RUELLE ZETA FUNCTIONS AND SPECTRA OF TRANSFER OPERATORS
FOR SOME AXIOM A FLOWS

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1 Introduction

Let $M$ be a $C^4$ complete non-compact connected Riemann manifold and let $\phi_t : M \rightarrow M \ (t \in \mathbb{R})$ be an Axiom A flow on $M$. A subset $\Lambda$ of $M$ is called a basic set for $\phi_t$ if $\Lambda$ is a locally maximal compact invariant hyperbolic subset of $M$ which is not a single closed orbit and $\phi_t$ is transitive on $\Lambda$.

Denote by $d(x, y)$ the distance between $x, y \in M$ and by $\| \cdot \|_z$ and $\langle \cdot, \cdot \rangle_z$ the norm and the inner product on $T_z M$ determined by the Riemann metric on $M$. Let $E^u(z)$ and $E^s(z)$ $(z \in \Lambda)$ be the tangent spaces to the strong unstable and stable manifolds $W^u_\epsilon(z)$ and $W^s_\epsilon(z)$, respectively (see Sect. 2). The diameter of a subset $A$ of $M$ will be denoted by $|A|$.

Throughout we assume that $M$, $\phi_t$ and $\Lambda$ satisfy the following conditions:

(i) $\Lambda$ is a basic set for $\phi_t$.

(ii) the strong stable and unstable families $W^s_\epsilon(z)$ and $W^u_\epsilon(z)$ $(z \in \Lambda)$ for $\phi_t$ at $\Lambda$ are Lipschitz.

(iii) there exists $\epsilon_0 > 0$ such that for any $0 < \delta < \epsilon \leq \epsilon_0$ there exists a constant $D = D(\delta, \epsilon) > 0$ with

$$|B^u_T(x, \epsilon)| \leq D |B^u_T(x, \delta)|$$

for all $x \in \Lambda$ and $T > 0$.

It should be mentioned immediately that the condition (iii) is only used to prove certain properties of cylinders defined by means of a Markov family for $\Lambda$ – see Lemma 3.5 below. In other words, (iii) can be replaced by any (weaker) condition that implies the conclusions of Lemma 3.5.

Apart from the above, we assume that the flow $\phi_t$ on $\Lambda$ satisfies a non-integrability condition (SNIC) stated in Sect. 2 below.

Under these conditions we prove spectral estimates for Ruelle transfer operators similar to the ones considered by Dolgopyat [D1] in the case of transitive Anosov flows on compact manifolds with $C^1$ jointly non-integrable horocycle foliations (see Theorem 2.1 below). Using this and an argument of Pollicott and Sharp [PoS] one derives valuable information about the Ruelle zeta function

$$\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1},$$

where $\gamma$ runs over the set of primitive closed orbits of $\phi_t : \Lambda \rightarrow \Lambda$ and $\ell(\gamma)$ is the least period of $\gamma$. In what follows $h_T$ denotes the topological entropy of $\phi_t$ on $\Lambda$.

Theorem 1.1. Under the conditions (i)-(iii) and (SNIC), the zeta function $\zeta(s)$ of the flow $\phi_t : \Lambda \rightarrow \Lambda$ has an analytic and non-vanishing continuation in a half-plane $\text{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Moreover, there exists $c \in (0, h_T)$ such that

$$\pi(\lambda) = \# \{ \gamma : \ell(\gamma) \leq \lambda \} = \text{li}(e^{h_T \lambda}) + O(e^{c\lambda})$$
as $\lambda \to \infty$, where $\text{li}(x) = \int_{2}^{x} \frac{du}{\log u}$.

As another consequence of Theorem 2.1 and the procedure described in [D1] one gets exponential decay of correlations for the flow $\phi_{t} : \Lambda \to \Lambda$.

Throughout, given $\alpha > 0$, we denote by $\mathcal{F}_{\alpha}(\Lambda)$ the set of Hölder continuous functions with Hölder exponent $\alpha$ and by $\|h\|_{\alpha}$ the Hölder constant of $h \in \mathcal{F}_{\alpha}(\Lambda)$.

**Theorem 1.2.** Assume that the flow $\phi_{t}$ on $\Lambda$ satisfies the conditions (1)-(iii) and (SNIC). Let $F$ be a Hölder continuous function on $\Lambda$ and let $\nu_{F}$ be the Gibbs measure determined by $F$ on $\Lambda$. For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that

$$\left| \int_{\Lambda} A(x) B(\phi_{t}(x)) \, d\nu_{F}(x) - \left( \int_{\Lambda} A(x) \, d\nu_{F}(x) \right) \left( \int_{\Lambda} B(x) \, d\nu_{F}(x) \right) \right| \leq C e^{-ct} \|A\|_{\alpha} \|B\|_{\alpha}$$

for any two functions $A, B \in \mathcal{F}_{\alpha}(\Lambda)$.

It should be stressed though that our main aim in this work is to obtain the results of Theorems 1.1 and 2.1 (see Sect. 2 below) rather than the one in Theorem 1.2. In fact, it is plausible that exponential decay of correlations could be established in a more general situation than the one considered here with other methods e.g. similar to the one developed by Liverani in [L2]. However it seems rather unlikely that the strong contraction properties of transfer operators established in Theorem 2.1 and their consequences stated in Theorem 1.1 would be fulfilled in the case of ‘very general’ Axiom A flows on basic sets.

There has been a considerable activity in recent times to establish exponential and other types of decay of correlations for various kinds of systems with some highly rated results of Chernov [Ch], Dolgopyat [D1], [D2], Liverani [L1], [L2], Young [Y]; see also the monograph of Baladi [Ba] which contains a lot of information and references in this direction. In [St1] a modification of the method from [D1] was used to proof results similar to Theorems 1.1 and 1.2 above for open billiard flows in the plane. Baladi and Vallée (see [BaV] and the references there) obtained Dolgopyat type estimates for transfer operators in the case of a suspension of an interval map.

In this paper we consider one particular application of the above results which concerns geodesic flows on non-compact connected manifolds of constant negative curvature.

Let $X$ be a complete (not necessarily compact) connected Riemann manifold of constant curvature $K = -1$ and dimension $\dim(X) = n + 1$, $n \geq 1$, and let $\phi_{t} : M = \mathbb{S}^{n}(X) \to M$ be the geodesic flow on the unit cosphere bundle of $X$. According to a classical result of Killing and Hopf (cf. e.g. Corollary 2.4.10 in [Wolf]), any such $X$ is a hyperbolic manifold, i.e. $X$ is isometric to $\mathbb{H}^{n+1}/\Gamma$, where $\mathbb{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 > 0\}$ is the upper half-space in $\mathbb{R}^{n+1}$ with the Poincaré metric $ds^2(x) = \frac{1}{x_1^2}(dx_1^2 + \ldots + dx_n^2)$ and $\Gamma$ is a discrete group of isometries (Möbius transformations) of $\mathbb{H}^{n+1}$ acting freely and discontinuously on $\mathbb{H}^{n+1}$. Moreover, $\mathbb{H}^{n+1}$ is isometric to the universal covering of $X$. See e.g. [Rate] for basic information on hyperbolic manifolds. Given a hyperbolic manifold $X = \mathbb{H}^{n+1}/\Gamma$, the **limit set** $L(\Gamma)$ is defined as the set of accumulation points of all $\Gamma$-orbits in $\partial\mathbb{H}^{n+1}$, the topological closure of $\partial\mathbb{H}^{n+1} = \{0\} \times \mathbb{R}^{n}$ including $\infty$. The **non-wandering set** $\Lambda$ of $\phi_{t} : M \to M$ is the image in $M$ of the set of all points of $\mathbb{S}^{n}(\mathbb{H}^{n+1})$ generating geodesics with end points in $L(\Gamma)$. In what follows we will assume that $\Lambda$ is compact and non-trivial, i.e. $L(\Gamma)$ has more than two points and $L(\Gamma) \neq \partial\mathbb{H}^{n+1}$; then $L(\Gamma)$ is a closed non-empty nowhere dense subset of $\partial\mathbb{H}^{n+1}$ without isolated points (see e.g. Sect. 12.1 in [Rate]). The compactness of $\Lambda$ is present for example when $\Gamma$ is convex cocompact (see e.g. [Su]). It turns
out that Theorems 1.1 and 1.2 (as well as Theorem 2.1 below) are applicable to the geodesic flow $\phi_t$ on $\Lambda = \Lambda(\Gamma)$.

**Theorem 1.3.** Let $X$ be a hyperbolic manifold and $\phi_t : M = S^*(X) \to M$ be its geodesic flow. If the non-wandering set $\Lambda$ of $\phi_t$ is compact and non-trivial, then $\phi_t : \Lambda \to \Lambda$ satisfies the conditions (i)-(iii) and (SNIC), and therefore the conclusions of Theorems 1.1 and 1.2 hold for $\phi_t$ on $\Lambda$.

Notice that the class of hyperbolic manifolds considered above contains all classical and non-classical Schottky manifolds (cf. e.g. Sect. 12.1 in [Ratc]). For Schottky surfaces the result of Theorem 1.3 was proved by Naud [N1]. Under somewhat stronger assumptions than these in Theorem 1.3, a first term asymptotic for the counting function $\tau_N$ was proved by Anantharaman [A] (using the results of [D1]) in the case of Anosov flows on two- and three-dimensional manifolds with $C^4$ jointly non-integrable horocycle foliations.

The motivation for this work comes from investigations on scattering resonances. In this area two particular types of chaotic systems have been studied extensively – geodesic flows on manifolds of constant negative curvature and open billiard flows. The latter arises in scattering by an obstacle which is a finite union of strictly convex bodies with smooth boundaries (cf. [I], [P], [St2]), while the former relates to studies on the distribution of resonances for convex co-compact hyperbolic surfaces ([Z], [N2]), or higher dimensional Schottky manifolds ([GLZ]).

Sec. 2 below contains some basic definitions and facts and the statement of the main result – Theorem 2.1. Sections 3-5 are devoted to the proof of Theorem 2.1. We use the general framework of Dolgopyat’s method from [D1] and its modification in [St1], however significant new development is necessary. The main difficulty comes from the fact that almost nothing is known about the structure of the set $\Lambda$, and therefore the argument has to work in all generality. It is worth mentioning that even in the case considered in Theorem 1.3, $\Lambda$ can have a fractal structure and topologically it can be as complicated as one could imagine (see e.g. [K] for some information in this direction).

In Sect. 3 some general properties of cylinders are studied. One defines cylinders basically in the same way as for one-sided subshifts, however instead of doing this on a symbol space, similarly to the approach in [Ch] and [D1], we define these direct on $U = \cup_{i=1}^k U_i$, where $U_i$ is a given Markov family of rectangles and for each $i$, $U_i$ is a fixed unstable leaf in $R_i$ (see Sect. 2). The corresponding shift map $\sigma : U \to U$ is defined by first shifting along the flow $\phi_t$ and then projecting along stable fibers. Using the condition (iii) we establish certain inequalities between diameters of cylinders and their subcylinders which are then used to prove the Federer property of Gibbs measures $\nu$ on $U$ (see Proposition 3.6 below).

Before we continue, it is probably worth making a few general remarks about Dolgopyat’s method. Given $f$, say in $C^1(U)$, one wants to show that large powers of the Ruelle transfer operator $L_{f^\tau} = L_{f^{-\tau} - (P + a + ib)\tau}$ are contracting for small $a \in \mathbb{R}$ and large $b \in \mathbb{R}$. Here $\tau$ is the first return time function defined by the Markov family and $P \in \mathbb{R}$ is the unique number so that the topological pressure of $f - P \tau$ is zero. For any integer $N > 0$ we have

$$L_{f}^N h(x) = \sum_{\sigma^N(y) = x} e^{g_N(y)} h(y), \tag{1.1}$$

where $g_N(y) = g(y) + g(\sigma(y)) + \ldots + g(\sigma^{N-1}(y))$. One of the main steps in Dolgopyat’s method is to define appropriately two $C^1$ inverses $v_1, v_2 : U_0 \to U$ of $\sigma^N$, i.e. $\sigma^N(v_i(x)) = x$ for $x$ in some ‘small’ open subset $U_0$ of $U$. For these one ultimately shows that for large $N$, small $|a|$ and
large $|b|$ (of magnitude depending on $N$, i.e. $|a| \leq a_0$, $|b| \geq 1/a_0$ for some $a_0 = a_0(N) > 0$) and $h \in C^1(U)$ with $h > 0$, $\|h\| \leq \text{Const}$, $\|dh\| \leq \text{Const} \ |b|$, one has

$$(1.2) \quad \left| e^{g_N(v_1(x))} h(v_1(x)) + e^{g_N(v_2(x))} h(v_2(x)) \right| \leq \lambda \left| \left[ e^{g_N(v_1(x))} h(v_1(x)) + e^{g_N(v_2(x))} h(v_2(x)) \right] \right|$$

for some $\lambda \in (0, 1)$, and this leads to a similar estimate for the whole sum in the right-hand side of (1.1). To make this cancellation mechanism\(^1\) work for general complex-valued functions $h$, one uses estimates involving $N[h]$, where $N$ is a specially defined operator acting on positive functions in $C^1(U)$ with bounded logarithmic derivatives. One of the main features of Dolgopyat’s operator $N$ is that it is an $L^2$-contraction with respect to the Gibbs measure $\nu$ determined by $f - P\tau$ on $U$ (and to prove this one needs the Federer property of $\nu$; see Sect. 3 below). The whole thing is rather more complicated than the above, and we refer the reader to Sect. 5 below for more details. It is worth mentioning though that it is the construction of the inverses $v_1$ and $v_2$ of $\sigma^N$ and the proof of their main properties where the joint non-integrability of the stable and unstable families is used. In [D1] this results in finding $C^1$ vector fields $e_1(z), \ldots, e_n(z)$ defined in a small neighbourhood $U_0$ of a point $z_0 \in U$ such that for large $N$,

$$\tag{1.3} \left| \partial_{u_i}(\tau_N(v_1(u)) - \tau_N(v_2(u))) \right| \geq \epsilon \ , \ \ u \in U_0 ,$$

$$\tag{1.4} \left| \partial_{u_i}(\tau_N(v_1(u)) - \tau_N(v_2(u))) \right| << \epsilon \ , \ i > 1 \ , \ u \in U_0 .$$

Notice that $e^{1b\tau_N(y)}$ is the ‘tricky’ part of the exponential term $e^{g_N(y)}$ in (1.1). With (1.3) and (1.4) one has that if $u, u' \in U_0$ are $Ce$-close however there is a $ce$-gap between their first coordinates (with respect to the vector field $e_1$) for some constants $C > c > 0$, then

$$\tag{1.5} \left| \tau_N(v_1(u)) - \tau_N(v_2(u)) \right| - \left| \tau_N(v_1(u')) - \tau_N(v_2(u')) \right| \geq \text{const} \ \epsilon .$$

This is what lies beneath the proof of (1.2).

In [St1] a modification of the above was used to deal with open billiard flows in the plane. In this case the unstable (and stable) manifolds are one-dimensional, so one has just one vector field $e_1(u)$, however $U$ is a Cantor set, so the construction of $v_1(u)$ and $v_2(u)$ is non-trivial. One of the main difficulties in [St1] was to show that there exist constants $c_2 > c_1 > 0$ such that for any $\epsilon > 0$, $U$ can be partitioned into intervals of lengths between $c_1\epsilon$ and $c_2\epsilon$ such that the intervals having common points with $\Lambda$ come into triads. (The same idea was later used in [N1] to deal with limit sets of convex co-compact hyperbolic surfaces.) Then the one-dimensionality of the unstable manifolds allows to construct the functions $v_1(u)$ and prove (1.3) and (1.5). The significance of the presence of triads is that for any of the small intervals $\Delta_1$ in the partition of $U$ intersecting $\Lambda$ one can find another interval $\Delta_2$ intersecting $\Lambda$ with a $c_1\epsilon$-gap between $\Delta_1$ and $\Delta_2$ such that (1.5) holds for all $u \in \Delta_1$ and all $u' \in \Delta_2$. The rest of Dolgopyat’s method was not so difficult to apply, although extra modifications were necessary to deal with the singularity of the Gibbs measures in this case.

Now to Sect. 4 of the present paper. The dimension $n$ of the unstable manifolds is of course arbitrary and, as mentioned above, the set $\Lambda$ can be ‘anything from a Cantor set to a manifold’. Therefore to construct vector fields $e_1(u)$ in a neighbourhood $U_0$ of a point $z_0 \in U$ with (1.3) and (1.4) is meaningless, unless we know that there are ‘many points’ of $\Lambda$ in the direction of $e_1(u)$ (or ‘very close’ to it) for a ‘large set’ of $u$’s in $U_0$. To establish something like this however seems impossible, at least without assuming anything in the spirit of the extra condition (SNIC) from Sect. 2 below. With (SNIC), assuming $Q_2 = \text{id}_2$ for simplicity, for some $z_0 \in U$, if $b \in E^u(z)$ is a

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\(^1\)To use the words of Liverani [L2] who is also using a similar idea.
direction of $\Lambda$-density at some $z \in U$ close to $z_0$, one constructs for large $N > 0$ a pair of inverses $v_1(u)$ and $v_2(u)$ of $\sigma^N$ such that the analogue of (1.3) holds on a neighbourhood of $z$ with $c_1$ replaced by unit vectors $a$ in a large cone whose axis is a parallel translate of $b$. Now the intervals appearing in triads from [St1] are replaced by cylinders, and the above gives that if $C_1$ and $C_2$ are two cylinders in the vicinity of $z$ of size $\leq \delta$, for some small $\delta > 0$, that can be ‘separated by a plane perpendicular to $b'$ (one can make sense of this by using an exponential map or some other parametrization of $U$ near $z$), then (1.5) holds for all $u \in C_1$ and all $u' \in C_2$. However, since $n > 1$ in general, just one direction $b$ does not give enough opportunities to separate cylinders. So, one needs to construct a finite number of directions $b_1, \ldots, b_{j_0}$ of $\Lambda$-density (at various points close to the initial point $z_0$), and for each $j = 1, \ldots, j_0$, a pair of inverses $v_1^{(j)}(u)$ and $v_2^{(j)}(u)$ of $\sigma^N$ defined on some small open subset $U_0$ of $U$ such that ‘sufficiently many’ pairs of cylinders in $U_0$ of size $\leq \delta$ can be separated by ‘planes perpendicular to some $b_j$', and this can be done for almost all $\delta$ in some interval $(0, \delta']$. This is the content of the main Lemma 4.5 below which is the most difficult part of the paper.

From Lemma 4.5 and the other constructions in Sect. 4 one gets some analogue of (1.5) involving different pairs of inverses $v_1^{(j)}(u)$ and $v_2^{(j)}(u)$, and this turns out to be enough to implement an appropriately modified analytic part of Dolgopyat’s method. This is done in Sect. 5. There are some extra technical difficulties to overcome due again to the unknown structure of $\Lambda$ and the fact that we deal with Lipschitz functions, not $C^1$ in general.

In Sect. 6 we prove Theorem 1.3 by showing that the conditions (i)-(iii) and (SNIC) are fulfilled for the geodesic flows on hyperbolic manifolds of the type considered. This amounts to some calculations on the universal cover $\mathbb{H}^{n+1}$.

Sect. 7 is an Appendix containing the proofs of two technical lemmas.

## 2 Preliminaries and Main Result

Throughout this paper $M$ denotes a $C^4$ complete (not necessarily compact) connected Riemann manifold, and $\phi_t : M \to M (t \in \mathbb{R})$ a $C^4$ flow on $M$. A $\phi_t$-invariant closed subset $\Lambda$ of $M$ is called hyperbolic if $\Lambda$ contains no fixed points and there exist constants $C > 0$ and $0 < \lambda < 1$ such that there exists a $d\phi_t$-invariant decomposition $T_xM = E^0(x) \oplus E^u(x) \oplus E^s(x)$ of $T_xM$ ($x \in \Lambda$) into a direct sum of non-zero linear subspaces, where $E^0(x)$ is the one-dimensional subspace determined by the direction of the flow at $x$, $\|d\phi_t(u)\| \leq C \lambda^t \|u\|$ for all $u \in E^s(x)$ and $t \geq 0$, and $\|d\phi_t(u)\| \leq C \lambda^{-t} \|u\|$ for all $u \in E^u(x)$ and $t \leq 0$.

