$

both follow the configuration $\beta_k$. It then follows from Theorem 5.1 that

$$\|q_{2k-1}^{(k)} - q_1\| < C\delta^{2k}, \quad \|q_{2k}^{(k)} - q_2\| < C\delta^{2k}, \quad \|q_{2k+3}^{(k)} - q_1\| < C\delta^{2k-4}. \quad (45)$$

for some global constant $C > 0$ independent of $k$.

Next, consider the following periodic configuration which is a cyclic permutation of $\alpha_{k-1}$:

$$\alpha_{k-1}' = (1, 2, 1, 2, \ldots, 1, 2, 3, 1, 2, 1, 2, \ldots, 1, 2, 1). \quad 2k-2 \text{ terms} \quad 2k-2 \text{ terms}$$

Clearly the successive reflection points of the corresponding periodic billiard trajectory are

$$q_{2k-1}^{(k-1)}, q_{2k}^{(k-1)}, \ldots, q_{4k-5}^{(k-1)}, q_{4k-4}^{(k-1)}, q_{4k-3}^{(k-1)}, q_1^{(k-1)}, q_1, q_2^{(k-1)}, \ldots, q_{2k-3}^{(k-1)}, q_{2k-2}^{(k-1)}, q_{2k-1}^{(k-1)}.$$
and the length of this trajectory is $T_{k-1}$. Another billiard trajectory which follows the same configuration $\alpha'_{k-1}$ is
\[ q_{2k-3}^{(k)}, q_{2k-4}^{(k)}, \ldots, q_{4k-1}^{(k)}, q_{4k+1}^{(k)}, q_{1}^{(k)}, q_{2}^{(k)}, \ldots, q_{2k-3}^{(k)}, q_{2k-2}^{(k)}, q_{2k-1}^{(k)}. \]
Denote the length of the latter by $T'_{k-1}$. It then follows from Theorem 5.1 that
\[ |T_{k-1} - T'_{k-1}| \leq \| q_{2k-1}^{(k)} - q_{2k+3}^{(k)} \| + \| q_{2k-1}^{(k)} - q_{2k-1}^{(k)} \|. \]
Using the latter and (45) for $k$ and $k - 1$, one gets $|T_{k-1} - T'_{k-1}| < 4C\delta^{2k-4}$. In fact, $T_{k-1}$ is the total minimum of the length function corresponding to the periodic configuration $\alpha'_{k-1}$ (cf. Sect. 3 in [St]), so $T_{k-1} < T'_{k-1}$. Thus,
\[ T_{k-1} < T'_{k-1} < T_{k-1} + 4C\delta^{2k-4}. \] (46)

On the other hand, we have $T_k = T'_{k-1} + \| q_{2k-1}^{(k)} - q_{2k+3}^{(k)} \| + \| q_{2k-1}^{(k)} - q_{2k-1}^{(k)} \|$, and (45) and (46) imply $T_{k-1} + 4d_0 < T_k < T_{k-1} + 4d_0 + 4C\delta^{2k-4}$. Since $d_0, T_{k-1}$ and $T_k$ are integers by assumption, the above is clearly impossible for large $k$—contradiction. This proves the assertion. $\square$

Appendix. Curvature of convex wavefronts
Let $X$ and $Y$ be smooth hypersurfaces in $\mathbb{R}^d$ endowed with smooth unit normal fields $n_X(x)$ and $n_Y(y)$, respectively. We assume that both $X$ and $Y$ are convex with respect to the given normal fields.

In what follows we use the notation $l(q, v) = \{q + tv : t \geq 0\}$, where $q, v \in \mathbb{R}^d$, $v \neq 0$.

Let $g_0 \in X$ be such that the ray $l(g_0, n_X(g_0))$ hits transversally $Y$ at some point $p_0$. Set $t_0 = \|p_0 - g_0\|$. Consider an arbitrary smooth parametrization $q = q(u), u \in U$ of $X$ near $g_0$. Here $U$ is some open subset of $\mathbb{R}^{d-1}$. Let $q_0 = q(u_0)$.

For every $u \in U$ close to $u_0$, the ray $l(q(u), n_X(u))$ hits transversally $Y$ at some point $p(u)$.

Denote by $f(u)$ the reflected direction of $n_X(u)$ at $p(u)$, i.e. $f(u) = n_X(u) - 2\langle n_X(u), n_Y(u) \rangle n_Y(u)$, where $n_X(u) = n_X(q(u)), n_Y(u) = n_Y(p(u))$. Then for the angle $\varphi(u)$ between $f(u)$ and $n_Y(u)$ we have $\cos \varphi(u) = \langle f(u), n_Y(u) \rangle$.