The flow $\phi_t$ is called an Axiom A flow on $M$ if the non-wandering set of $\phi_t$ (see e.g. [KH]) is a disjoint union of a finite set consisting of fixed hyperbolic points and a compact hyperbolic subset containing no fixed points in which the periodic points are dense.

A non-empty compact $\phi_t$-invariant hyperbolic subset $\Lambda$ of $M$ which is not a single closed orbit is called a basic set for $\phi_t$ if $\phi_t$ is transitive on $\Lambda$ (see e.g. [KH]) and $\Lambda$ is locally maximal, i.e. there exists an open neighbourhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \cap_{t \in \mathbb{R}} \phi_t(V)$. According to Smale’s spectral decomposition theorem (see e.g. [KH]), if the non-wandering set $\Omega$ of an Axiom A flow $\phi_t$ is compact and $F$ is the (finite) set of fixed hyperbolic points of the flow, then $\Omega \setminus F$ is a finite disjoint union of basic sets.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, d(\phi_t(x), \phi_t(y)) \to_{t \to -\infty} 0 \},$$

$$W^u_\epsilon(x) = \{ y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, d(\phi_t(x), \phi_t(y)) \to_{t \to -\infty} 0 \}$$
be the (strong) stable and unstable manifolds of size $\epsilon$. For any $A \subset M$ and $I \subset \mathbb{R}$ denote $\phi_t(A) = \{ \phi_t(y) : y \in A, t \in I \}$.

It follows from the hyperbolicity of $\Lambda$ (cf. [KH]) that if $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W^s_\epsilon(x)$ and $\phi_{[-\epsilon, \epsilon]}(W^s_\epsilon(y))$ intersect at exactly one point $[x, y] \in \Lambda$. That is, there exists a unique $t \in [-\epsilon, \epsilon]$ such that $\phi_t([x, y]) \in W^s_\epsilon(y)$. Setting $\Delta(x, y) = t$, defines the so called temporal distance function ([KB], [Ch], [D1]) which will be used significantly throughout this paper. For $x, y \in \Lambda$ with $d(x, y) < \delta$, define the projection $\pi_y : W \rightarrow \phi_{[-\epsilon, \epsilon]}(W^s_\epsilon(y))$ along local stable manifolds by $\pi_y(x) = [x, y] = W^s_\epsilon(x) \cap \phi_{[-\epsilon, \epsilon]}(W^s_\epsilon(y))$.

Given $z \in \Lambda$, denote by $\exp_z : E^u(z) \rightarrow W^u_{\epsilon_0}(z)$ the (locally defined near 0) exponential map. A vector $b \in E^u(z) \setminus \{0\}$ will be called a direction of $\Lambda$-density at $z$ if there exist infinite sequences $\{u(m)\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp_z(t_m u(m)) \in \Lambda \cap W^u_{\epsilon_0}(z)$ for all $m$, $\lim_{m \rightarrow \infty} u(m) = b$ and $t_m \rightarrow 0$ as $m \rightarrow \infty$.

We are now ready to state the strong non-integrability condition for $\phi_t$ and $\Lambda$ mentioned in Sect. 1:

(SNIC) There exist $z_0 \in \Lambda$, $\epsilon_0 > 0$ and for any $z \in \Lambda \cap W^u_{\epsilon_0}(z_0)$ a linear operator $Q_z : E^u(z) \rightarrow E^u(z)$ depending continuously on $z$ such that:

(i) $Q_{z_0}$ is positive definite;

(ii) for any $\theta > 0$, any $\tilde{z} \in \Lambda \cap W^u_{\epsilon_0}(z_0)$ and any direction $b \in E^u(\tilde{z}) \setminus \{0\}$ of $\Lambda$-density at $\tilde{z}$ there exist $\tilde{z} \in \Lambda \cap W^u_{\epsilon_0}(z_0)$ arbitrarily close to $\tilde{z}$, $\tilde{y} \in \Lambda \cap W^u_{\epsilon_0}(z) \setminus \{\tilde{z}\}$ arbitrarily close to $\tilde{z}$, $\delta = \delta(\theta, \tilde{z}, \tilde{z}, \tilde{y}) > 0$ and $\epsilon' = \epsilon'(\theta, \tilde{z}, \tilde{z}, \tilde{y}) > 0$ such that

$$\Delta(\exp_z(h a), \pi_{\tilde{y}}(z))| \geq \delta h$$

for all $z \in W^u_{\epsilon'}(\tilde{z}) \cap \Lambda$, $a \in E^u(z)$ with $\|a\|_z = 1$ and $\langle a, Q_z b_z \rangle_z \geq \theta$, and $h \in \mathbb{R}$ with $0 < h < \epsilon'$ and $\exp_z(h a) \in \Lambda$, where $b_z$ is the parallel translation of $b$ along the geodesic in $W^u_{\epsilon_0}(z_0)$ from $\tilde{z}$ to $z$.

Remark. In the application considered in Sect. 6 below we have $Q_z = \text{id}$ for all $z$. The reason we consider the more general situation described above is that it occurs naturally in some other models where the results of this paper might be applicable (e.g. open billiard flows in $\mathbb{R}^n$).

We will now briefly recall the notion of a Markov family; see e.g. Bowen [B] for more details. Given $A \subset \Lambda$ we will denote by $\text{Int}_A(A)$ and $\partial_A A$ the interior and the boundary of the subset $A$ of $\Lambda$ in the topology of $\Lambda$.

Let $D$ be a smooth submanifold of $M$ diffeomorphic to a closed ball in $\mathbb{R}^{\dim(M)-1}$ which is transversal to the flow $\phi_t$ and has a small diameter. Such $D$ will be called a cross-sectional disk. For small $\epsilon > 0$ the projection along the flow defines a smooth map $\pr_D : \phi_{[-\epsilon, \epsilon]}(D) \rightarrow D$. Let $R$ be a closed subset of $D \cap \Lambda$ such that $R \subset \text{Int}_A(D) = D \cap \partial D$ and $R = \text{Int}_A(R)$. Assuming $\diam(R)$ is sufficiently small, compared to $d(R, \partial D)$, the map $R \times R \ni (x, y) \mapsto \pr_D([x, y]) \in D \cap \Lambda$ is well-defined. Such a set $R$ is called a rectangle if $\pr_D([x, y]) \in R$ for all $x, y \in R$. For a rectangle $R$, we will use the notation

$$\langle x, y \rangle_R = \pr_D([x, y]) \text{, } W^u_R(z) = \{ \langle x, z \rangle_R : x \in R \} \text{, } W^s_R(z) = \{ \langle z, y \rangle_R : y \in R \},$$

where $z \in R$. We will also use the notation $W^u_D(z) = \pr_D(W^u_\epsilon(z))$ and $W^s_D(z) = \pr_D(W^s_\epsilon(z))$ for $z \in \Lambda \cap D$ and a sufficiently small $\epsilon > 0$.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a family of rectangles such that each $R_i$ is contained in a $C^1$ cross-sectional disk $D_i$ to the flow $\phi_t$, and $R_i = \{U_i, S_i\}R_i = \{ \langle x, y \rangle_{R_i} : x \in U_i, y \in S_i \}$, where $U_i$
and $S_i$ are closed subsets of $W^u(z_i) \cap \Lambda$ and $W^s(z_i) \cap \Lambda$, respectively, for some $z_i \in \Lambda$ with $W^u(z_i) \subset D_i$. Set $R = \bigcup_{i=1}^k R_i$. The family $\mathcal{R}$ is called complete if there exists $T > 0$ such that for every $x \in \Lambda$, $\phi_t(x) \in R$ for some $t \in (0, T]$. Thus, $\tau(x) > 0$ is the smallest positive time with $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, and $\mathcal{P} : R \rightarrow R$ is the Poincaré map related to the family $\mathcal{R}$. The function $\tau : R \rightarrow [0, \infty)$ is called the first return time associated with $\mathcal{R}$.

Following [B] we will say that a complete family $\mathcal{R} = \{R_i\}_{i=1}^k$ of rectangles in $\Lambda$ is a Markov family of size $\chi > 0$ for the flow $\phi_t$ if:

(a) for each $i$, $U_i = \text{Int}_\Lambda(U_i)$ and $S_i = \text{Int}_\Lambda(S_i)$ in the topology of $W^u_{R_i}(z_i) \cap \Lambda$, and there exists a cross-sectional disk $D_i$ containing $R_i$ with $\text{diam}(D_i) < \chi$.

(b) for any $i \neq j$ and any $x \in R_i \cap \mathcal{P}^{-1}(R_j)$ we have

$$\mathcal{P}(W^u_{R_i}(x)) \subset W^s_{R_j}(\mathcal{P}(x)), \quad \mathcal{P}(W^u_{R_i}(x)) \supset W^u_{R_j}(\mathcal{P}(x)).$$

(c) for any $i \neq j$ at least one of the sets $D_i \cap \phi_{[0, \chi]}(D_j)$ and $D_j \cap \phi_{[0, \chi]}(D_i)$ is empty; in particular $D_i \cap D_j = \emptyset$.

(d) for any $i = 1, \ldots, k$ and any $x \in R_i$ the function $\tau$ is constant on the set $W^u_{R_i}(x)$.

For later convenience and without loss of generality we will also assume that each $D_i$ is part of a larger cross-sectional disk $D'_i$ such that $D_i \subset \text{Int}(D'_i)$, $d(R_i, \partial D'_i) > \chi$ and the projection along the flow $\mathcal{P}_{D_i} : \phi_{[2\chi, 2\chi]}(D'_i) \rightarrow D'_i$ is well-defined and $C^1$.

The existence of a Markov family $\mathcal{R}$ of an arbitrarily small size $\chi > 0$ for $\phi_t$ follows from the construction of Bowen [B] (cf. also Ratner [Ra]).

From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for $\phi_t$. Set $U = \bigcup_{i=1}^k U_i$. The shift map $\sigma : U \rightarrow U$ is given by $\sigma = \pi(U) \circ \mathcal{P}$, where $\pi(U) : R \rightarrow U$ is the projection along stable leaves.

Given $F \in \mathcal{F}_\alpha(\Lambda)$ for some $\alpha > 0$, consider the function

$$f_1(x) = \int_0^{\tau(x)} F(\phi_s(x)) \, ds, \quad x \in R.$$ 

As in the proof of Sinai’s Lemma (see e.g. Proposition 1.2 in [PP]), define

$$f_2(x) = \sum_{j=0}^\infty [f_1(\mathcal{P}^j(x)) - f_1(\mathcal{P}^j(\pi(U)(x)))] \quad f(x) = f_1(x) + f_2(x) - f_2(\mathcal{P}(x)), \quad x \in R.$$ 

Then $f(\pi(U)(x)) = f(x)$ for $x \in R$, so $f$ can be considered as a function on $U$. Moreover, one can show that $f_2$ is Hölder continuous on a residual subset of $R$ and has an extension $f_2 \in \mathcal{F}_{\alpha'}(R)$ for some $\alpha' \leq \alpha$ (see Remark 3.4 below). Let $\mu$ be the Gibbs measure related to $F$ with respect to the flow $\phi_t$ on $\Lambda$, and let $P = \text{Pr}_{\phi_t}(F)$ be the topological pressure of $F$ with respect to $\phi_t$. Then $\text{Pr}_{\sigma}(f - P\tau) = 0$. Denote by $\nu$ the Gibbs measure on $U$ related to $f - P\tau$ (see [PP]).

A function $g : U \rightarrow \mathbf{C}$ will be called Lipschitz on cylinders if the restriction of $g$ to any non-empty set of the form $C_\Lambda[i, j] = \text{Int}_\Lambda(U_i) \cap \sigma^{-1}(\text{Int}_\Lambda(U_j))$ is Lipschitz. For such $g$ set

$$\text{Lip}(g) = \inf \{ C > 0 : |g(x) - g(y)| \leq C \, d(x, y), \quad x, y \in C_\Lambda[i, j], \quad 1 \leq i \neq j \leq k \}.$$ 

Similar terminology and notation will also be used for functions on $R$, and for maps on $U$ and $R$. Clearly in general $\tau$ is not continuous on $U$, however $\tau$ is Lipschitz on cylinders. The same applies to $\sigma : U \rightarrow U$. It is easy to derive from this that $f$ is Hölder continuous on cylinders with a Hölder exponent $\alpha' \leq \alpha$. 

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In what follows we will assume that \( f \) is a **fixed function on \( U \) which is Lipschitz on cylinders**; the general case of a function which is Hölder continuous on cylinders can be derived from this via a standard approximating procedure.

Denote by \( C^{\text{Lip}}(U) \) the space of Lipschitz functions \( g : U \to \mathbb{C} \) and by \( \text{Lip}(g) \) the Lipschitz constant of \( g \). We will also need the space \( B(U) \) of bounded functions \( g : U \to \mathbb{C} \) with its standard norm \( \|g\| = \sup_{x \in U} |g(x)| \). Given a function \( g \in B(U) \), the Ruelle operator \( L_g : B(U) \to B(U) \) is given by

\[
(L_g h)(u) = \sum_{\sigma(v) = u} e^{\rho(v)} h(v) .
\]

If \( g \) is Lipschitz on cylinders, then \( L_g \) preserves the space \( C^{\text{Lip}}(U) \) in the sense that for each \( h \in C^{\text{Lip}}(U) \) the function \( L_g h \) is Lipschitz on \( \text{Int}_A(U) = \bigcup_{i=1}^k \text{Int}_A(U_i) \), and therefore has a Lipschitz extension to \( U \).

As an operator on \( C^{\text{Lip}}(U) \), \( L_g \) has an eigenvalue equal to the spectral radius of \( L_g \) and a corresponding (positive) eigenfunction. Given a real number \( a \) (with \( |a| \) small), denote by \( \lambda_a \) the largest eigenvalue of \( L_{f-(P+a)\tau} \) and by \( h_a \in C^{\text{Lip}}(U) \) the corresponding (positive) eigenfunction such that \( \sup_{u \in U} h_a(u) = 1 \). Since \( \text{Pr}(f-(P+a)\tau) = 0 \), it follows from the main properties of pressure (cf. e.g. Ch. 3 in [PP]) that \( |\text{Pr}(f-(P+a)\tau)| \leq |\tau| \tilde{a} |a| \). Moreover, for small \( |a| \) the maximal eigenvalues \( \lambda_a \) and the eigenfunctions \( h_a \) depend analytically on \( a \). In particular, there exist constants \( a_0' > 0 \) and \( C_0 > 0 \) such that \( \lambda_a \geq 1 - C_0 |a| \) on \( U \) for \( |a| \leq a_0' \).

For \( |a| \leq a_0' \), as in [D1], consider the function

\[
f^{(a)}(u) = f(u) - (P + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a
\]

and the operators

\[
L_{ab} = L_{f^{(a)}-b\tau} : C^{\text{Lip}}(U) \to C^{\text{Lip}}(U), \quad \mathcal{M}_a = L_{f^{(a)}} : C^{\text{Lip}}(U) \to C^{\text{Lip}}(U).
\]

One checks that \( \mathcal{M}_a 1 = 1 \) and \( |(L_{ab}^m h)(u)| \leq (\mathcal{M}_a^m |h|)(u) \) for all \( u \in U \), \( h \in C^{\text{Lip}}(U) \) and \( m \geq 0 \). For \( h \in C^{\text{Lip}}(U) \) and \( b \in \mathbb{R} \setminus \{0\} \), define the norm \( \|h\|_{\text{Lip},b} \) on \( C^{\text{Lip}}(U) \) by \( \|h\|_{\text{Lip},b} = \|h\|_0 + \frac{\|L_b h\|}{|b|} \).

The following is the main result in this paper.

**Theorem 2.1.** For every \( \epsilon > 0 \) there exist constants \( \rho \in (0, 1) \), \( a_0 \in (0, 1) \) and \( C > 0 \) such that for every integer \( m > 0 \) and every \( h \in C^{\text{Lip}}(U) \), if \( a, b \in \mathbb{R} \), are such that \( |a| \leq a_0 \) and \( |b| \geq 1/a_0 \), then

\[
\|L_{f^{-(P+a+b)\tau}}^m h\|_{\text{Lip},b} \leq C \rho^m |b|^\epsilon \|h\|_{\text{Lip},b} .
\]

In particular, the spectral radius of \( L_{f^{-(P+a+b)\tau}} \) on \( C^{\text{Lip}}(U) \) does not exceed \( \rho \).

Such a result was first established by Dolgopyat [D1] in the case of transitive Anosov flows on compact manifolds with \( C^1 \) jointly non-integrable stable and unstable foliations. A similar result was proved in [St1] for open billiard flows in the plane (in which case \( U \) is a Cantor set).

It is well known that \( \sigma^m \) is an expanding map, i.e. there exist constants \( c_0 \in (0, 1) \) and \( \gamma_1 > \gamma > 1 \) such that

\[
(2.2) \quad c_0 \gamma^m d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma^m}{c_0} d(u_1, u_2)
\]

whenever \( \sigma^j(u_1) \) and \( \sigma^j(u_2) \) belong to the same \( \text{Int}_A(U_i) \) for all \( j = 0, 1, \ldots, m \). From now on we will assume that \( c_0, \gamma_0, \gamma \) and \( \gamma_1 \) are fixed constants with the above properties. For later convenience we will also assume that \( \gamma_1 \geq \frac{1}{16} \).
Set $\tilde{\tau} = \max\{\|\tau\|_0, \text{Lip}(\tau)\}$. Assuming that the constant $a'_0 > 0$ is sufficiently small, there exists $T = T(a'_0)$ such that

$$
T \geq \max\{\|f^{(a)}\|_0, \text{Lip}(f^{(a)}), \tilde{\tau}\}
$$

for all $|a| \leq a'_0$. Fix $a'_0 > 0$ and $T > 0$ and with these properties.

## 3 Some properties of cylinders

Let again $\mathcal{R} = \{R_i\}_{i=1}^k$ be a fixed Markov family satisfying the conditions (a)-(d) of Sect. 2, and let $A = (A_{ij})_{i,j=1}^k$ be the matrix given by $A_{ij} = 1$ if $P(\text{Int}_A(R_i)) \cap \text{Int}_A(R_j) \neq \emptyset$ and $A_{ij} = 0$ otherwise. According to [BR] (see Sect. 2 there), we may assume that $\mathcal{R}$ is chosen in such a way that the corresponding subshift of finite type $\sigma_A : \Sigma^+_A \to \Sigma^+_A$ is topologically mixing, i.e. there exists an integer $M > 0$ such that $A^M > 0$ (all entries of the $M$-fold product of $A$ by itself are positive).

Given a finite string $i = (i_0, i_1, \ldots, i_m)$ of integers $i_j \in \{1, \ldots, k\}$, we will say that $i$ is admissible if for any $j = 0, 1, \ldots, m - 1$ we have $A_{ij_{j+1}} = 1$. Given an admissible string $i$, denote by $\hat{C}_A[i]$ the set of those $x \in U$ so that $\sigma^j(x) \in \text{Int}_A(U_{i_j})$ for all $j = 0, 1, \ldots, m$. The set $C[i] = \hat{C}_A[i] \subset A$ will be called a cylinder of length $m$ in $U$. It follows from the properties of the Markov family that $\hat{C}_A[i]$ is an open dense subset of $C[i]$ and $\nu(C[i] \setminus \hat{C}_A[i]) = 0$ ([B]). Any cylinder of the form $C[i_0, i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+q}]$ will be called a subcylinder of $C[i]$ of co-length $q$. For any admissible sequence $i = (i_0, i_1, \ldots, i_m)$ and any $j = 0, 1, \ldots, m$ we will denote by $\hat{\sigma}_j^i$ the continuous extension of $\sigma^j : \hat{C}_A[i] \to U_{i_j}$ to $C[i]$.

In what follows the cylinders considered are always defined by finite admissible strings.

Denote by $\hat{U}$ the core of $U$, i.e. the set of those $x \in U$ such that $\sigma^m(x) \in \text{Int}_A(U)$ for all $m \geq 0$. It is well-known ([B]) that $\hat{U}$ is a residual subset of $U$ and $\nu(\hat{U}) = 1$.

For any $X \subset M$ we will denote by $|X|$ the diameter of $X$. Given $x \in U_i$ for some $i$ and $r > 0$ we will denote by $B_U(x, r)$ the set of all $y \in U_i$ with $d(x, y) < r$.

Recall the constants $c_0 > 0$ and $\gamma_1 > \gamma > 1$ from Sect. 2, and fix an integer $p_0 \geq 1$ such that

$$
\rho_0 = \frac{1}{c_0 \gamma^{p_0}} < \min\left\{\frac{|U_i|}{|U_j|} : i, j = 1, \ldots, k\right\}.
$$

Then clearly $\rho_0 < 1$. Set

$$
\rho_1 = \rho_0^{1/p_0}, \quad C' = \max\{|D'_1|, \ldots, |D'_k|\},
$$

where $D'_j \supset D_j$ are the cross-sectional disks containing the rectangle $R_j$ (see Sect. 2).

**Fix a constant** $r_0 > 0$ and for each $i = 1, \ldots, k$ a point $\hat{z}_i \in U_i$ such that

$$
B_U(\hat{z}_i, r_0) \subset U_i \quad \text{and} \quad d(\hat{z}_i, \partial A(U_i)) \geq r_0.
$$

The following is an immediate consequence of the expanding property of the map $\sigma$ (cf. [B]).

**Lemma 3.1.** There exists a global constant $C'_1 > 0$ such that for any cylinder $C[i]$ of length $m$ we have $|C[i]| \leq C'_1 \rho_0^m$ and $|C[i]| \geq \frac{c_0 \gamma^m}{\gamma^m}$.

**Proof of Lemma 3.1.** Let $i = (i_0, \ldots, i_m)$. It follows from the definition of $r_0$ that there exist $x', y' \in U_{i_m}$ with $d(x', y') \geq r_0$. Then the properties of the Markov family show that $\sigma^m(x) = x'$ and $\sigma^m(y) = y'$ for some $x, y \in C[i]$. Now (2.2) gives $d(x, y) \geq \frac{c_0}{\gamma^m} d(x', y') \geq \frac{c_0 \gamma^m}{\gamma^m}$, so $|C[i]| \geq \frac{c_0 \gamma^m}{\gamma^m}$.
Next, assume that \( m \geq p_0 \). Set \( \ell = \lfloor m/p_0 \rfloor \) and \( C_s = C[i_{m-sp_0}, i_{m-sp_0+1}, \ldots, i_m] \) for any \( s = 1, \ldots, \ell \). Then \( \sigma^{p_0}(C_{s+1}) \subset C_s \), and it follows from (2.2) and the choice of \( p_0 \) that

\[
|C_{s+1}| \leq \min \left\{ |U_{i_{m-(s+1)p_0}}|, \frac{1}{c_0^{s+1}p_0} |C_s| \right\} = \rho_0 |C_s|.
\]

A simple induction now proves the lemma with \( C'_1 = C'(\rho_0)^{-p_0+1} \).  

Recall that \( \nu \) is the Gibbs measure on \( U \) determined by the function \( g = f - P\tau \). It follows from the properties of Gibbs measures (cf. [Si], [R] or [PP]) and \( \text{Pr}_{\sigma}(g) = 0 \) that there exist constants \( \alpha > 0 \) and \( \beta > 0 \) such that

\[
\alpha \leq \frac{\nu(C[y])}{e^{\beta m(y)}} \leq \beta
\]

for any cylinder \( C[y] \) of length \( m \) in \( U \) and any \( y \in \hat{\sigma}_x \).

**Definition 3.2.** We will say that two cylinders \( C[y] = C[i_0, \ldots, i_m] \) and \( C'[y] = C[i'_0, \ldots, i'_{m'}] \) are matching if \( i_0 = i'_0 \), \( C[y] \cap C'[y] \neq \emptyset \), and for any \( x \in C[y] \cap C'[y] \) for the sequences of successive times \( t_0(x) = 0 < t_1(x) < \ldots < t_m(x) \) and \( t'_0(x) = 0 < t'_1(x) < \ldots < t'_{m'}(x) \) with \( \phi_{t_j}(x) \in R_i \) and \( \phi_{t'_j}(x) \in R'_j \), respectively, we have \( |t_m(x) - t'_{m'}(x)| \leq \chi \).

The following lemma contains some properties of matching cylinders that will be needed later on. It should be noted that at this stage the condition (iii) from Sect. 1 is still not being used.

**Lemma 3.3.** (a) For any \( i = 1, \ldots, k \) and any points \( v \neq v' \) in \( \tilde{U} \cap U_i \) there exist cylinders \( C[y] = C[i_0, \ldots, i_m, i_{m+1}] \) and \( C'[y'] = C[i'_0, \ldots, i'_{m'}, i'_{m'+1}] \) such that \( v \in C[y], v' \in C'[y'], C[y] \cap C'[y'] = \emptyset \), and \( C[i_0, \ldots, i_m] \) and \( C[i'_0, \ldots, i'_{m'}] \) are matching.

(b) There exists a constant \( C_2 \geq 1 \) such that for any pair \( C[y] = C[i_0, \ldots, i_m], C'[y'] = C[i'_0, \ldots, i'_{m'}] \) of matching cylinders we have

\[
\frac{1}{C_2} \leq \frac{\nu(C[y])}{\nu(C'[y'])} \leq C_2.
\]

**Proof of Lemma 3.3.** (a) Let \( v \neq v' \) belong to \( \tilde{U} \cap U_i \) for some \( i \). There exist infinite admissible sequences \( [i_0 = i, i_1, i_2, \ldots] \) and \( [i'_0 = i, i'_1, i'_2, \ldots] \) such that \( \sigma_j(v) \in U_{i_j} \) and \( \sigma'_j(v') \in U'_{i'_j} \) for all \( j \geq 0 \).

Clearly there exist \( m \geq 0 \) and \( m' \geq 0 \) such that the cylinders \( C[i_0, \ldots, i_m] \) and \( C[i'_0, \ldots, i'_{m'}] \) are matching (take e.g. \( m = m' = 0 \)). Let \( m \) be the largest integer for which a number \( m' \) with this property exists. (By Lemma 3.1, \( m \) and \( m' \) cannot be arbitrarily large.) Fix \( m \) and then take \( m' \) to be the largest integer such that \( C = C[i_0, \ldots, i_m] \) and \( C' = C[i'_0, \ldots, i'_{m'}] \) are matching.

Since \( C \) and \( C' \) are matching, there exists \( x \in C \cap C' \) with the properties described in Definition 3.2. First, assume that \( t_m(x) < t'_{m'}(x) \). It then follows from property (c) of the Markov family \( \{R_i\} \) that \( t_m(x') < t'_{m'}(x') \) for any \( x' \in C \cap C' \). Now the choice of \( m \) implies \( C[i_0, \ldots, i_m, i_{m+1}] \cap C' = \emptyset \). Indeed, assuming the latter is not true, for any \( x' \in C[i_0, \ldots, i_m, i_{m+1}] \cap C' \) we have \( x' \in C \cap C' \), so by the above and Definition 3.2, \( t_m(x') < t'_{m'}(x') \) and \( t'_{m'}(x') - t_m(x') \leq \chi \). It then follows that \( t_{m+1}(x') - t'_{m'}(x') \leq \chi \), which implies \( C[i_0, \ldots, i_m, i_{m+1}] \cap C' = \emptyset \).

Next, assume that \( t_m(x) > t'_{m'}(x) \). Arguing as above, reversing the roles of \( m \) and \( m' \) and using the maximality of \( m' \), one gets \( C \cap C[i'_0, \ldots, i'_{m'}, i'_{m'+1}] = \emptyset \), which implies \( C[i_0, \ldots, i_m, i_{m+1}] \cap C[i'_0, \ldots, i'_{m'}, i'_{m'+1}] = \emptyset \).  

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The case $t_m(x) = t'_m(x)$ is trivial, since then we must have $i_m = i'_m$.

(b) Assume $C[i]$ and $C[i']$ are matching, and let $x \in C[i] \cap C[i']$. Take $y \in \hat{C}_A[i]$ and $y' \in \hat{C}_A[i']$ arbitrarily close to $x$. By (3.3), to prove the statement it is enough to establish the existence of a constant $C' > 0$ independent of the cylinders $C[i]$ and $C[i']$ such that $|g_m(y) - g_m(y')| \leq C'$. Since $\tau_m(y)$ and $\tau_m(y')$ can be made arbitrarily close to $t_m(x)$ and $t'_m(x)$, respectively, we may assume that $|\tau_m(y) - \tau_m(y')| \leq 2\chi$. So, it remains to estimate $|f_m(y) - f_m(y')|$.

Recall from Sect. 1 that the function $f$ was initially defined on $R$ in such a way that its restriction to any stable fibre in any $R_i$ is constant. Thus, $f(\sigma^j(z)) = f(\pi^j(U)(\mathcal{P}^j(z))) = f(\mathcal{P}^j(z))$ for any $z \in \text{Int}_A(U)$. Consequently, $f_m(y) = \sum_{j=0}^{m-1} f(\mathcal{P}^j(y))$, and therefore

$$f_m(y) = \sum_{j=0}^{m-1} \int_0^{\tau_m(y)} F(\phi_s(\mathcal{P}^j(y))) \, ds = \int_0^{\tau_m(y)} F(\phi_s(y)) \, ds \rightarrow \int_0^{t_m(y)} F(\phi_s(x)) \, ds$$

as $y \rightarrow x$. Similarly, $f_m(y') \rightarrow \int_0^{t'_m(x)} F(\phi_s(x)) \, ds$, so choosing $y$ and $y'$ sufficiently close to $x$, we have $|f_m(y) - f_m(y')| \leq 2\chi \|F\|_0$. This proves (b).

Remark 3.4. It follows from Lemma 3.3(a) and Lemma 3.1 that there exist constants $0 < \tilde{\rho} < \tilde{\rho}_1 < 1$ and $0 < \tilde{c} < 1 < \tilde{C}$ such that for any $\nu \neq \nu'$ in $U$ there exists $m \geq 0$ with $\tilde{c} \tilde{\rho}_1^m \leq d(\nu, \nu') \leq \tilde{C} \tilde{\rho}_1^m$. Modifying slightly the argument in the proof of Lemma 3.3(a) gives similar estimates for $\nu \neq \nu'$ lying in the same unstable leaf in some $R_i$. It then follows easily that the function $f_2(x) = \sum_{j=0}^{\infty} [f_1(\mathcal{P}^j(x)) - f_1(\mathcal{P}^j(\pi^j(U)(x)))$ is H"older continuous on the core $R$ of $R$, i.e. the set of all $x \in R$ such that $\mathcal{P}^m(x) \in R \setminus \partial_A(R)$ for all $m \in \mathbb{Z}$. Thus, $f_2$ has a H"older continuous extension to $R$.

It follows from the condition (iii) in Sect.1 that there exists an integer $m_0 > 0$ such that, possibly choosing a larger constant $D_0 > 0$ and assuming $\chi > 0$ is chosen sufficiently small, for any cylinder $C[i]$ of length $m \geq m_0$ we have

$$\frac{d(z,x)}{d(z,y)} \leq D_0 \frac{d(\sigma^m(z),\sigma^m(x))}{d(\sigma^m(z),\sigma^m(y))} , \quad x, y, z \in C[i] , \quad z \neq y .$$

Given $i = 1, \ldots, k$, according to the choice of the Markov family $\{R_i\}$, the projection $\text{pr}_{D'_i} : W_i = \phi_{[-2\chi,2\chi]}(D'_i) \rightarrow D'_i$ along the flow $\phi_t$ is well-defined and $C^1$. Since the projection $\pi_i : D'_i \rightarrow W_{D'_i}(z_i)$ along stable leaves is Lipschitz, the map $\psi_i = \pi_i \circ \text{pr}_{D'_i} : W_i \rightarrow W_{D'_i}(z_i)$ is also Lipschitz. Thus, there exists a constant $C'' > 0$ such that $d(\psi_i(u), \psi_i(v)) \leq C'' d(u, v)$ for all $u, v \in W_i$ and all $i = 1, \ldots, k$. Set

$$a = \min \{d(\psi_i(U_p), \psi_i(U_q)) : 1 \leq i, p, q \leq k , U_p \cup U_q \subset W_i , \psi_i(U_p) \cap \psi_i(U_q) = \emptyset\}$$

if the set in the right-hand side of (3.5) is not empty, and $a = 1$ otherwise. Clearly $a > 0$.

The following lemma describes the main consequences of condition (iii) that will be needed later on.

Lemma 3.5. There exist global constants $0 < \rho < 1$ and $C_1 > 0$ such that:

(a) For any cylinder $C[i] = C[i_0, \ldots, i_m]$ and any subcylinder $C[i'] = C[i_0, i_1, \ldots, i_{m+1}]$ of $C[i]$ of co-length 1 we have $\rho |C[i]| \leq |C[i']|$. Moreover any subcylinder $C[i'] = C[i, i'_{m+1}, \ldots, i'_{m+1}]$ of $C[i]$ of co-length $p$ has $|C[i']| \leq C_1 \rho^p |C[i]|$.

(b) If $C[i] = C[i_0, \ldots, i_m]$ and $C[i'] = C[i_0', \ldots, i_m']$ are matching cylinders, then $|C[i']| \geq \rho |C[i]|$. 

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(c) If \( C[i] = C[i_0, \ldots, i_m] \) and \( C[i'] = C[i'_0, \ldots, i'_{m'}] \) are matching cylinders, and \( i_{m+1} \neq i'_{m'+1} \) are such that \( C[i, i_{m+1}] \cap C[i', i'_{m+1}] = \emptyset \), then \( d(C[i, i_{m+1}], C[i', i'_{m+1}]) \geq \rho |C[i]| \).

(d) Let \( p_1 \) be a positive integer such that \( C_1 p_1 \leq r_2 \). Then for every cylinder \( C[i] = C[i_0, \ldots, i_m] \) there exists a subcylinder \( C[i'] = C[i, i'_{m+1}], \ldots, i'_{m+p_1} \) of \( C[i] \) of co-length \( p_1 \) such that \( d(C[i'], \partial \Lambda(C[i])) \geq \rho |C[i]| \).

**Proof of Lemma 3.5.** Notice that the properties of the Markov family (and the fact that \( \nu \) is admissible) imply \( \sigma^m_\nu(C[i]) = U_{i_m} \).

(a) Set \( z = \hat{z}_{i_{m+1}} \) for brevity, and let \( x \in C[i'] \) be the point such that \( \sigma^m(x) = z \). Let \( y \in \partial \Lambda(C[i']) \) be such that \( d(x, y) = d(x, \partial \Lambda(C[i'])) \).

Given any \( u \in C[i] \setminus \{ x \} \), we have \( d(\sigma^m(u), \sigma^m(x)) \leq |U_{i_m}| \leq C^\prime \). Now (3.4) gives

\[
\frac{d(u, x)}{d(y, x)} \leq D_0 \frac{d(y, u), \sigma^m(x)}{d(y, \sigma^m(x))} \leq \frac{D_0 C'}{d(y, \sigma^m(x))}.
\]

On the other hand, \( y \in \partial \Lambda(C[i']) \) shows that \( \sigma^m(y) \in \partial \Lambda(U_{i_{m+1}}) \), and in particular \( d(\sigma^m(x), \sigma^m(y)) = d(z, \sigma^m(y)) \geq r_0 \). Combining this with (2.2), gives \( d(\sigma^m(x), \sigma^m(y)) \leq \frac{c_0 r_0}{\gamma_1} d(\sigma^m(x), \sigma^m(y)) \geq \frac{c_0 r_0}{\gamma_1} \). Thus,

\[
\frac{d(u, x)}{d(y, x)} \leq \frac{D_0 C'}{c_0 r_0} \frac{\gamma_1}{\gamma_1} \quad \text{and so } \quad d(u, x) \leq \frac{\gamma_1}{c_0 r_0} \frac{D_0 C'}{d(y, \sigma^m(x))}.
\]

Hence it is enough to take \( \rho \leq \frac{c_0 r_0}{D_0 C'} \).

To prove the second part of (a), assume that \( C[i'] = C[i, i'_{m+1}, \ldots, i'_{m+p}] \) is a subcylinder of \( C[i] \) of co-length \( p \). Then \( \sigma^m_\nu(C[i']) \) is a cylinder of length \( p \), so by Lemma 3.1, \( |\sigma^m_\nu(C[i'])| \leq C_1 \rho_1^p \). Given \( x \neq y \) in \( C[i'] \), there exists a point \( v \in U_{i_m} = \sigma^m_\nu(C[i]) \) with \( d(v, \sigma^m(x)) \geq |U_{i_m}|/2 \). There exists \( u \in C[i] \) with \( \sigma^m_\nu(u) = v \), so \( d(\sigma^m(x), \sigma^m(u)) \geq |U_{i_m}|/2 \geq r_0/2 \). Hence by (3.4)

\[
\frac{d(x, y)}{d(x, u)} \leq D_0 \frac{d(\sigma^m(x), \sigma^m(y))}{d(\sigma^m(x), \sigma^m(u))} \leq \frac{D_0 |\sigma^m_\nu(C[i'])|}{r_0} \leq \frac{2D_0 C_1 \rho_1^p}{r_0}.
\]

Thus, \( d(x, y) \leq C_1 \rho_1^p |C[i]| \) for any \( x, y \in C[i'] \), where \( C_1 = \frac{2D_0 C_1}{r_0} > 0 \), and therefore \( |C[i']| \leq C_1 \rho_1^p |C[i]| \).

(b)-(c) Assume that \( C[i] \) and \( C[i'] \) are matching and let \( x \in C[i] \cap C[i'] \) be as in Definition 3.2. Assume e.g. that \( t_m(x) \leq t_m'(x) \) and set \( D = D_{t_m'} \) for brevity. Since \( t_m'(x) - t_m(x) \leq \chi \), the projection \( pr_D \) along the flow \( \phi_t \) is well-defined on \( R_{t_m} \). Set \( z = \hat{z}_{i_{m'}} \), \( W = W_{i_{m'}} \) and \( \psi = \psi_{i_{m'}} \).

Let \( x' \in C[i'] \) be such that \( \sigma^m_\nu(x') = \hat{z}_{i_{m'}} \). Setting \( t = t_{m'}(x) \), \( \hat{x} = \hat{\sigma}^m_\nu(x') \in U_{i_{m'}} \subset W_{t_m}(z) \), we have \( \hat{x} = \psi(\phi_t(x')) \).

To prove (b), we will assume \( i \neq i' \) (the case \( i = i' \) is trivial). We then have \( t_m(x) < t_{m'}(x) \) and \( x \in \partial \Lambda(C[i]) \cap \partial \Lambda(C[i']) \), so \( \hat{x} = \hat{\sigma}^m_\nu(x) \in \partial \Lambda(U_{i_{m'}}) \). Using this and (3.2) gives

\[
r_0 \leq d(\hat{x}, \hat{z}_{i_{m'}}) = d(\psi(\phi_t(x)), \psi(\phi_t(x'))) \leq C'' d(\psi(\phi_t(x)), \psi(\phi_t(x'))) = \rho^m d(\phi_t(x), \phi_t(x')).
\]
It now follows from the condition (III) that for any $y \in C[i]$ we have

$$\frac{d(x,y)}{d(x,x')} \leq D_0 \frac{d(\hat{\phi}_t(x), \hat{\phi}_t(y))}{d(\hat{\phi}_t(x), \hat{\phi}_t(x'))} \leq \frac{D_0 (C'+2\chi)}{r_0/C''}.$$ 

Thus, $d(x,y) \leq \frac{D_0 (C'+2\chi)}{r_0/C''} |C'[\nu]|$ for any $y \in C[i]$, so $|C[i]| \leq \frac{2D_0 (C'+2\chi)}{r_0} |C'[\nu]|$. Hence it is enough to take $\rho \leq \frac{2D_0 (C'+2\chi)}{r_0}$.

To prove (c) we will use the above notation, however we will no longer assume that $t \neq t'$. So now $\hat{x}$ may be an interior point of $U_{i_m}$. Assume that $\hat{C} = C[i, i_{m+1}]$ and $\hat{C}' = C[i', i'_{m+1}]$ have no common points. Set $i = [i, i_{m+1}]$ and $i' = [i', i'_{m+1}]$. We then have $\hat{\sigma}_i^{m+1}(\hat{C}) = U_{i_{m+1}}$ and $\hat{\sigma}_i^{m+1}(\hat{C}') = U_{i'_{m+1}}$. Since $\psi$ is one-to-one on every unstable leaf $W_u^u(v)$, $v \in D$, it follows easily from $\hat{C} \cap \hat{C}' = \emptyset$ that $\psi(U_{i_{m+1}}) \cap \psi(U_{i'_{m+1}}) = \emptyset$, so (3.5) implies $d(\psi(U_{i_{m+1}}), \psi(U_{i'_{m+1}})) \geq a$.