Let $\gamma(u)$ be the (billiard) trajectory formed by the segment $[q(u), p(u)]$ and the ray $l(p(u), n_X(u))$. For $t \geq 0$, denote by $S_t(u)$ the shift of $q(u)$ along $\gamma(u)$ after time $t$. Set $X_t = \{S_t(u) : u \in U\}$.

Clearly for $t < t_0$ and $t > t_0$, $X_t$ is a smooth (local) hypersurface. Denote by $\kappa(t)$ the minimal sectional curvature of $X_t$ at the point $S_t(u_0)$ and set $\kappa_-(t_0) = \lim_{t \downarrow t_0} \kappa(t)$, $\kappa_+(t_0) = \lim_{t \uparrow t_0} \kappa(t)$.

Finally, let $k_0$ be the minimal sectional curvature of $Y$ at $p_0$ and let $\varphi_0 = \varphi(u_0)$.

The following fact is a folklore. We prove it for the sake of completeness.

**Proposition A.1.** $\kappa_+(t_0) \geq \kappa_-(t_0) + 2 \cos \varphi_0 k_0$.

**Proof.** Fix for a moment $t'$ and $t''$ with $0 \leq t' < t_0 < t''$ and set $q'(u) = S_{t'}(u)$, $q''(u) = S_{t''}(u)$, $X' = X_{t'}$ and $X'' = X_{t''}$.

Given $a = (a_1, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$, consider the vectors $v' = \sum_{i=1}^{d-1} a_i \frac{\partial q'}{\partial u_i}(u_0) \in T_{q'(u_0)}X'$, $v'' = \sum_{i=1}^{d-1} a_i \frac{\partial q''}{\partial u_i}(u_0) \in T_{q''(u_0)}X''$, $w = \sum_{i=1}^{d-1} a_i \frac{\partial p}{\partial u_i}(u_0) \in T_{p(u_0)}Y$.

We are going to express the sectional curvatures of $X', X''$ and $Y$ at the points $q'(u_0)$, $q''(u_0)$ and $p(u_0)$ with respect to the sections determined by $q'$, $q''$ and $w$, respectively.
Let $I'$ and $II'$ be the first and the second fundamental forms of $X'$ at $q'(u_0)$. Then $I'(v') = \sum_{j=1}^{d-1} v_i' v_j' g_{ij}'(u_0)$ and $II'(v') = \sum_{j=1}^{d-1} v_i' v_j' b_{ij}'(u_0)$, where $g_{ij}'(u) = \langle \frac{\partial^2 q}{\partial u_i \partial u_j}(u), \frac{\partial^2 q}{\partial u_i \partial u_j}(u) \rangle$.

The corresponding quantities for $X''$ will be denoted by $I''$, $II''$, $g''_{ij}$, $b''_{ij}$ and those for $Y$ by $I''(Y)$, $II''(Y)$, $g''_{ij}(Y)$ and $b''_{ij}(Y)$.

We have $p(u) = q'(u) + (t(u) - t')n_X(u)$ and $q''(u) = p(u) + (t'' - t(u))f(u)$. From these equalities one derives with some standard computations (see Sect. 10.1 in [PS] for more details) that $I''(v'') = I''(Y)(w) - \langle n_Y(u_0), w \rangle^2 + O(t'' - t_0)$ and $II''(v'') = 2 \cos \varphi_0 II''(Y)(w) + II''(v') - O(t_0 - t') + O(t'' - t_0)$. On the other hand, $I''(Y)(w) = I'(v') + \langle n_X(u_0), w \rangle^2 + O(t_0 - t')$. Combining the last three equalities gives

$$-\frac{II''(v'')}{I''(v'')} = \frac{-2 \cos \varphi_0 II''(Y)(w) - II'(q')}{II''(Y)(w) - \langle n_Y(u_0), w \rangle^2 + O(t'' - t_0)} + O(t_0 - t') + O(t'' - t_0)$$

$$\geq 2 \cos \varphi_0 \left( -\frac{II''(Y)(w)}{I''(Y)(w) + O(t'' - t_0)} - \frac{II'(v')}{I'(v')} \right) + O(t_0 - t') + O(t'' - t_0).$$

Now letting $t' \nearrow t_0$ and $t'' \searrow t_0$, we get

$$-\frac{II''(v'')}{I''(v'')} \geq -2 \cos \varphi_0 \frac{II''(Y)(w)}{I''(Y)(w)} - \frac{II'(v')}{I'(v')} \geq 2 \cos \varphi_0 k_0 + \kappa_-(t_0).$$

This holds for every $v'' \in T_{q'(u_0)}X''$, so the assertion follows. □

**References**