Now we proceed very much as in part (a). Let $y \in \hat{C}$ and $y' \in \hat{C}'$ be such that $d(y,y') = d(\hat{C}, \hat{C}')$. By (3.4), for any $u \in C[i] \setminus \{y\}$ we have

$$\frac{d(y,u)}{d(y,y')} \leq D_0 \frac{d(\hat{\phi}_t(y), \hat{\phi}_t(u))}{d(\hat{\phi}_t(y), \hat{\phi}_t(y'))} \leq \frac{D_0 (C' + 2\chi) C''}{d(\psi(U_{i_{m+1}}), \psi(U_{i'_{m+1}}))} \leq \frac{D_0 (C' + 2\chi) C''}{a}.$$ 

Thus, $d(y,u) \leq \frac{D_0 (C' + 2\chi) C''}{a} d(\hat{C}, \hat{C}')$ for any $u \in C[i]$, so $|C[i]| \leq \frac{2D_0 (C' + 2\chi) C''}{a} d(\hat{C}, \hat{C}')$. Hence it is enough to take $\rho \leq \frac{2D_0 (C' + 2\chi) C''}{r_0}$.

(d) Let $x \in C[i]$ be the point such that $\sigma^m(x) = \hat{z}_{i_m}$. Since $\hat{z}_{i_m} \in \Lambda$, it belongs to an arbitrarily long cylinders. Let $C' = C[i_m, i_{m+1}, \ldots, i_{m+p_1}]$ be a cylinder of length $p_1$ containing $\hat{z}_{i_m}$; then $C'[\nu] = C[i_0, i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+p_1}]$ is a subcylinder of $C[i]$ of co-length $p_1$. It follows from Lemma 3.1 that $|C'[\nu]| \leq C^1 p_1^{p_1}$, so by the choice of $p_1$, $|C'| \leq r_0/2$, and therefore $|\hat{\sigma}_i^m(C'[\nu])| \leq r_0/2$. The choice of $r_0$ now implies

$$d(\hat{\sigma}_i^m(C'[\nu]), \partial \Lambda(U_{i_m})) \geq \frac{r_0}{2}. \tag{3.6}$$

Fix for a moment $x' \in C[i']$ and $y \in \partial \Lambda(C[i])$; then $\hat{\sigma}_i^m(y) \in \partial \Lambda(U_{i_m})$. Consequently, for any $u \in C[i] \setminus \{x'\}$, (3.4) and (3.6) give

$$\frac{d(u,x')}{d(y,x')} \leq D_0 \frac{d(\hat{\sigma}_i^m(u), \hat{\sigma}_i^m(x'))}{d(\hat{\sigma}_i^m(y), \hat{\sigma}_i^m(x'))} \leq \frac{D_0 C'}{r_0/2}.$$ 

This implies $|C[i]| \leq \frac{4D_0 C'}{r_0} d(y,x')$. Now this is true for all $x' \in C[i']$ and $y \in \partial \Lambda(C[i])$, so $\rho |C[i]| \leq d(C[i], \partial \Lambda(C[i]))$, provided $\rho \leq \frac{r_0}{4D_0 C'}$. \hfill \blacksquare

**Fix an integer** $p_1 \geq p_0$ such that

$$C_1 p_1^{p_1} \leq \min \left\{ \frac{r_0}{2}, \frac{\rho_0}{2} \right\}. \tag{3.7}$$

Our final aim in this section is to prove that the Gibbs measure $\nu$ on $U$ is a *Federer measure*, that is, it has the following property.

**Proposition 3.6.** For every $m \geq 1$ there exists a constant $d_m > 0$ such that

$$\nu(B_U(z, mr)) \leq d_m \nu(B_U(z, r))$$

for all $z \in U$ and all $r > 0$. 

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Proof of Proposition 3.6. Generally speaking we use the idea of the proof of Proposition 5.1 in [St1]. However, the present situation is technically more difficult to deal with since different cylinders may have common points.

Let \( m \geq 1, \ z \in U \) and \( r > 0 \). Without loss of generality we may assume \( m \geq 2 \) and \( z \in \hat{U} \).

Let \( C[i] = C[i_0, i_1, \ldots, i_q] \) be a cylinder of minimal length such that \( z \in C[i] \subset B_U(z, r) \). The minimality of \( q \) shows that \( C[i_0, i_1, \ldots, i_{q-1}] \) is not contained in \( B_U(z, r) \), so \( |C[i_0, i_1, \ldots, i_{q-1}]| \geq r \). This and Lemma 3.5(a) imply
\[
(3.8) \quad \rho r \leq |C[i]| \leq 2r.
\]

**Claim.** There exists a constant \( c(m) \geq 1 \) such that for every \( y \in \hat{U} \cap B_U(z, mr) \setminus C[i] \) there exists a cylinder \( C' \) containing \( y \) with
\[
(3.9) \quad \rho^2 r \leq |C'| \leq \frac{mr}{\rho}, \quad \frac{\nu(C')}{\nu(C[i])} \leq c(m).
\]

To prove the Claim, fix for a moment a point \( y \in \hat{U} \cap B_U(z, mr) \setminus C[i] \). There is an (uniquely determined) infinite admissible sequence \( i_0 = i_0, i_1, i_2, \ldots \) such that \( \sigma_j(y) \in U_{i_j} \) for all \( j \geq 0 \). Clearly there exist \( \ell = 0, 1, \ldots, q \) and \( p \geq 0 \) such that the cylinders \( C[i_0, \ldots, i_{\ell}] \) \( \) and \( C[i_0^p, \ldots, i_{p}^p] \) are matching (take e.g. \( \ell = p = 0 \)). Let \( \ell \) be the largest integer \( \leq q \) for which a number \( p \) with this property exists. Fix \( \ell = \ell(y) \) and then take \( p \) to be the largest integer such that \( C = C[i_0, \ldots, i_{\ell}] \) \( \) and \( C' = C[i_0^p, \ldots, i_{p}^p] \) are matching. It then follows from Lemma 3.5(b) and Lemma 3.3 that
\[
(3.10) \quad \rho |C| \leq |C'| \leq \frac{|C|}{\rho}, \quad \frac{\nu(C')}{\nu(C)} \leq C_2.
\]

This and (3.8) imply \( |C'| \geq \rho^2 r \).

**Case 1.** \( \ell = q \). According to (3.8) and \( m \geq 2 \), \( C' \) satisfies (3.9) with \( c(m) = C_2 \).

**Case 2.** \( \ell < q \). Then using the argument from the proof of Lemma 3.3(a) it follows that
\[
C[i_0, \ldots, i_{\ell}, i_{\ell+1}] \cap C[i_0^p, \ldots, i_{p}^p, i_{p+1}] = \emptyset.
\]

Since \( z \in C[i_0, \ldots, i_{\ell}, i_{\ell+1}] \) and \( y \in C[i_0^p, \ldots, i_{p}^p, i_{p+1}] \cap B_U(z, mr) \), it follows from Lemma 3.5(c) that
\[
\rho r \geq d(z, y) \geq d(C[i_0, \ldots, i_{\ell}, i_{\ell+1}], C[i_0^p, \ldots, i_{p}^p, i_{p+1}]) \geq \rho |C'|.
\]

Hence \( |C'| \leq mr/\rho \). Similarly \( |C| \leq mr/\rho \), and combining this with (3.8) and Lemma 3.5(a), one gets \( \rho r \leq |C[i]| \leq C_1 \rho^{1-\ell} |C| \leq C_1 \rho^{1-\ell} mr \), which implies \( \rho^{1-\ell} \geq \frac{\nu(C)}{\nu(C[i])} \leq \frac{C_1 \rho^{1-\ell} mr}{\nu(C[i])} \), i.e. \( q - \ell \leq b(m) = \frac{\ln(C, m) - 2 \ln \rho}{\ln \rho_1} \). This and (3.5) imply \( \frac{\nu(C)}{\nu(C[i])} \leq \frac{\beta}{\alpha} e^{g_0 b(m)} \), where \( g_0 = \sup_{z \in U} |g(z)| \), while Lemma 3.3 gives \( \nu(C')/\nu(C) \leq C_2 \). Thus, (3.9) holds with \( c(m) = \frac{C_2 \beta}{\alpha} e^{g_0 b(m)} \). This concludes the proof of the Claim.

It follows from the Claim that there exists a finite number of distinct cylinders \( C_1 = C[i], C_2, \ldots, C_s \) in \( U \) such that each of them has a common point with \( \hat{U} \cap B_U(z, mr) \), their union covers \( \hat{U} \cap B_U(z, mr) \), and \( \rho^2 r \leq |C[j]| \leq \frac{mr}{\rho} \) and \( \nu(C[j]) \leq c(m) \nu(C[i]) \) \( \) for all \( j = 1, \ldots, s \). By Lemma 3.5(d), each \( C_j \) has a subcylinder \( C'_j \) of co-length \( p_1 \) such that \( d(C'_j, \partial \lambda(C_j)) \geq \rho |C_j| \geq \rho^2 r \). Choosing an arbitrary point \( x_j \in C'_j \), we then have that the ball \( B_j = B_U(x_j, \rho^2 r) \) is entirely in the interior of \( C_j \) and therefore \( B_j \cap B_i = \emptyset \) for \( j \neq i \) (\( C_j \) and \( C_i \) being distinct cylinders, their interiors have no common points). Moreover, \( B_j \subset C_j \subset B_U(z, 2mr/\rho) \). Thus, we have \( s \) disjoint balls of the same radius \( \rho^2 r \) situated in a ball of radius \( 2mr/\rho \). Since all this is contained
in an \( n \)-dimensional manifold which is Lipschitz homeomorphic to a subset of \( \mathbb{R}^n \), it follows that there exists a constant \( N(m) \) (depending also on \( n, \rho \) and \( U \) but independent of \( z \) and \( r \)) such that \( s \leq N(m) \) in the situation described above.

We now get \( \nu(B_U(z, mr)) = \nu(B_U(z, mr) \cap \tilde{U}) \leq \sum_{j=1}^{s} \nu(C_j) \leq c(m) N(m) \nu(B_U(z, r)) \), so we can take \( d_m = c(m)N(m) \). ■

4 The temporal distance function

We are now going to use the strong local non-integrability condition (SNIC) from Sect. 2. Fix a point \( z_0 \in \Lambda, \epsilon_0 > 0 \) and a continuous family of non-singular linear operators \( Q_z : E^u(z) \rightarrow E^u(z) \) with the properties in (SNIC). Without loss of generality we will assume that \( z_0 = z_1 \in \text{Int}_\Lambda(U_1) \), where \( R_i = \langle U_i, S_i \rangle R_i \) are the members of the Markov family \( \{ R_i \}_{i=1}^k \) (see Sect. 2). Recall that \( U = \cup_{i=1}^k U_i \).

Fix an arbitrary orthonormal basis \( u_1, \ldots, u_n \) in \( E^u(z_0) \) and a \( C^1 \) parametrization \( r(s) = \exp_{z_0}(s), s \in V'_0 \), of a small neighbourhood \( \hat{U}_0 \) of \( z_0 \) in \( W^u_{\epsilon_0}(z_0) \) such that \( V'_0 \) is a convex compact neighbourhood of \( 0 \) in \( \mathbb{R}^n \approx \text{span}(u_1, \ldots, u_n) = E^u(z_0) \). Then \( r(0) = z_0 \) and \( \frac{\partial r_i}{\partial s_j} r(s)_{|s=0} = u_i \) for all \( i = 1, \ldots, n \). Set \( V'_0 = \hat{U}_0 \cap \Lambda \). Shrinking \( \hat{U}_0 \) (and therefore \( V'_0 \) as well) if necessary, we may assume that \( \hat{U}_0 \subset \text{Int}_\Lambda(U_1) \) and \( \langle \frac{\partial r_i}{\partial s_j}(s), \frac{\partial r_i}{\partial s_j}(s) \rangle_{r(s)} - \delta_{ij} \) is uniformly small for all \( i, j = 1, \ldots, n \) and \( s \in V'_0 \), so that

\[
\frac{1}{2} \| s - s' \| \leq d(r(s), r(s')) \leq 2 \| s - s' \| , \quad s, s' \in V'_0 .
\]

Here \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) are the standard norm and inner product in the parameter space \( \mathbb{R}^n \).

For each \( s \in V'_0 \) with \( r(s) \in \Lambda \), let \( \tilde{Q}_s : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the linear operator such that \( dr(s) \circ \tilde{Q}_s = Q_{r(s)} \circ dr(s) \). Then \( Q_s \) depends continuously on \( s \in V'_0 \cap r^{-1}(\Lambda) \), so shrinking \( V'_0 \) if necessary, we may assume that there exists a constant \( \omega > 0 \) such that

\[
\| \tilde{Q}_s(a) \| \geq \omega , \quad a \in S^{n-1} , \quad s \in V'_0 \cap r^{-1}(\Lambda) .
\]

Set \( \hat{Q} = \tilde{Q}_0 = Q_{z_0} \). By (i) in (SNIC), we may also assume that \( \omega \) is chosen in such a way that \( \langle \hat{Q}a, a \rangle \geq \omega \) for all \( a \in S^{n-1} \).

Recall the constants \( p_1, \rho, p_0 \) and \( p_1 \) from Sect. 3, and fix \( \omega \in (0,1) \) with the above properties and a positive integer \( q_0 \) with

\[
(p_1)^{q_0} \leq \frac{\rho^2 \omega}{16 C_1 \| \hat{Q} \|} .
\]

Set

\[
\theta_0 = \frac{\rho^{p_0 (q_0+2)/2} \omega}{4} .
\]

Finally, shrinking again \( V'_0 \) if necessary, we may assume that

\[
\| \tilde{Q} - \hat{Q} \| < \frac{\theta_0}{4} , \quad s \in V'_0 \cap r^{-1}(\Lambda) .
\]
4.1. Definitions. (a) For a cylinder \( C \subset U'_0 \) and a non-zero vector \( a = (a_1, \ldots, a_n) \) in the parameter space \( \mathbb{R}^n = E^u(z_0) \) we will say that a separation by an a-plane occurs in \( C \) if there exist two distinct subcylinders \( C' \) and \( C'' \) of \( C \) of co-lengths \( p_0/q_0 \) and \( c \in \mathbb{R} \) such that \( \sum_{\ell=1}^n a_\ell s_\ell \geq c \) on \( r^{-1}(C') \) and \( \sum_{\ell=1}^n a_\ell s_\ell \leq c \) on \( r^{-1}(C'') \).

Let \( S_n \) be the family of all cylinders \( C \subset U'_0 \) such that a separation by an a-plane occurs in \( C \).

(b) A subset \( V \) of \( U \) will be called regular if there exist finitely many cylinders \( D_1, \ldots, D_p \) in \( U \) with \( V \subset \bigcup_{j=1}^p D_j \) and \( \nu(\bigcup_{j=1}^p D_j \setminus V) = 0 \). (Then clearly \( V = \bigcup_{j=1}^p D_j \).)

(c) Given a regular subset \( V \) of \( U'_0 \) and \( \delta > 0 \), let \( C_1, \ldots, C_p \) \((p = p(\delta) \geq 1)\) be a fixed family of maximal cylinders in \( \mathcal{V} \) with \( |C_j| \leq \delta \) such that \( \mathcal{V} = \bigcup_{j=1}^p C_j \). Set
\[
\mathcal{M}_n^{(\delta)}(V) = \bigcup\{C_j : C_j \in S_{Q_a}, 1 \leq j \leq p\}.
\]

Next, we need the following two elementary lemmas. The proof of the first is given in the Appendix, while that of the second is omitted.

**Lemma 4.2.** Let \( \{u_m : m = 0, 1, 2, \ldots\} \) be an infinite sequence of points in \( \mathbb{R}^n \setminus \{0\} \) such that \( u_m \to 0 \) as \( m \to \infty \). Then there exist a \( C^1 \) curve \( z(t) \) \((0 \leq t \leq a)\) in \( \mathbb{R}^n \) for some \( a > 0 \) with \( z(0) = 0 \), a subsequence \( \{u_{m_p}\} \) of \( \{u_m\} \) and a sequence of positive numbers \( a \geq t_1 \geq t_2 \geq \ldots \geq t_p \geq \ldots \) with \( t_p \to 0 \) as \( p \to \infty \) such that \( z(t_p) = u_{m_p} \) for all \( p \geq 1 \).

**Lemma 4.3.** Let \( a > 0 \) and \( f : (0, a] \to [0, \infty) \) be an arbitrary function such that for any sequence \( \delta_m \searrow 0 \) in \( (0,a] \) we have \( \sum_{m=1}^{\infty} f(\delta_m) < \infty \). Then there exists \( a' \in (0,a] \) such that the set \( \{\delta \in (0,a') : f(\delta) > 0\} \) is finite or countable. ■

The following lemma is a consequence of the assumption (ii) in Sect. 1. A proof of its is easily obtained using for example arguments from [PSW].

**Lemma 4.4.** For any \( z_0 \in \Lambda \) and any \( \delta > 0 \) there exists \( \epsilon > 0 \) such that for any \( y \in W^s_{e_0}(z_0) \cap \Lambda \) with \( d(z_0, y) < \epsilon \) we have \( |\Delta(x, p_y(z))| < \delta d(x, z) \) for all \( x, z \in W^u_{e_0}(z_0) \cap \Lambda \). ■

In what follows we will construct, amongst other things, a sequence of unit vectors \( b_1, b_2, \ldots, b_j_0 \in \mathbb{R}^n \). For each \( \ell = 1, \ldots, j_0 \) set
\[
(4.7) \quad B_\ell = \{a \in \mathbb{S}^{n-1} : \langle a, \hat{Q}_\ell \rangle \geq \theta_0\}.
\]

Below we use the notation \( I_{v,t}g(s) = \frac{g(s+t v) - g(s)}{t}, \quad t \neq 0 \).

Clearly if \( x, y \in \mathbb{S}^{n-1} \) are such that \( \langle x, y \rangle \leq 1/2 \), then \( \|x - y\| \geq 1 \). Thus there exists a positive integer \( j_0' \), depending on \( n \) only, such that for any finite set \( \{x_1, \ldots, x_k\} \subset \mathbb{S}^{n-1} \) with \( \langle x_i, x_j \rangle \leq 1/2 \) for all \( i \neq j \) we have \( k \leq j_0' \). **Fix \( j_0' \) with this property.**

As another preparatory step, fix \( 2j_0' \) distinct points \( \lambda_i^{(j)} \in \text{Int}_\Lambda(U_1) \setminus \overline{U_0} \) \((i = 1, \ldots, j_0', i = 1, 2)\) and for each \( \lambda_i^{(j)} \) **fix a small open neighbourhood** \( V_i^{(j)} \) of its in \( \text{Int}_\Lambda(U_1) \) such that the sets \( \overline{V_i^{(j)}} \) \((j = 1, \ldots, j_0'; i = 1, 2)\) are disjoint and contained in \( \text{Int}_\Lambda(U_1) \). **Fix a constant** \( c_1 > 0 \) such that
\[
(4.8) \quad \text{dist}(\hat{V}_i^{(j)}, \hat{V}_i^{(j)'}) \geq c_1, \quad \text{dist}(\hat{V}_i^{(j)}, \partial_\Lambda(U_1)) \geq c_1
\]
for all \((i,j)\) and all \((i',j') \neq (i,j)\).
Lemma 4.5. (Main Lemma) There exist an integer $j_0 \geq 1$, open subsets $U_0^{(j_0)} \subset \ldots \subset U_0^{(1)}$ of $U_0$, and for each $\ell = 1, \ldots, j_0$, an integer $m_\ell \geq 1$ and a point $\zeta_\ell \in U_0^{(\ell)} \cap \bar{U}$ such that the following hold:

(i) for each $\ell = 1, \ldots, j_0$ and each $i = 1, 2$, there exists a contracting map $w_\ell^{(i)} : U_0^{(\ell)} \rightarrow \bar{V}_\ell^{(i)}$ such that $\sigma^{m_\ell}(w_\ell^{(i)}(x)) = x$ for all $x \in U_0^{(\ell)}$ and $w_\ell^{(i)} : U_0^{(\ell)} \rightarrow w_\ell^{(i)}(U_0^{(\ell)})$ is a homeomorphism;

(ii) for each $\ell = 1, \ldots, j_0$ there exist a number $\delta_\ell \in (0, \delta_0)$ and a vector $b_\ell \in \mathbb{S}^{n-1}$ such that

$$\begin{align*}
\left\langle b_\ell, \tilde{Q} b_{\ell'} \right\rangle &\leq \frac{\omega}{2\|Q\|}, \quad 1 \leq \ell' < \ell,
\end{align*}$$

and

$$\begin{align*}
\inf_{0<|h|\leq \delta_\ell} |I_{\alpha,\beta}||\tau_{m_\ell}(w_2^{(\ell)}(r(s))) - \tau_{m_\ell}(w_1^{(\ell)}(r(s)))| &\geq \frac{\delta_\ell}{|\delta_\ell| |h|}, \quad r(s) \in U_0^{(\ell)}, \ a \in B_\beta,
\end{align*}$$

for all $\ell = 1, \ldots, j_0$, where the inf is taken over $h$ with $r(s + ha) \in U_0^{(\ell)}$.

(iii) for any regular open neighbourhood $V$ of $\zeta_{j_0}$ in $U_0^{(j_0)}$ there exist a constant $\delta' = \delta'(V) \in (0, \delta_0)$ and a finite or countable subset $E$ of $(0, \delta']$ such that

$$M^{(\delta)}_b(V) \cup M^{(\delta)}_{b_1}(V) \cup \ldots \cup M^{(\delta)}_{b_{j_0}}(V) \supset V, \quad \delta \in (0, \delta'] \setminus E.$$ 

Proof of Lemma 4.5. We will construct the required objects by induction.

**Step 1.** Since $z_0 \in U_1$ and $U_1$ has no isolated points, there exists a sequence $\{y_m\} \subset U_1 \setminus \{z_0\}$ such that $y_m \rightarrow z_0$. Now Lemma 4.2 shows that there exists a $C^1$ curve $y(t)$, $0 \leq t \leq \delta$, in $W^{1;m}_0(z_0)$ for some $\delta > 0$ such that $y(0) = z_0$, a subsequence $\{y_{m_p}\}$, and a sequence of positive numbers $t_p \rightarrow 0$ such that $y(t_p) = y_{m_p} \in U_1 \subset \Lambda$. Changing the parametrization of the curve if necessary, we may assume that $b_1 = (d\delta(0))^{-1}(y(0)) \in \mathbb{S}^{n-1}$. Clearly $b_1$ is a direction of $\Lambda$-density in $E^{n}(z_0)$.

It follows from the condition (SNIC) and the choice of $z_0$ that there exist $\bar{z} = r(\bar{s}) \in U_0^{(1)}$, $\bar{y} \in W^{1;m}_{R|s}(\bar{z}) \setminus U_1$ (so $\bar{y} \in \Lambda$), $\delta' > 0$ and $\epsilon'_1 > 0$ such that $|\Delta(r(s + h a), \pi(y)(r(s)))| \geq \delta'_1 |h|$ for all $r(s) \in U_0^{(1)}$ with $d\bar{z}(\bar{s}, \bar{r}(s)) < \epsilon'_1$, $a \in B_1$ and $h \in \mathbb{R}$ with $|h| < \epsilon'_1$ and $r(s + ha) \in U_0^{(1)}$. We will assume that $\epsilon'_1 > 0$ so that $B_U(z, \epsilon'_1) \subset U_0^{(1)}$.

Since $\bar{V}_1^{(1)}$ and $\bar{V}_2^{(1)}$ are open subsets of $U$ having common points with $\Lambda$, it follows that $\mathcal{P}^{m}(\bar{V}_1^{(1)})$ and $\mathcal{P}^{m}(\bar{V}_2^{(1)})$ fill in $R_1$ densely as $m \rightarrow \infty$. Using this and Lemma 4.4, it follows that taking $m_1 \geq 1$ large enough we can find $y_1 \in R_1 \cap \mathcal{P}^{m_1}(\bar{V}_1^{(1)}) \setminus U_1$ arbitrarily close to $\bar{y}$ and $y_1' \in \mathcal{P}^{m_1}(\bar{V}_1^{(1)})$ arbitrarily close to $\bar{z}$ such that $\pi^{(U)}(y_1) = \pi^{(U)}(y_1') \in U_0^{(1)} \cap \bar{U}$ and there exists an open neighbourhood $U_0^{(1)}$ of $\zeta_1 = \pi^{(U)}(y_1) = \pi^{(U)}(y_1')$ in $B_U(z, \epsilon'_1)$ with

$$|\Delta(r(s + h a), \pi_{y_1}(r(s))) - \Delta(r(s + h a), \pi_{y_1'}(r(s)))| \geq \frac{\delta_1}{|h|}$$

whenever $r(s) \in U_0^{(1)}$, $a \in B_1$, $|h| \leq \delta_1$ and $r(s + ha) \in U_0^{(1)}$, where $\delta_1 = \min\{\delta'_1/2, \epsilon'_1\}$. Fix $m_1 \geq 1$, $y_1$ and $y_1'$ with these properties.

Since $y_1 \in R_1 \cap \mathcal{P}^{m_1}(\bar{V}_1^{(1)})$, we have $W^{1;m}_{R_1}(y_1) \cap \mathcal{P}^{m_1}(\bar{V}_1^{(1)}) \neq \emptyset$. Let $O_1^{(1)}$ be a small open neighbourhood of $y_1$ in $W^{1;m}_{R_1}(y_1) \cap \mathcal{P}^{m_1}(\bar{V}_1^{(1)})$ and let $f_1^{(1)} : O_1^{(1)} \rightarrow f_1^{(1)}(O_1^{(1)}) \subset \bar{V}_1^{(1)}$ be a Lipschitz homeomorphism (local inverse of $\mathcal{P}^{m_1}$) such that $\mathcal{P}^{m_1}(f_1^{(1)}(z)) = z$ for all $z \in O_1^{(1)}$. Shrinking $U_0^{(1)}$ if necessary, we may assume that $\pi_{y_1}(U_0^{(1)}) \subset O_1^{(1)}$. Now define a Lipschitz homeomorphism $w^{(1)}_0 : U_0^{(1)} \rightarrow w^{(1)}_0(U_0^{(1)}) \subset \bar{V}_1^{(1)}$ by $w^{(1)}_0(x) = f_1^{(1)}(\pi_{y_1}(x))$. We then have $\mathcal{P}^{m_1}(w^{(1)}_0(x)) = \pi_{y_1}(x)$.
and therefore \( \sigma^m_1(w^{(1)}_1(x)) = x \) for all \( x \in U^{(1)}_0 \). Moreover, \( \text{Lip}(w^{(1)}_1) \leq \frac{1}{\alpha \gamma_m^{1}} \), so assuming \( m_1 \) is sufficiently large, \( w^{(1)}_1 \) is contracting.

In the same way one constructs a Lipschitz homeomorphism \( w^{(1)}_2 : U^{(1)}_0 \rightarrow w^{(1)}_2(U^{(1)}_0) \subset \hat{V}^{(1)}_2 \) (shrinking \( U^{(1)}_0 \) again if necessary) with \( \mathcal{P}^m_1(w^{(1)}_2(x)) = \pi y'_1(x) \) for all \( x \in U^{(1)}_0 \). Then \( \sigma^m_1(w^{(1)}_2(x)) = x \) for all \( x \in U^{(1)}_0 \).

Now for \( z = r(s) \in U^{(1)}_0 \), \( a \in B_1 \) and \( h \in \mathbb{R} \) with \( r(s + ha) \in U^{(1)}_0 \) we get

\[
[\tau^m_1(w^{(1)}_2(r(s + ha))) - \tau^m_1(w^{(1)}_2(r(s + ha)))] - [\tau^m_1(w^{(1)}_2(r(s))) - \tau^m_1(w^{(1)}_2(r(s)))] = \Delta(\mathcal{P}^m_1(w^{(1)}_1(r(s + ha))), \mathcal{P}^m_1(w^{(1)}_1(r(s))) - \Delta(\mathcal{P}^m_1(w^{(1)}_2(r(s + ha))), \mathcal{P}^m_1(w^{(1)}_2(r(s))))
\]

\[
= \Delta_{\mathcal{P}^m_1}(w^{(1)}_1(r(s + ha))), \pi y'_1(r(s)) - \Delta(\pi y'_1(r(s + ha)), \pi y'_1(r(s)))
\]

\[
= \Delta(r(s + ha), \pi y'_1(r(s)))
\]

(Here we used the fact that \( \Delta(x, y) = \Delta(\pi(U)(x), y) \) for any \( x, y \in \gamma_1 \) ) Now (4.12) gives

\[
|\Delta(r(s + ha), \pi y'_1(r(s))) - \Delta(r(s + ha), \pi y'_1(r(s)))| \geq \delta_1 |h|
\]

so \( |I_{a,b}[\tau^m_1(w^{(1)}_2(r(s))) - \tau^m_1(w^{(1)}_2(r(s)))]| \geq \delta_1 \) whenever \( r(s) \in U^{(1)}_0 \), \( a \in B_1 \), \( 0 < |h| \leq \epsilon'_1 \) and \( r(s + ha) \in U^{(1)}_0 \).

In this way we have completed the first step in our recursive construction. Whether we need to make more steps or not depends on which of the following two alternatives takes place.

**Alternative 1.A.** For any regular open neighbourhood \( V \) of \( \zeta_1 \) in \( V_0^{(1)} \) there exist a constant \( \delta_1 \in (0, \delta_0) \) and a finite or countable subset \( \mathcal{E}_1 \) of \( (0, \delta_1) \) such that \( M_{b_1}^{(\delta)}(V) \supset V \) for all \( \delta \in (0, \delta_1] \setminus \mathcal{E}_1 \).

**Alternative 1.B.** Alternative 1.A does not hold.

If the case of Alternative 1.A we simply terminate the recursive construction at this stage.

If Alternative 1.B takes place, we need to make at least one more step in the inductive construction.

**Inductive Step.** Suppose that for some \( j \geq 1 \) we have constructed open subsets \( U^{(j)}_0 \subset \ldots \subset U^{(1)}_0 \) of \( U'_0 \), and for each \( \ell = 1, \ldots, j \), an integer \( m_\ell \geq 1 \) and a point \( \zeta_\ell \in U^{(j)}_0 \cap U'_0 \) such that the conditions (i), (ii) and (iii) in the lemma are fulfilled with \( j_0 \) replaced by \( j \).

There are two alternatives again.

**Alternative j.A.** For any regular open neighbourhood \( V \) of \( \zeta_j \) in \( U^{(j)}_0 \) there exist \( \delta_j \in (0, \delta_{j-1}) \) and a finite or countable subset \( \mathcal{E}_j \) of \( (0, \delta_j] \) such that

\[
M_{b^j_1}(V) \cup M_{b^j_2}(V) \cup \ldots \cup M_{b^j_\ell}(V) \supset V, \quad \delta \in (0, \delta_j] \setminus \mathcal{E}_j.
\]

**Alternative j.B.** Alternative j.A does not hold.

If the case of Alternative j.A we terminate the recursive construction at this stage.

Next, assume that Alternative j.B takes place. One then needs to complete

**Step j+1.** Construct an open subset \( U^{(j+1)}_0 \) of \( U^{(j)}_0 \), an integer \( m_{j+1} \geq 1 \) and a point \( \zeta_{j+1} \in U^{(j+1)}_0 \) such that the conditions (i), (ii) and (iii) in the lemma are fulfilled with \( j_0 \) replaced by \( j + 1 \).
It follows from Alternative j.B that there exists a regular open neighbourhood $V$ of $\zeta_j$ in $U_0^{(j)}$ and a sequence $\mu_m \searrow 0$ such that for every $m$,

$$A_\delta(V) = M_{b_1}^{(\delta)}(V) \cup M_{b_2}^{(\delta)}(V) \cup \ldots \cup M_{b_j}^{(\delta)}(V)$$

does not contain $V$ for uncountably many $\delta \in (0, \mu_m]$. Thus, $\nu(V \setminus A_\delta(V)) > 0$ for uncountably many $\delta \in (0, \mu_m]$, i.e. for the function $f(\delta) = \nu(V \setminus A_\delta(V))$, $\delta \in (0, \delta_j]$, we have that for any $m \geq 1$ there are uncountably many $\delta \in (0, \mu_m)$ so that $f(\delta) > 0$. It now follows from Lemma 4.3 that there exists a sequence $\mu'_m \searrow 0$ in $(0, \delta_j]$ such that $\sum_{m=1}^{\infty} f(\mu'_m) = \infty$. Then Borel-Cantelli’s Lemma implies that there exists $\hat{\delta} \in V \setminus A_{\mu'_m}(V)$ for infinitely many $m$. Fix a neighbourhood $V$ of $\zeta_j$ in $U_0^{(j)}$ and $\hat{\delta} \in V \setminus \hat{U}$ with these properties. Choosing an appropriate subsequence of $\{\mu'_m\}$, we will assume that $\hat{\delta} \notin \bigcup_{\ell=1}^{j} M_{b_\ell}^{(\mu'_m)}(V)$ for all $\ell$. Consequently, for any $m \geq 1$ there exists a cylinder $C^{(m)} \subset V$ containing $\hat{\delta}$ with $|C^{(m)}| \leq \mu'_m$ and $C^{(m)} \notin \mathcal{S}_{Qb}$ for any $\ell = 1, \ldots, j$.

Fix $m$ for a moment, and let $C^{(m)} = C[i_0, i_1, \ldots, i_{k_m}]$ for some $k_m$. There exists a subcylinder $C' = C[i_0, i_1, \ldots, i_{k_m}, i'_{k_m+1}]$ of $C^{(m)}$ of co-length 1 with $\hat{\delta} \in C'$. Consider an arbitrary subcylinder $C'' = C[i_0, i_1, \ldots, i_{k_m}, i'_{k_m+1}]$ of $C^{(m)}$ of co-length 1 such that $i'_{k_m+1} \neq i_{k_m+1}$. It follows from Lemma 3.5(d) that there exists $x_m \in C''$ with $d(x_m, \partial(C'')) \geq \rho |C''|$. Fix a point $x_m$ with this property; then Lemma 3.5(a) implies

\begin{equation}
(4.13)\quad d(x_m, \hat{\delta}) \geq d(x_m, \partial(C'')) \geq \rho |C''| \geq \rho^2 |C^{(m)}|.
\end{equation}

Let $C_1, \ldots, C_q$ be all subcylinders of $C^{(m)}$ of co-length $p_0 q_0$. We may assume $\hat{\delta} \in C_1$, $x_m \in C_2$, and $C_1 \neq C_2$. Fix an arbitrary $\ell = 1, \ldots, j$ for a moment. Since $C^{(m)} \notin \mathcal{S}_{Qb}$, there exists a hyperplane $\alpha_\ell \in \mathbb{R}^n$ with $\tilde{Q}b_\ell \perp \alpha_\ell$ such that each of the sets $r^{-1}(C_1)$ and $r^{-1}(C_2)$ has common points with both half-spaces determined by $\alpha_\ell$. If $z'$ and $y$ are the orthogonal projections in $\alpha_\ell$ of $r^{-1}(\hat{\delta})$ and $r^{-1}(x_m)$, respectively, then obviously $\|r^{-1}(\hat{\delta}) - z'\| \leq \|r^{-1}(C_1)\|$, and Lemma 3.5(a) and (4.2) imply $\|r^{-1}(\hat{\delta}) - z'\| \leq 2 |C_1| \leq 2 C_1 (p_1)^{q_0} |C^{(m)}|$. Similarly, $\|r^{-1}(x_m) - y\| \leq 2 C_1 (p_1)^{q_0} |C^{(m)}|$, and combining these with (4.13), (4.2) and (4.3), gives that for the angle $\beta_m$ between the vector $r^{-1}(x_m) - r^{-1}(\hat{\delta})$ and the plane $\alpha_\ell$ we have

$$\sin \beta_m \leq \frac{4 C_1 (p_1)^{q_0} |C^{(m)}|}{\rho^2 |C^{(m)}|/2} = \frac{8 C_1 (p_1)^{q_0} \rho^2}{\rho^2} \leq \frac{\omega}{2 \|\tilde{Q}\|}.\n$$

That is,

$$\left\langle \frac{r^{-1}(x_m) - r^{-1}(\hat{\delta})}{\|r^{-1}(x_m) - r^{-1}(\hat{\delta})\|}, \frac{\tilde{Q}b_\ell}{\|\tilde{Q}b_\ell\|} \right\rangle \leq \frac{\omega}{2 \|\tilde{Q}\|},\n$$

for all $\ell = 1, \ldots, j$.

Clearly, $x_m \to \hat{\delta}$ as $m \to \infty$, and choosing a subsequence, we may assume that

$$b_{j+1} = \lim_{m \to \infty} \frac{r^{-1}(x_m) - r^{-1}(\hat{\delta})}{\|r^{-1}(x_m) - r^{-1}(\hat{\delta})\|} \in \mathbb{S}^{n-1}$$

exists. Let $\tilde{\delta} = r(\hat{\delta})$; then $dr(\hat{\delta}) \cdot b_{j+1}$ is a direction of $\Lambda$-density at $\hat{\delta}$, and according to the above,

$$\langle b_{j+1}, \frac{\tilde{Q}b_\ell}{\|\tilde{Q}b_\ell\|} \rangle \leq \frac{\omega}{2 \|\tilde{Q}\|} \quad \text{for all } \ell = 1, \ldots, j.$$

It follows from the condition (SNIC) that there exist $\bar{\zeta} = r(\hat{\delta}) \in V \subset U_0^{(j)}$, $\bar{y} \in W_{R_1}^{(\bar{\zeta})}(\bar{\zeta}) \setminus U_1$, $\delta_{j+1}' \in (0, \delta_j)$ and $\epsilon_{j+1}' > 0$ such that $|\Delta(r(s + h a), \pi\bar{y}(r(s)))| \geq \delta_{j+1}' \|h\|$ for all $r(s) \in V$ with
\[ d(\tilde{z}, r(s)) < \epsilon'_{j+1}, a \in B_{j+1} \text{ and } h \in \mathbb{R} \text{ with } |h| < \epsilon'_{j+1} \text{ and } r(s + ha) \in U^{(j)}_0. \] We will assume that \( \epsilon'_{j+1} > 0 \) is so small that \( B_U(\tilde{z}, \epsilon'_{j+1}) \subset V. \)

Since \( V^{(j+1)}_1 \) and \( V^{(j+1)}_2 \) are open subsets of \( U \) having common points with \( U \), it follows that \( \mathcal{P}^{m}(V^{(j+1)}_1) \) and \( \mathcal{P}^{m}(V^{(j+1)}_2) \) fill in \( R_1 \) densely as \( m \to \infty \). As in Step 1, using this and Lemma 4.4, choosing \( m_{j+1} \geq 1 \) sufficiently large, we can find \( y_{j+1} \in R_1 \cap \mathcal{P}^{m_{j+1}}(V^{(j+1)}_1) \setminus U_1 \) arbitrarily close to \( \tilde{y} \) and \( y'_{j+1} \in \mathcal{P}^{m}(V^{(j+1)}_2) \) arbitrarily close to \( \tilde{z} \) such that \( \pi^{(U)}(y_{j+1}) = \pi^{(U)}(y'_{j+1}) \in U_0^{(j)} \cap \hat{U} \) and there exists an open neighbourhood \( U^{(j+1)}_0 \) of \( \zeta_{j+1} = \pi^{(U)}(y_{j+1}) = \pi^{(U)}(y'_{j+1}) \) in \( U_0^{(j)} \cap B_U(\tilde{z}, \epsilon'_{j+1}) \) with

\[ |\Delta(r(s + ha), \pi_{y_{j+1}}(r(s))) - \Delta(r(s + ha), \pi_{y'_{j+1}}(r(s)))| \geq \hat{\delta}_{j+1} |h| \]

whenever \( r(s) \in U_0^{(j+1)}, a \in B_{j+1} \) and \( |h| \leq \hat{\delta}_{j+1} \) with \( r(s + ha) \in U_0^{(j+1)} \), where \( \hat{\delta}_{j+1} = \min\{\epsilon'_{j+1}/2, \epsilon'_{j+1}\} \). Fix \( m_{j+1} \geq 1, y_{j+1} \) and \( y'_{j+1} \) with these properties.

Again as in Step 1, shrinking the neighbourhood \( U_0^{(j+1)} \) of \( \zeta_{j+1} \) if necessary, one constructs contracting homeomorphisms \( w_i^{(j+1)} : U_0^{(j+1)} \to w_i^{(j+1)}(U_0^{(j+1)}) \subset V_i^{(j+1)} (i = 1, 2) \) with \( \sigma^{m_{j+1}}(w_i^{(j+1)}(x)) = x \) for all \( x \in U_0^{(j+1)} \), and derives from (4.14) that

\[ |I_{a,h}[\tau_{m_{j+1}}(w_2^{(j+1)}(r(s))) - \tau_{m_{j+1}}(w_1^{(j+1)}(r(s)))]| \geq \hat{\delta}_{j+1} \]

for all \( r(s) \in U_0^{(j+1)}, a \in B_{j+1}, 0 < |h| \leq \hat{\delta}_{j+1} \) with \( r(s + ha) \in U_0^{(j+1)}. \)

This completes Step \( j + 1. \)

To show that this inductive procedure terminates after not more than \( j_0' \) steps, it is enough to observe that for the vectors \( v_\ell = \sqrt{Q}b_\ell \| \sqrt{Q}b_\ell \| \) we have \( \langle v_\ell, v_\ell' \rangle \leq 1/2 \) whenever \( \ell \neq \ell' \). Indeed, assuming \( \ell' < \ell \), we have \( \langle b_\ell, \sqrt{Q}b_\ell \| \sqrt{Q}b_\ell \| \rangle \leq \omega/(2\|\sqrt{Q}\|) \), which implies

\[ \left( \frac{\sqrt{Q}b_\ell}{\|\sqrt{Q}b_\ell\|}, \frac{\sqrt{Q}b_{\ell'}}{\|\sqrt{Q}b_{\ell'}\|} \right) \leq \frac{\omega \|\sqrt{Q}b_\ell\|}{2\|\sqrt{Q}b_\ell\| \|\sqrt{Q}b_{\ell'}\|} \leq \frac{\omega}{2\|\sqrt{Q}b_\ell\| \|\sqrt{Q}b_{\ell'}\|}. \]

By the choice of \( \omega \), for any \( v \in S^{n-1} \) we have \( \omega \leq \langle \sqrt{Q}v, v \rangle = \langle \sqrt{Q}v, \sqrt{Q}v \rangle = \|\sqrt{Q}v\|^2 \), so \( \|\sqrt{Q}v\| \geq \sqrt{\omega} \), and the above gives \( \langle v_\ell, v_\ell' \rangle \leq 1/2. \)

The choice of \( j_0' \) now shows that the inductive construction can not have more than \( j_0' \) steps, so it will terminate at some step \( j_0 \leq j_0' \). That is, for some \( j_0 \leq j_0' \) the Alternative \( j_0.A \) holds, and then we terminate the construction at that step. ■

Set \( \hat{\delta} = \min_{1 \leq j \leq j_0} \hat{\delta}_j, n_0 = \max_{1 \leq j \leq j_0} m_j \) and \( \hat{z}_0 = \zeta_{j_0} \in U_0^{(j_0)} \cap \hat{U} \).

Lemma 4.6. There exist an open neighbourhood \( U_0 \) of \( \hat{z}_0 \) in \( U_0^{(j_0)} \) and an integer \( n_1 \geq 1 \) such that \( \text{Int}_\Lambda(U) = \sigma^{n_1}(U_0) \) and \( \sigma^{n_1} : U_0 \to \sigma^{n_1}(U_0) \) is a homeomorphism.

Proof of Lemma 4.6. Since \( \hat{z}_0 \in \hat{U} \), there exists a cylinder \( C[i] = C[i_0, \ldots, i_m] \) such that \( \hat{z}_0 \in C[i] \subset U_0^{(j_0)}. \) For the matrix \( A \) we have \( A^M > 0 \), so for each \( j = 1, \ldots, k \) there exists an admissible string \( s^{(j)} = (i_{m+1}^{(j)}, \ldots, i_{m+M}^{(j)}) \) such that \( i_{m+M}^{(j)} = j \) and \( A_{i_{m+1}^{(j)} i_{m+1}^{(j)}} = 1. \) Fix an arbitrary string \( s^{(j)} \) with this property.

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Now set $n_1 = m + M$ and $U_0 = \bigcup_{j=1}^{k} \hat{C}_\Lambda [i; s^{(j)}]$. It is the easy to show that
\begin{equation}
\text{Int}_\Lambda(U) = \bigcup_{j=1}^{k} \sigma^{n_1} \left( \hat{C}_\Lambda [i; s^{(j)}] \right) = \sigma^{n_1}(U_0),
\end{equation}
which proves the lemma. ■

Using the above lemma, **fix an open neighbourhood $U_0$ of $\hat{z}_0$ in $U^{(j_0)}_0$ and $n_1 > 0$ such that**
\begin{equation}
U = \sigma^{n_1}(U_0) = \text{Int}_\Lambda(U) = U_0
\end{equation}
and $\sigma^{n_1} : U_0 \rightarrow U$ is a homeomorphism. Consider the inverse homeomorphism
\begin{equation}
\psi : U \rightarrow U_0 \text{ such that } \sigma^{n_1}(\psi(x)) = x, \ x \in U.
\end{equation}
Then $\tilde{r}(s) = \sigma^{n_1}(r(s))$, $s \in V_0$, where $V_0 = r^{-1}(U_0) \subset V'_0$, gives a Lipschitz parametrization of $U$ with $\psi(\tilde{r}(s)) = r(s)$ for all $s \in V_0$. Finally, set
\begin{equation}
V_i^{(j)} = w_i^{(j)}(U_0) \subset \hat{V}_i^{(j)}, \ i = 1, 2; \ j = 1, \ldots, j_0.
\end{equation}
The following two lemmas are proved essentially by using arguments from [D1] and Lemma 4.4 above. We omit the proof of the first and sketch that of the second.

**Lemma 4.7.** For every $\delta'' > 0$ there exists an integer $n_2 > 0$ such that for any $m > n_2$ and any $j = 1, \ldots, j_0$ and $i = 1, 2$ there exist contracting maps $\tilde{v}_i^{(j)} : V_i^{(j)} \rightarrow U$ with $\sigma^m(\tilde{v}_i^{(j)}(w)) = w$ for all $w \in V_i^{(j)}$ such that
\begin{equation}
\text{Lip}(\tau_m \circ \tilde{v}_i^{(j)}) \leq \delta'' \text{ on } V_i^{(j)}.
\end{equation}

Set $\delta'' = \frac{c_0 \delta}{8}$, fix $n_2 = n_2(\delta'') > 0$ with the properties listed in Lemma 4.7, and denote $N_0 = n_0 + n_1 + n_2$.

**Lemma 4.8.** For any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(j)}, v_2^{(j)} : U \rightarrow U$ ($j = 1, \ldots, j_0$) such that $\sigma^N(v_i^{(j)}(x)) = x$ for all $x \in U$ and
\begin{equation}
I_{a,h} [\tau_N(v_2^{(j)}(\tilde{r}(s))) - \tau_N(v_1^{(j)}(\tilde{r}(s)))] \geq \frac{\delta}{2}
\end{equation}
for all $j = 1, \ldots, j_0$, $s \in V_0$, $0 < |h| \leq \delta$ and $a \in B_j$ such that $s$ and $s + h a$ are in $V_0$.

**Proof of Lemma 4.8.** Let $N \geq N_0$. For any $j = 1, \ldots, j_0$, $m_j \leq n_0$ implies $m = N - m_j - n_1 > n_2$, so for each $i = 1, 2$ there exists a contracting homeomorphisms
\begin{equation}
\tilde{v}_i^{(j)} : V_i^{(j)} \rightarrow \tilde{v}_i^{(j)}(V_i^{(j)}) \subset U \text{ with } \sigma^{N - m_j - n_1}(\tilde{v}_i^{(j)}(w)) = w, \ w \in V_i^{(j)},
\end{equation}
such that (4.19) holds with $m = N - m_j - n_1$ and $\delta''$ as above. Now define Lipschitz maps
\begin{equation}
v_i^{(j)} : U \rightarrow U, \ v_i^{(j)}(x) = \tilde{v}_i^{(j)}(w_i^{(j)}(\psi(x))).
\end{equation}
Let $a \in B_j$, and let $s$ and $s + ha$ (for some $h$ with $0 < |h| \leq \hat{\delta}$) belong to $V_0$. Since $\text{Lip}(w_t^{(j)}(x)) \leq 1/(c_0 \gamma^m) < 1/c_0$, as in [D1] (see also [St1]), using (4.2) and (4.19) with $m = N - m_j - n_i$, it follows that

$$
\left| I_{a,h} \left[ \tau_{N-m_j-n_i} (\tilde{v}_t^{(j)}(w_t^{(j)}(r(s)))) \right] \right| \leq \text{Lip} \left[ \tau_{N-m_j-n_i} (\tilde{v}_t^{(j)}) \right] \cdot \text{Lip}(w_t^{(j)}) \cdot \text{Lip}(r)
$$

$$
\leq \hat{\delta}'' \frac{2}{c_0} \leq \frac{\hat{\delta}}{4}.
$$

Combining this with the above and (4.10), gives

$$
\left| I_{a,h} \left[ \tau_N (v_2^{(j)}(\omega(s))) - \tau_N (v_1^{(j)}(\omega(s))) \right] \right| \geq \hat{\delta} - 2 \frac{\hat{\delta}}{4} = \frac{\hat{\delta}}{2}.
$$

This proves the assertion. $\blacksquare$

## 5 Dolgopyat’s operator

The central point here is to prove the $L^1$-contraction property of Dolgopyat’s normalized operator $L_{ab}$ with respect to the norm $\|h\|_{\text{Lip},b}$.

**Theorem 5.1.** There exist a positive integer $N$ and constants $\hat{\rho}_1 \in (0, 1)$ and $a_0 > 0$ such that for any $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $|b| \geq 1/a_0$ and every $h \in C^{\text{Lip}}(U)$ with $\|h\|_{\text{Lip},b} \leq 1$ we have $\int_U |L_{ab}^N h|^2 \, dv \leq \hat{\rho}_1^m$ for every positive integer $m$, where $\nu$ is the Gibbs measure determined by $f - \mu$ on $U$.

Theorem 2.1 is derived from the above in the same way as in [D1] (see also the proof of Corollary 3.3(a) in [St1]).

Given $A > 0$, denote by $K_A(U)$ the set of all functions $h \in C^{\text{Lip}}(U)$ such that $h > 0$ and $|h(u) - h(u')| \leq A d(u, u')$ for all $u, u'$ that belong to the same $U_i$ for some $i = 1, \ldots, k$. Notice that $h \in K_A(U)$ implies $|\ln h(u) - \ln h(v)| \leq A d(u, v)$ and therefore $e^{-A d(u, v)} \leq h(u)/h(v) \leq e^{A d(u, v)}$ for any $u, v \in U_i$, $i = 1, \ldots, k$.

To prove Theorem 5.1 we will use the following lemma which is similar to Lemma 10′′ in [D1].

**Lemma 5.2.** There exist a positive integer $N$ and constants $\hat{\rho} \in (0, 1)$, $a_0 > 0$ and $E \geq 1$ such that for every $a, b \in \mathbb{R}$ with $|a| \leq a_0$, $1/|b| \leq a_0$, there exists an operator $N : C^{\text{Lip}}(U) \rightarrow C^{\text{Lip}}(U)$ (depending on $a$ and $b$) with the following properties:

(a) The operator $N$ preserve the cone $K_{E|b|}(U)$;

(b) For all $H \in K_{E|b|}(U)$ we have $\int_U |NH|^2 \, dv \leq \hat{\rho} \int_U H^2 \, dv$.

(c) If $h, H \in C^{\text{Lip}}(U)$ are such that $H \in K_{E|b|}(U)$, $|h(u)| \leq H(u)$ for all $u \in U$ and $|h(u) - h(u')| \leq E|b|H(u') d(u, u')$ whenever $u, u' \in U_i$ for some $i = 1, \ldots, k$, then $|L_{ab}^N h(u)| \leq (NH)(u)$ for all $u \in U$ and $|(L_{ab}^N h)(u) - (L_{ab}^N h)(u')| \leq E|b|(NH)(u') d(u, u')$ whenever $u, u' \in U_i$ for some $i = 1, \ldots, k$.

As in [D1], Theorem 5.1 is an easy consequence of Lemma 5.2.

The remainder of this section if devoted to the proof of Lemma 5.2. We begin with a technical lemma whose proof is given in the Appendix.
Lemma 5.3. There exists a constant $A_0 > 0$ such that for all $a \in \mathbb{R}$ with $|a| \leq a_0'$ the following hold:

(a) If $H \in K_B(U)$ for some $B > 0$, then
\[
\left| (\mathcal{M}_a^m H)(u) - (\mathcal{M}_a^m H)(u') \right| \leq A_0 \left[ \frac{B}{\gamma^m} + \frac{T}{\gamma - 1} \right] d(u, u')
\]
for all $m \geq 1$ and all $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If $h, H \in C(U)$ and $B > 0$ are such that $H > 0$ on $U$ and $|h(v) - h(v')| \leq BH(v') d(v, v')$ for any $v, v' \in U_i$, $i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b \in \mathbb{R}$ with $|b| \geq 1$ we have
\[
\left| L_{ab}^m h(u) - L_{ab}^m h(u') \right| \leq A_0 \left[ \frac{B}{\gamma^m} (\mathcal{M}_a^m H)(u') + |b| (\mathcal{M}_a^m |h|)(u') \right] d(u, u')
\]
whenever $u, u' \in U_i$ for some $i = 1, \ldots, k$.

Throughout we will frequently use the objects constructed in the previous section, notably $\hat{\delta} > 0$, the integers $n_0, n_1 \geq p_1$ and $n_2$, the sets $U_0 \subset U_1$ and $U = \sigma^{n_1}(U_0) = \text{Int}_\Lambda(U)$, etc.

We will now impose certain condition on the numbers $N$, $\epsilon_1$, $b$ and $\mu$ that will be used frequently throughout. Where these conditions come from will become clear later on.

Set
\[
E = \max \left\{ 4A_0, \frac{2A_0 T}{\gamma - 1} \right\},
\]
where $A_0$ is the constant from Lemma 5.3, fix an integer $N > N_0 = n_0 + n_1 + n_2$ such that
\[
\gamma^N \geq \max \left\{ 6A_0, \frac{200 \gamma_i^{n_1}}{c_0}, \frac{32 \gamma_i^{n_1} E}{c_0 \delta \rho^{n_1 + 3}} \right\},
\]
and define $\bar{v}_i^{(j)}$ and $v_i^{(j)}$ by (4.20) and (4.21). Then the conclusion of Lemma 4.8 holds. Set
\[
\epsilon(0) = \min \left\{ \frac{1}{32C_0}, c_1, \frac{1}{4E}, \frac{1}{\delta \rho^{n_1 + 2}}, \frac{c_0 r_0}{\gamma_i^{n_1}} \right\}.
\]

It follows from Lemma 4.5 with $V = U_0$ that there exist a constant $\delta' = \delta'(U_0) \in (0, \delta_0)$ and a finite or countable subset $\mathcal{E}$ of $(0, \delta']$ such that
\[
M_{b_1}^{(\delta)}(U_0) \cup \ldots \cup M_{b_0}^{(\delta)}(U_0) \supset U_0, \quad \delta \in (0, \delta'] \setminus \mathcal{E},
\]
Fix $\delta'$ and $\mathcal{E}$ with the above properties. Let $\epsilon_1 > 0$ and $b \in \mathbb{R}$ be such that $|b| \geq 1$,
\[
\frac{\epsilon(0)}{2} \leq \epsilon_1 < \epsilon(0),
\]
and
\[
\frac{\epsilon_1}{|b|} \in (0, \delta'] \setminus \mathcal{E}.
\]
In what follows most of the time $\epsilon_1$ and $b$ will stay fixed, however at the end of the section we will vary them so that (5.4) and (5.5) hold.

Let $C_j = C_j^{(p_1/|b|)}$ ($1 \leq j \leq p$) be the fixed family of maximal cylinders in $\overline{U_0}$ with $|C_j| \leq \frac{c_1}{|b|}$ such that $U_0 \subset \cup_{j=1}^p C_j$ and $\overline{U_0} = \cup_{j=1}^p C_j$ (see Definition 4.1). Then $\nu(U_{j=1}^p C_j \setminus U_0) = 0$. It follows from
(5.5), (5.2) and Lemma 3.1, that the length of each $C_j$ is not less than $n_1$, so $\sigma^{n_1}$ is expanding on $C_j$. Moreover, Lemma 3.5(a) implies that $|C_j| \geq \rho \frac{\epsilon_1}{|b|}$ for all $j$, so

$$\rho \frac{\epsilon_1}{|b|} \leq |C_j| \leq \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq p.$$

Another important property of the cylinders $C_j$ which follows from their choice and the construction of the maps $w_i^{(l)}$, $\hat{v}_i^{(l)}$, and $v_i^{(l)}$ (see the proof of Lemma 4.8) is that $v_i^{(l)}(C_j)$ is contained in a cylinder of length $N$ for any $l = 1, \ldots, j_0$ and $i = 1, 2$, i.e. we have that (2.2) holds for any $u_1, u_2 \in v_i^{(l)}(C_j)$ and any $m = 0, 1, \ldots, N$.

Next, let $D_1, \ldots, D_q$ be the list of all cylinders in $U_0$ that are subcylinders of co-length $p_0 q_0$ of some $C_j$ ($1 \leq j \leq p$). That is, if $k_j$ is the length of $C_j$, we consider the subcylinders of length $k_j + p_0 q_0$ of $C_j$, and we do this for any $j = 1, \ldots, p$. Then

$$U_0 = C_1 \cup \ldots \cup C_p = D_1 \cup \ldots \cup D_q.$$

Moreover, it follows from the properties of $C_j$ and Lemma 3.5 that

$$\rho^{p_0 q_0 + 1} \frac{\epsilon_1}{|b|} \leq |D_j| \leq (\rho_1)^{q_0} \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq q.$$

We will say that $D_j$ and $D_{j'}$ are adjacent if they are subcylinders of the same $C_i$. If for some $\ell = 1, \ldots, j_0$ there exists a hyperplane $\alpha$ in $\mathbb{R}^n$ that separates the sets $r^{-1}(D_j)$ and $r^{-1}(D_{j'})$ and $\alpha \perp \tilde{Q}b_{\ell}$, we will say that $D_j$ and $D_{j'}$ are $\tilde{Q}b_{\ell}$-separable (see the beginning of Sect. 4 for the definition of $\tilde{Q}$).

For each $j = 1, \ldots, q$, fix a subcylinder $D_j'$ of $D_j$ of co-length $p_1$ and a subcylinder $D_j''$ of $D_j'$ of co-length $p_1$ such that

$$d(D_j', \partial_l(D_j)) \geq \rho |D_j|, \quad d(D_j'', \partial_l(D_j')) \geq \rho |D_j'|.$$

The existence of such subcylinders follows from Lemma 3.5(d). Notice that combining (5.9), Lemma 3.5(a) and (5.8) gives $d(D_j'', \partial_l(D_j')) \geq \rho^{p_0 q_0 + p_1 + 2} \frac{\epsilon_1}{|b|}$. Thus, there exists a Lipschitz function $\eta_j : U \rightarrow [0, 1]$ such that $\eta_j = 0$ on $U_2 \cup \ldots \cup U_k$, $\eta_j = 0$ on $U_1 \setminus D_j'$, $\eta_j = 1$ on $D_j''$, and

$$|\eta_j(x) - \eta_j(y)| \leq \frac{C_3 |b|}{\epsilon_1} d(x, y), \quad x, y \in U_1,$$

for some constant $C_3 > 0$ (depending on $\rho, p_0, q_0$), which we choose so that $C_3 \geq \max \{C_0, C_1, \frac{1}{\rho^{p_0 q_0 + 3}}\}$.

For each $j$ fix a function $\eta_j$ with the above properties.

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, j_0$ and $i = 1, 2$, denote

$$Y_{j,\ell} = w_i^{(l)}(D_j), \quad Z_j = \sigma^{m_1}(D_j), \quad Z_j' = \sigma^{m_1}(D_j'), \quad Z_j'' = \sigma^{m_1}(D_j''), \quad X_{j,\ell} = \{v_i^{(l)}(u) : u \in Z_j\}.$$

See the diagram below.
It then follows that $D_j = \psi(Z_j)$, $X_{j,\ell} = v_i^{(\ell)}(Y_{j,\ell})$, and $U = \cup_{j=1}^b Z_j$.

Let $J$ be a subset of the set

$$
\Xi = \{(i,j,\ell) : 1 \leq i \leq 2, 1 \leq j \leq q, 1 \leq \ell \leq j_0\}.
$$

Fix for a moment $\mu = \mu(N, \epsilon_1) \in (0,1)$ such that

$$
0 < \mu \leq \min \left\{ \frac{1}{2}, \frac{c_0 \rho^{\rho_0+2}}{4 C_3 \gamma_1^N}, \frac{1}{4 e^2 A_{1N}}, \sin^2 \left( \frac{\delta \rho^{\rho_0+3} \epsilon_0}{32} \right), \min_{i \neq j} d(U_i, U_j) \right\},
$$

where $A_1 = T + (P + 1)\tau + 1$, and define the function $\beta = \beta_{\mu,b,J} : U \rightarrow [0,1]$ by

$$
\beta(v) = \begin{cases} 1 - \mu \eta_j(\sigma^{N-n_1}(v)), & v \in X_{j,\ell}^i \text{ for some } (i,j,\ell) \in J, \\ 1, & \text{otherwise}. \end{cases}
$$

Notice that $v \in X_{j,\ell}^i$ is equivalent to $v = v_i^{(\ell)}(u)$ for some $u \in Z_j$; then $\sigma^N(v)$ is well-defined and $\sigma^N(v) = \sigma^N(v_i^{(\ell)}(u)) = u$. Thus, $\sigma^{N-n_1}(v) = \sigma^{N-n_1}(\psi(v_i^{(\ell)}(u))) = \psi(v_i^{(\ell)}(u))$, so

$$
\sigma^{N-n_1}(v) = \sigma^m(\psi(v_i^{(\ell)}(u))) = \psi(u) = \psi(\sigma^N(v)) \in D_j.
$$

This implies $\beta(v) = 1 - \mu \eta_j(\sigma^{N-n_1}(v))$ for $v \in X_{j,\ell}^i$. Since the sets $D_j$ have no common interior points and $\eta_j$ is zero near $\partial_{\Lambda}(D_j)$, the definition of $\beta$ is correct and $\beta$ is Lipschitz. Notice that if $\sigma^{N-n_1}(v) \in D'_{j'}$ for some $(i,j,\ell) \in J$, then $\eta_j(\sigma^{N-n_1}(v)) = 1$, so $\beta(v) = 1 - \mu$.

We will now prove that

$$
\text{Lip}_{U_i}(\beta) \leq \frac{2 \mu C_3 \gamma_1^N}{c_0 \rho^{\rho_0+2}} \cdot \frac{|b|}{\epsilon_1}, \quad i = 1, \ldots, k.
$$

For $i > 1$ this is obvious, so consider $v, v' \in U_1$. It is enough to consider the case $v, v' \in \bar{U}$. We have to show that

$$
|\beta(v) - \beta(v')| \leq \frac{2 \mu C_3 \gamma_1^N}{c_0 \rho^{\rho_0+2}} \cdot \frac{|b|}{\epsilon_1} \cdot d(v, v').
$$

If $\beta(v) = \beta(v') = 1$ this is trivial, so assume $\beta(v) < 1$. Then $v \in X_{j,\ell}^i$ for some $(i,j,\ell) \in J$, so $\eta_j(\sigma^{N-n_1}(v)) > 0$ and therefore $\sigma^{N-n_1}(v) \in D'_j$. Let $C[i] = C[i_0, i_1, \ldots, i_{N-n_1}]$ be the (unique) cylinder of length $N - n_1$ containing $v$. Then $\sigma^{N-n_1}(v) \in D'_j \subset D_j \subset U_1$ implies $i_{N-n_1} = 1$ and moreover by (5.9) and (5.8),

$$
d(\sigma^{N-n_1}(v), \partial_{\Lambda}(U_1)) \geq d(D'_j, \partial_{\Lambda}(D_j)) \geq \rho^{\rho_0+2} \cdot \frac{\epsilon_1}{|b|}.
$$

Therefore by (2.2),

$$
d(v, \partial_{\Lambda}(C[i])) \geq \frac{c_0}{\gamma_1^{N-n_1}} \cdot \frac{\epsilon_1}{|b|} \geq \frac{c_0 \rho^{\rho_0+2}}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|}.
$$

We will consider two cases for $v'$.

**Case 1.** $v' \in C[i]$. Then (2.2) is applicable and gives

$$
d(\sigma^{N-n_1}(v), \sigma^{N-n_1}(v')) \leq \frac{\gamma_1^N}{c_0} \cdot d(v, v').
$$

**Subcase 1.1.** $\sigma^{N-n_1}(v') \in D_j$. Then $\sigma^{N-n_1}(v') \notin D'_j$, for any $j' \neq j$, so $\beta(v') = 1 - \mu \eta_j(\sigma^{N-n_1}(v'))$, regardless whether $v' \in X'_{j',\ell}$, for some $(i',j',\ell') \in J$ or not. Combining this with (5.10) and the above gives

$$
|\beta(v) - \beta(v')| \leq \mu |\eta_j(\sigma^{N-n_1}(v)) - \eta_j(\sigma^{N-n_1}(v'))| \\
\leq \mu \frac{C_3 |b|}{\epsilon_1} \cdot d(\sigma^{N-n_1}(v), \sigma^{N-n_1}(v')) \leq \mu \frac{C_3 |b| \gamma_1^N}{\epsilon_1 c_0} \cdot d(v, v'),
$$

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so (5.13) holds.

Subcase 2.2. \( \sigma^{N-n_1}(v') \notin \mathcal{D}_j \). Then by (5.9) and (5.8), \( d(\sigma^{N-n_1}(v), \sigma^{N-n_1}(v')) \geq \rho^{\rho_0+2} \frac{\epsilon_1}{|b|} \), so \( \frac{\gamma_1^N}{c_0} d(v,v') \geq \rho^{\rho_0+2} \frac{\epsilon_1}{|b|} \), which implies

\[
|\beta(v) - \beta(v')| \leq 2\mu \frac{\gamma_1^N}{c_0 \rho^{\rho_0+2}} \frac{|b|}{\epsilon_1} d(v,v') \leq \frac{2\mu C_3 \gamma_1^N}{c_0 \rho^{\rho_0+2}} \frac{|b|}{\epsilon_1} d(v,v') .
\]

Thus (5.13) holds again.

Case 2. \( v' \notin C[i] \). Then (5.14) gives \( d(v,v') \geq d(v, \partial \Lambda(C[i])) \geq \frac{c_0 \rho^{\rho_0+2}}{\gamma_1^N} \cdot \frac{\epsilon_1}{|b|} \). Consequently,

\[
|\beta(v) - \beta(v')| \leq 2\mu \frac{\gamma_1^N}{c_0 \rho^{\rho_0+2}} \frac{|b|}{\epsilon_1} d(v,v') .
\]

This completes the proof of (5.13).

Next, define the operator \( \mathcal{N} = \mathcal{N}_a^{(J,\mu)} : C^{\text{Lip}}(U) \to C^{\text{Lip}}(U) \) by

\[
(\mathcal{N} h) (u) = (\mathcal{M}_a^N (\beta \cdot h)) (u) , \quad u \in U .
\]

The following lemma contains statements similar to Proposition 6 and Lemma 11 in [D1]. The proofs are almost the same, so we omit them.

**Lemma 5.4.** Under the above conditions for \( N \) and \( \mu \) the following hold:

(a) \( \mathcal{N} h \in K_{E|b|}(U) \) for any \( h \in K_{E|b|}(U) \);

(b) If \( h \in C^{\text{Lip}}(U) \) and \( h \in K_{E|b|}(U) \) are such that \( |h| \leq H \) in \( U \) and \( |h(v) - h(v')| \leq E|b| H(v') d(v,v') \) for any \( v, v' \in U_j \), \( j = 1, \ldots, k \), then \( |(L^N_{ab} h)(u) - (L^N_{ab} h)(u')| \leq E|b| (N H)(u') d(u,u') \) whenever \( u, u' \in U_i \) for some \( i = 1, \ldots, k \). ■

In what follows for each integer \( m > 0 \), \( d_m \) will denote a fixed constant with the property described in Proposition 3.6.

Given \( t > 0 \) and \( m > 0 \), a subset \( W \) of \( U \) will be called \((t,m)\)-dense in \( U \) if for every \( u \in U \) there exists \( v \in U \) such that \( B_U(v,t) \cap U \subset W \cap B_U(u,m t) \).

Recall the parametrization \( \tilde{r}(s) = \sigma^{n_1}(r(s)) \), \( s \in V_0 \), of \( U = \sigma^{n_1}(U_0) \) (see Sect. 4). Set

\[
L_0 = \text{Lip}(\tilde{r}) , \quad L_0' = \text{Lip}(\tilde{r}^{-1}) , \quad L_n = (2[L_0 L_0' \sqrt{n}] + 3)^n .
\]

The following is a consequence of Proposition 3.6; its proof is almost the same as in [D1] (cf. also [St1]).

**Lemma 5.5** Let \( A > 0 \), \( m > 0 \) and let \( \epsilon = \epsilon (m,A) = \frac{1}{L_n e^{4m^2 A^2} d_{2m}} \). Then for any \( t > 0 \), any \((t,m)\)-dense subset \( W \) of \( U \) and any \( H \in K_{A/t}(U) \), we have

\[
\int_W H^2 \ dv \geq \epsilon \int_U H^2 \ dv .
\]

We will say that the subset \( J \) of \( \Xi \) is dense if for any \( j = 1, \ldots, q \) there exists \((i,j',\ell) \in J \) such that \( \mathcal{D}_J \) and \( \mathcal{D}_{j'} \) are adjacent. (Compare this with the corresponding definitions in [D1] and [St1]).
The following lemma is similar to Lemma 12 in [D1].

**Lemma 5.6.** Given the numbers $N$ and $\mu$, there exist $a_0 = a_0(N, \mu) > 0$ and $\epsilon_2 = \epsilon_2(N, \mu) \in (0, 1)$ such that

$$\int_U (NH)^2 d\nu \leq (1 - \epsilon_2) \int_U H^2 d\nu$$

whenever $J$ is dense, $H \in K_{E|b|}(U)$, and

$$|a| \leq a_0 \quad , \quad 1/|b|^{\alpha} \leq a_0.$$ 

More precisely, we can take

$$a_0 = \min \left\{ \left( a_0' \right)^{1/\alpha}, \frac{-\ln(1 - \epsilon' \mu e^{-NT})}{4NT}, \frac{\epsilon' \mu e^{-NT}}{4C_0} \right\}, \quad \epsilon_2 = \frac{\epsilon' \mu e^{-NT}}{4},$$

where

$$\epsilon' = \frac{1}{L_n d_{2m'} e^{4(E\gamma_n \epsilon(0)/c_0)^2}}, \quad m' = \frac{\gamma_n}{c_0^2 \gamma_n \rho^2 p_1 + p_0 q_0 + 2}.$$ 

**Sketch of Proof of Lemma 5.6.** The definition of $N$ and the Cauchy-Schwartz inequality imply

$$(NH)^2(u) = (\mathcal{M}_a^N(\beta H))^2(u) \leq (\mathcal{M}_a^N \beta^2(u) \cdot (\mathcal{M}_a^N H^2(u))$$

for all $u \in U$.

Denote $W = \cup_{(i,j,\ell) \in J} Z''_j$. Then $u \in W$ means that there exists $(i, j, \ell) \in J$ with $\psi(u) \in D''_j$, and so $\beta(v_1^f(u)) = 1 - \mu$.

We will now show that $W$ is $(t, m')$-dense in $U$, where

$$t = c_0 \gamma_n \rho^2 p_1 + p_0 q_0 + 2, \quad \frac{c_1}{|b|}, \quad m' = \frac{\gamma_n}{c_0^2 \gamma_n \rho^2 p_1 + p_0 q_0 + 2}.$$ 

For any $j = 1, \ldots, q$, by Lemma 3.5(d) there exists a subcylinder $D''_j$ of $D''$ of co-length $p_1$ such that $d(D''_j, \partial(D''_j)) \geq \rho|D''_j|$. Let $u \in U$. Then $u \in Z''_j$ for some $j' = 1, \ldots, q$, i.e. $\psi(u) \in D''_{j'}$. Since $J$ is dense, there exists $(i, j, \ell) \in J$ so that $D_j$ and $D_{j'}$ are adjacent, i.e. $D_j$ and $D_{j'}$ are contained in the same $C_m$ for some $m$. Choosing an arbitrary $v' \in D''_j$, by (5.9), Lemma 3.5(a) and (5.8) we have

$$d(v', \partial(D''_j)) \geq \rho|D''_j| \geq \rho^2 p_1 + p_0 q_0 + 2. \quad \frac{c_1}{|b|} = t'. $$

Thus, $B_U(v', t') \subset D''_j$, so for $v = \sigma^{n_1}(v') \in Z_j$, using (2.2) and the fact that $\sigma^{n_1}$ is expanding on $C_m$, we get

$$d(v, \partial(\sigma^{n_1}(C_m))) \geq c_0 \gamma_n d(v', \partial(\sigma^{n_1}(C_m))) \geq c_0 \gamma_n d(v', \partial(D''_j))) \geq c_0 \gamma_n t' = t.$$ 

Thus, $B_U(v, t) \subset \sigma^{n_1}(C_m)$, and moreover $B_U(v, t) \subset \sigma^{n_1}(B_U(v', t')) \subset \sigma^{n_1}(D''_j) = Z''_j \subset W$. On the other hand, for any $x \in B_U(v, t)$, using $u, x \in \sigma^{n_1}(C_m)$ and (5.6) gives $d(u, x) < |\sigma^{n_1}(C_m)| \leq \frac{\gamma_n}{c_0} \cdot \frac{\gamma_n}{c_0} = m't$, so $x \in B_U(u, m't)$. Thus, $B_U(v, t) \subset B_U(u, m't) \cap W$, which proves that $W$ is a $(t, m')$-dense subset of $U$.

Let $H \in K_{E|b|}(U)$. Setting $A = E c_0 \gamma_n \rho^2 p_1 + p_0 q_0 + 2 \epsilon_1$, we have $H \in K_{A}(U)$. By (5.4), $\epsilon_1 < \epsilon(0)$, so $\epsilon' \leq \frac{1}{L_n \epsilon(\mu A^2) d_{2m'}}$, and therefore $\epsilon' \leq \epsilon(m', E)$, the number defined in Lemma 5.5

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with $m$ replaced by $m'$. It then follows from Lemma 5.5 that $\int_W H^2 \, dv \geq \epsilon' \int_U H^2 \, dv$, and as in [D1] (cf. also [St1]) one derives that

$$
\int_U (NH)^2 \, dv \leq (1 - \epsilon' \mu e^{-NT}) \int_U M_a^N(H^2) \, dv.
$$

(5.17)

This, $\lambda_a = e^{Pr(f-(P+a)r)} \geq e^{-T|a|}$ and $1 \geq h_a \geq 1 - C_0|a|$ for $|a| \leq a_0$ yield $\int_U (NH)^2 \, dv \leq (1 - \epsilon_2) \int_U H^2 \, dv$, since $(1 - \epsilon' \mu e^{-NT}) e^{2\epsilon \mu e^{-NT}} \leq 1 - \epsilon_2$. 

In what follows we assume that $h, H \in C^{\text{Li}}(U)$ are such that

$$
H \in K_{E|b}(U) , \quad |h(u)| \leq H(u) , \quad u \in U ,
$$

and

$$
|h(u) - h(u')| \leq E|b| H(u') \, dv(u, u') \quad \text{whenever } u, u' \in U_i , \ i = 1, \ldots, k .
$$

(5.18)

(5.19)

Given $\mu \in (0, 1/2)$, following [D1], define the functions $\chi^{(i)}_{\ell} : U \rightarrow \mathbb{C}$ ($\ell = 1, \ldots, j_0, \ i = 1, 2$) by

$$
\chi^{(1)}_{\ell}(u) = \frac{e^{f_{\ell}(u) + i\tau_N(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{f_{\ell}^{(2)}(u) + i\tau_N(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))}{(1 - \mu) e^{f_{\ell}(u)} h(v_1^{(\ell)}(u)) + e^{f_{\ell}^{(2)}(u)} h(v_2^{(\ell)}(u))} ,
$$

$$
\chi^{(2)}_{\ell}(u) = \frac{e^{f_{\ell}(u) + i\tau_N(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{f_{\ell}^{(2)}(u) + i\tau_N(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u))}{e^{f_{\ell}(u)} h(v_1^{(\ell)}(u)) + (1 - \mu) e^{f_{\ell}^{(2)}(u)} h(v_2^{(\ell)}(u))} .
$$

Lemma 5.7. Assume $\epsilon_1$ and $b$ are chosen in such a way that (5.4) and (5.5) hold. Then for any $j = 1, \ldots, q$ there exist $i = 1, 2$, $j' = 1, \ldots, q$ and $\ell = 1, \ldots, j_0$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi^{(i)}_{\ell}(u) \leq 1$ for all $u \in Z_{j'}$.

The proof of this lemma makes use of the following two lemmas, the first of which coincides with Lemma 14 in [D1] and its proof is almost the same.

Lemma 5.8. If $h$ and $H$ satisfy (5.18)-(5.19), then for any $j = 1, \ldots, q$, $i = 1, 2$ and $\ell = 1, \ldots, j_0$ we have:

(a) $\frac{1}{2} \leq \frac{H(v_i^{(\ell)}(u'))}{H(v_i^{(\ell)}(u''))} \leq 2$ for all $u', u'' \in Z_j$;

(b) Either for all $u \in Z_j$ we have $|h(v_i^{(\ell)}(u))| \leq \frac{3}{4} H(v_i^{(\ell)}(u))$, or $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$ for all $u \in Z_j$. 

For any $\ell = 1, \ldots, j_0$ consider the function

$$
\gamma_{\ell}(u) = b[\tau_N(v_1^{(\ell)}(u)) - \tau_N(v_2^{(\ell)}(u))] \quad u \in U .
$$

Lemma 5.9. Let $j, j' \in \{1, 2, \ldots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\tilde{Q}_{b_1}$-separable for some $\ell = 1, \ldots, j_0$. Then $|\gamma_{\ell}(u) - \gamma_{\ell}(u')| \geq c_2 \epsilon_1$ for all $u \in Z_{j}$ and $u \in Z_{j}'$, where $c_2 = \frac{\hat{\delta} \rho^{\rho_{b_1} + 3}}{2}$. 

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Proof of Lemma 5.9. Let \( u \in Z_j \) and \( u' \in Z_{j'} \); then \( v = \psi(u) \in D_j \) and \( v' = \psi(u') \in D_{j'} \). Also \( u = \hat{r}(s) \), \( u' = \hat{r}(s') \) for some \( s, s' \in V_0 \), hence \( v = r(s) \) and \( v' = r(s') \).

Since \( D_j \) and \( D_{j'} \) are \( \tilde{Q}b_e \)-separable, there exists a hyperplane \( \alpha \in \mathbb{R}^n \) separating \( r^{-1}(D_j) \) and \( r^{-1}(D_{j'}) \) and such that \( \langle \tilde{Q}b_e, a \rangle > 0 \). Without loss of generality we will assume that \( r^{-1}(D_j) \) is in the upper half-space (with respect of the normal \( \tilde{Q}b_e \) to \( \alpha \)), so that for \( a = \frac{s - s'}{\|s - s'\|} \in S^{n-1} \) we have \( \langle \tilde{Q}b_e, a \rangle > 0 \). Moreover, \( D_j \) and \( D_{j'} \) are adjacent, so they are contained in the same cylinder \( C_m \) for some \( m \).

Let \( w \) and \( w' \) be the orthogonal projections in \( \alpha \) of \( s \) and \( s' \), respectively. Since \( v \in D_j \), the choice of \( D_j' \), (4.2), Lemma 3.5(a) and (5.9) imply

\[
\|s - w\| \geq d(s, r^{-1}(\partial \Lambda(D_j))) \geq \frac{1}{2} d(v, \partial \Lambda(D_j)) \geq \frac{\rho}{2} |D_j| \geq \frac{\rho_{p_0+2}}{2} |C_m|.
\]

Similarly, \( \|s' - w'\| \geq \frac{\rho_{p_0+2}}{2} |C_m| \). On the other hand, \( \|s - s'\| \leq 2 d(v, v') \leq 2 |C_m| \), so

\[
\left\langle \langle \tilde{Q}b_e \rangle_{\|Qb_e\|} \cdot a \right\rangle = \frac{\|s - w\| + \|s' - w'\|}{\|s - s'\|} \geq \frac{\rho_{p_0+2}}{2} |C_m| = \frac{\rho_{p_0+2}}{2}.
\]

Thus, using (4.3) and (4.5), \( \langle \tilde{Q}b_e, a \rangle \geq \frac{\rho_{p_0+2}}{2} \|Qb_e\| \geq \frac{\rho_{p_0+2}}{2} \omega = \theta_0 \), so by (4.7), \( a \in B_\ell \). According to the choice of \( v_1^{(\ell)} \) and \( v_2^{(\ell)} \), the conclusions of Lemma 4.8 hold, so

\[
\left| \tau_N(v_2^{(\ell)}(\tilde{r}(s))) - \tau_N(v_1^{(\ell)}(\tilde{r}(s))) \right| \geq \frac{\delta}{2}
\]

for all \( s \in V_0 \) and \( h \neq 0 \) such that \( s + ha \in V_0 \). Since \( \hat{s}, \hat{s}' \in V_0 \) and \( \hat{s} = \hat{s}' + ha \) with \( h = \|s - s'\| \), it follows that

\[
\left| \frac{\tau_N(v_2^{(\ell)}(\tilde{r}(s))) - \tau_N(v_1^{(\ell)}(\tilde{r}(s)))}{\|s - s'\|} \right| \geq \frac{\delta}{2} \left( \|s - w\| + \|s' - w'\| \right) \geq \frac{\delta}{2} \rho_{p_0+2} |C_m| \geq \frac{\delta \rho_{p_0+3}}{2 |b|} c_1.
\]

Hence \( |\gamma_\ell(u) - \gamma_\ell(u')| \geq |b| \frac{\delta \rho_{p_0+3} e_1}{2 |b|} = c_2 \epsilon_1 \). ■

Proof of Lemma 5.7. Given \( j = 1, \ldots, q \), let \( m = 1, \ldots, p \) be such that \( D_j \subset C_m \). By (5.5), \( \delta = \epsilon_1 / |b| \in (0, \delta] \setminus \mathcal{E} \), so it follows from (5.3) that \( C_m \subset M_{b(t)}(U_0) \) for some \( \ell = 1, \ldots, j_0 \). This means that there exists two subcylinders \( D_j \) and \( D_{j'} \) of \( C_m \) that are \( \tilde{Q}b_e \)-separable. (Notice that we may have \( j' = j \) or \( j'' = j \).)

Fix \( \ell, j' \) and \( j'' \) with the above properties, and set \( \tilde{Z}_j = Z_j \cup Z_{j'} \cup Z_{j''} \). If there exist \( t \in \{j, j', j''\} \) and \( i = 1, 2 \) such that the first alternative in Lemma 5.8 (b) holds for \( Z_t \), \( \ell \) and \( i \), then \( \chi^{(\ell)}_t(u) \leq 1 \) for any \( u \in Z_t \) follows immediately.

Assume that for every \( t \in \{j, j', j''\} \) and every \( i = 1, 2 \) the second alternative in Lemma 5.8 (b) holds for \( Z_t \), \( \ell \) and \( i \), i.e.

\[
(5.20) \quad |h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u)) , \quad u \in \tilde{Z}_j.
\]

Given \( u, u' \in \tilde{Z}_j \subset C_m \) we have \( \sigma^{N-m_1}(v_i^{(\ell)}(u)), \sigma^{N-m_1}(v_i^{(\ell)}(u')) \in C_m \). Thus, using (5.18), (5.19), (5.20), (5.6), (5.1) and the choice of \( C_m \) (see also the remark after (5.6)), and assuming
Thus, the angle between the complex numbers \( h(v_i^{(\ell)}(u)) \) and \( h(v_i^{(\ell)}(u')) \) (regarded as vectors in \( \mathbb{R}^2 \)) is less than \( \pi/3 \). In particular, we can choose real continuous functions \( \theta_i(u), u \in \mathbb{Z}_j, i = 1, 2, \) such that \( h(v_i^{(\ell)}(u)) = e^{i\theta_i(u)}h(v_i^{(\ell)}(u)) \) for all \( u \in \mathbb{Z}_j \). Using the above and some elementary geometry yields
\[
|\theta_i(u) - \theta_i(u')| \leq 2 \sin \frac{|\theta_i(u) - \theta_i(u')|}{2} < 2 \frac{c_2 \epsilon_1}{8} = \frac{c_2 \epsilon_1}{4}.
\]

Consider the function
\[
\Gamma_{\ell}(u) = \text{Arg} \left( e^{ib\tau N(v_i^{(\ell)}(u))} h(v_i^{(\ell)}(u)) \right) - \text{Arg} \left( e^{ib\tau N(v_i^{(\ell)}(u))} h(v_i^{(\ell)}(u)) \right),
\]
and let \( u' \in Z_j' \) and \( u'' \in Z_j'' \). Since \( D_j' \) and \( D_j'' \) are adjacent and \( \mathcal{Q}_b \)-separable, it follows from Lemma 5.9 and the above that
\[
|\Gamma_{\ell}(u') - \Gamma_{\ell}(u'')| \geq |\gamma(v') - \gamma(v'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq c_2 \epsilon_1 - \frac{c_2 \epsilon_1}{4} = \frac{c_2 \epsilon_1}{2},
\]

Thus, \( |\Gamma_{\ell}(u') - \Gamma_{\ell}(u'')| \geq \frac{c_2 \epsilon_1}{4} \) for all \( u' \in Z_j' \) and \( u'' \in Z_j'' \). Hence either \( |\Gamma_{\ell}(u')| \geq \frac{c_2 \epsilon_1}{4} \) for all \( u' \in Z_j' \) or \( |\Gamma_{\ell}(u')| \geq \frac{c_2 \epsilon_1}{4} \) for all \( u'' \in Z_j'' \).

Assuming for example that \( |\Gamma_{\ell}(u)| \geq \frac{c_2 \epsilon_1}{4} \) for all \( u \in Z_j' \), one shows as in [D1] (cf. also [St1]) that \( \chi^{(i)}_{\ell}(u) \leq 1 \) for all \( u \in Z_j', i = 1, 2 \). This proves the lemma. ■

**Proof of Lemma 5.2.** As before, we assume that \( N \) and \( \mu > 0 \) satisfy (5.1) and (5.11).

Let \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \setminus \{0\} \) satisfy (5.15), where \( a_0 > 0 \) is defined by (5.16). Fix for a moment \( \epsilon_1 \) with (5.4) and assume that \( b \) satisfies (5.5). Define the subset \( J = J(b) \) of \( \Xi \) in the following way. First, include in \( J \) all \( (1, j, \ell) \in \Xi \) such that \( \chi^{(1)}_{\ell}(u) \leq 1 \) for all \( u \in Z_j' \). Then for any \( j = 1, \ldots, q \) and \( \ell = 1, \ldots, j_0 \) include \( (2, j, \ell) \) in \( J \) if and only if \( (1, j, \ell) \) has not been included in \( J \) (that is, \( \chi^{(1)}_{\ell}(u) > 1 \) for some \( u \in Z_j' \)) and \( \chi^{(2)}_{\ell}(u) \leq 1 \) for all \( u \in Z_j' \). It follows from Lemma 5.7 that \( J \) is dense. Consider the operator \( \mathcal{N} = \mathcal{N}^{J}_{ab} : \text{C}^{\text{Lip}}(U) \longrightarrow \text{C}^{\text{Lip}}(U) \). Then Lemma 5.4 (a) implies property (a) in Lemma 5.2, while Lemma 5.6 gives property (b) in Lemma 5.2 with \( \hat{\rho} = 1 - \epsilon_2 = \hat{\rho}(\mathcal{N}) \).

To check (c) in Lemma 5.2, assume that \( h, H \in \text{C}^{\text{Lip}}(U) \) satisfy (5.18) and (5.19). Then Lemma 5.4 (b) implies \( |(L_{ab}^{N} h)(u) - (L_{ab}^{N} h)(u')| \leq E|b|\mathcal{N}(H)(u) d(u, u') \) whenever \( u, u' \in U \) for some \( i = 1, \ldots, k \). So, it remains to show that
\[
|(L_{ab}^{N} h)(u)| \leq (\mathcal{N} H)(u) , \quad u \in U .
\]

Let \( u \in U \). If \( u \notin Z_j' \) for any \( (i, j, \ell) \in J \), then \( \beta(v) = 1 \) whenever \( \sigma^N v = u \) (since \( v \in X^{(i)}_{j,\ell} \) implies \( u = \sigma^{N} v \in Z_j \), and \( |(L_{ab}^{N} h)(u)| \leq (\mathcal{M}^{\hat{\rho}}_{ab}(\beta H))(u) = (\mathcal{N} H)(u) \) follows immediately. If \( u \in Z_j' \) for some \( (i, j, \ell) \in J \) and \( i = 1 \) (the case \( i = 2 \) is similar), one shows as in [D1] (cf. also [St1]) that \( |(L_{ab}^{N} h)(u)| \leq (\mathcal{N} H)(u) \), which proves (5.22).
Since for $\epsilon_1$ we can choose arbitrary values in the interval $[\epsilon(0)/2, \epsilon(0)]$, the above shows that the operator $\mathcal{N}$ has the properties (a), (b) and (c) in Lemma 5.2 for all $a$ and $b$ satisfying (5.15) and $|b| \geq \epsilon(0)/\delta'$. This completes the proof of Lemma 5.2. \[\square\]

6 Geodesic flows on manifolds of constant negative curvature

In this section Theorem 1.3 is proved.

Let $P : \mathbb{H}^{n+1} \rightarrow X = \mathbb{H}^{n+1}/\Gamma$ and $\hat{P} : S^*(\mathbb{H}^{n+1}) \rightarrow M = S^*(X)$ be the natural projections. Consider the geodesic flow $\varphi_t : S^*(\mathbb{H}^{n+1}) \rightarrow S^*(\mathbb{H}^{n+1})$ on $\mathbb{H}^{n+1}$. Recall that the geodesics in $\mathbb{H}^{n+1}$ are either straight lines perpendicular to $\partial \mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1} : x_1 = 0\}$ or semi-circles with centers in $\partial \mathbb{H}^{n+1}$ whose planes are perpendicular to $\partial \mathbb{H}^{n+1}$.

It is known that the non-wandering set $\Lambda \subset M$ of $\phi_t$ has the form $\Lambda = \hat{P}(\hat{\Lambda})$, where $\hat{\Lambda}$ is the set of those $x \in S^*(\mathbb{H}^{n+1})$ such that both $\lim_{t \rightarrow \infty} \varphi_t(x)$ and $\lim_{t \rightarrow -\infty} \varphi_t(x)$ belong to the limit set $L(\Gamma)$ of the group $\Gamma$. The assumptions made in Theorem 1.3 imply that $L(\Gamma)$ is a non-empty $\Gamma$-invariant closed subset of $\partial \mathbb{H}^{n+1}$ without isolated points (see Ch. 12 in [Rate]).

A horosphere in $\mathbb{H}^{n+1}$ is either an $n$-sphere in $\mathbb{H}^{n+1}/\Gamma$ tangent to $\partial \mathbb{H}^{n+1}$, or an $n$-plane in $\mathbb{H}^{n+1}$ parallel to $\partial \mathbb{H}^{n+1}$. Let $S$ be a horosphere and $x \in S \cap \mathbb{H}^{n+1}$. If $S$ is an $n$-plane, denote by $\nu_S(x)$ the outward normal to $S$ at $x$ with $||\nu_S(x)|| = 1/x_1$, while if $S$ is an $n$-plane, set $\nu_S(x) = -\frac{1}{x_1} e_1 = \frac{1}{x_1} (-1,0,\ldots,0)$. The stable and unstable manifolds for $z = (x,\nu_S(x))$ in $S^*(\mathbb{H}^{n+1})$ are given by

$$W^s(x) = \{(y, -\nu_S(y)) : y \in S \cap \mathbb{H}^{n+1}\} \quad \text{and} \quad W^u(x) = \{(y, \nu_S(y)) : y \in S \cap \mathbb{H}^{n+1}\},$$

so obviously the local stable and unstable foliations are smooth. The projections of the latter via $\hat{P}$ give the local stable and unstable foliations in $M$, so condition (ii) of Sect. 1 is clearly satisfied.

To check the condition (iii) it is again enough to work on the universal cover $\mathbb{H}^{n+1}$. Let $\hat{z} = (z,\zeta) \in S^*(\mathbb{H}^{n+1})$ and $t > 0$. Since the isometry group of $\mathbb{H}^{n+1}$ is both point and direction transitive, we may assume that $z = (1,0,\ldots,0)$ and $\zeta = -e_1$. Then

$$W^u(\hat{z}) = \{(y, -e_1) : y_1 = 1\} \quad \text{and} \quad W^u(\varphi_t(\hat{z})) = \{(w, -e^{-t} e_1) : w_1 = 1 - e^{-t}\}.$$ Obviously, for any smooth curve $\gamma$ in $W^u(\hat{z})$ of length $\ell_\gamma$, the length of $\phi_t(\gamma)$ is exactly $e^t \cdot \ell_\gamma$. Thus, for any $\hat{x}, \hat{y} \in W^u(\hat{z}) \setminus \{\hat{z}\}$ we have $d(\hat{x},\hat{y}) = d(\phi_t(\hat{x}),\phi_t(\hat{y}))$. Since $\hat{P}$ is a local isometry conjugating the geodesic flows $\varphi_t$ and $\hat{\varphi}_t$, it follows that $\phi_t$ satisfies the condition (iii).

It remains to check the condition (SNIC) of Sect. 2. Again we will work on the universal cover $\mathbb{H}^{n+1}$. Let $\theta_0 > 0$ and assume $\epsilon_0 < 1$. Fix an arbitrary $z(0) \in \hat{\Lambda}$ and let $\Gamma : T_2 M \rightarrow T_2 M$ be the identity operator for any $z \in M$. Replacing the group $\Gamma$ by a conjugate of its, we may assume that $z(0) = (x(0),-e_1)$, where $x_1(0) = 1$ and $e_1 = (1,0,\ldots,0) \in \mathbb{H}^{n+1}$. Then $W^u_\epsilon(z(0))$ is a subset of $W = \{(x,-e_1) \in S^*(\mathbb{H}^{n+1}) : x_1 = 1\} \times \mathbb{R}^n$. In what follows for any $x = (x_1,\ldots,x_n) \in \mathbb{H}^{n+1}$ we denote $x' = (0,x_2,\ldots,x_n)$. Consider an arbitrary $\hat{\zeta} = (\hat{x},-e_1) \in \hat{\Lambda} \cap W$ close to $z(0)$, and let $b \in \partial \mathbb{H}^{n+1}$ be a direction of $\hat{\Lambda}$-density at $\hat{\zeta}$, $\|b\|_{\hat{\zeta}} = 1$. Setting

$$\epsilon = \min\{\epsilon_0/4, \theta_0/15\} < \frac{1}{4},$$

the above implies the existence of $\omega = (\hat{y},-e_1) \in \hat{\Lambda} \cap W \setminus \{\hat{\zeta}\}$ such that

$$\|\hat{y} - \hat{x}\| < \epsilon \quad \text{and} \quad \|\frac{\hat{y} - \hat{x}}{\|\hat{y} - \hat{x}\|} - b\| < \epsilon.$$
Next, there exists $\gamma = (p, \tilde{e}) \in \hat{\Lambda}$ arbitrarily close to $\hat{z}$ such that $\gamma$ lies on the axis of a hyperbolic transformation $\tilde{g} \in \Gamma$ (see e.g. Sect. 12.1 in [Ratc]). Shifting $\omega$ along the flow, we may assume that $p_1 = 1$. Fix $\gamma$ so close to $\hat{z}$ that

\begin{equation}
\|p - \hat{x}\| < \epsilon \|\hat{y} - \hat{x}\|, \quad \|\tilde{e} - e_1\| < \epsilon \|\hat{y} - \hat{x}\|.
\end{equation}

Changing the coordinate system in $\mathbb{H}^{n+1}$ if necessary we may assume that $p = (1, 0, \ldots, 0)$ and $\tilde{e} = (-\sqrt{1 - \mu^2}, \mu, 0, \ldots, 0)$ for some small $\mu > 0$. Then the axis of $\tilde{g}$ is a semi-circle $K$ with centre $\hat{d} = (0, -\mu, 0, \ldots, 0)$ for some $d > 0$ passing through $p$ with $K \perp \partial \mathbb{H}^{n+1}$. Let $\gamma_\epsilon$ be the isometry on $\mathbb{H}^{n+1}$ with respect to $K$ small. Then $\gamma_\epsilon$ is an isometry on $\mathbb{H}^{n+1}$ of radius 1. For $\gamma_\epsilon$ sufficiently small, one checks that $
abla K_\gamma < \epsilon$ and $K_\gamma' = 0$, so by (6.2), $H_{\gamma_\epsilon} - H_{\gamma_0} = 0$. Thus, $H_{\gamma_\epsilon} - H_\gamma = H_{\gamma_0} - H_{\gamma_\epsilon}_0\gamma_\epsilon$, and therefore $H_{\gamma_\epsilon} - H_\gamma$ increases by $H_{\gamma_0} - H_{\gamma_\epsilon}_0\gamma_\epsilon$.

Next, there exists $\gamma_\epsilon = \frac{\epsilon}{4} \|\hat{y}\|$ so by (6.2), $\mu < \epsilon/16$. From (6.2) and (6.3) one derives that

\begin{equation}
\|\hat{y}\| - \frac{\epsilon}{2} \|\hat{y}\| < \epsilon \|\hat{y} - \hat{x}\| < \epsilon \|\hat{y} - \hat{x}\| + \epsilon \|\hat{x}\|.
\end{equation}

Thus, (6.4) implies

\begin{equation}
0 < \mu < \frac{\epsilon}{4} \|\hat{y}\|.
\end{equation}

For the radius $r$ of $K$ we have $r = \sqrt{1 + d^2} = 1/\mu$. Let $\hat{K}$ be the sphere in $\mathbb{H}^{n+1}$ with center $\hat{d} = (0, -1/\mu - d, 0, \ldots, 0) \in \partial \mathbb{H}^{n+1}$ and radius $2/\mu$. Let $\tau_1$ be the inversion in $\mathbb{H}^{n+1}$ with respect to $\hat{K}$, and let $\tau_2$ be the translation in $\mathbb{H}^{n+1}$ along the vector $(0, -(1/\mu - d), 0, \ldots, 0)$ Then $\varphi = \tau_2 \circ \tau_1$ is an isometry on $\mathbb{H}^{n+1}$. Since $\tau_1$ is internally tangent to $\hat{K}$ at the point $w = (0, 1/\mu - d, 0, \ldots, 0)$, it follows that $\tau_1(K) = w + \mathbb{R}_+ \cdot e_1$, so $\varphi(K) = \mathbb{R}_+ \cdot e_1$. Assuming that $\mu > 0$ is sufficiently small, one checks that $\|\varphi^\pm(x)\| \leq 2$ for $x \in \mathbb{H}^{n+1}$ with $\|x\| \leq 1$, $\|\varphi^\pm(x)\| \leq \frac{1}{2}$ for $\|x\| \leq 1/2$, and moreover $\|\varphi^\pm(x) - \varphi^\pm(y)\| \leq 2 \|x - y\|$ for $\|x\|, \|y\| \leq 1$.

Also, since $g = \varphi \circ \tilde{g} \circ \varphi^{-1}$ is a hyperbolic transformation, $\mathbb{R}_+ \cdot e_1$, it has the form $g = \varphi^0 \Lambda A$ for some $k > 0$, $\Lambda \neq 1$, and an orthogonal transformation $A$ in $\mathbb{H}^{n+1}$ with $Ae_1 = e_1$. Replacing $\tilde{g}$ by $\tilde{g}^{-1}$ if necessary, we may assume that $k > 1$. Thus,

$\tilde{g} = k \cdot \varphi^{-1} \circ \Lambda A \circ \varphi$.

Since $(w, -e_1) = \lim_{t \to -\infty} \phi_t(\gamma)$, it follows that $w \in L(\Gamma)$, so setting $\tilde{x} = (1, x')$ with $x' = w$, we get $\tilde{z} = (\tilde{x}, -e_1) \in \Lambda \cap W$. Then the inward unit (with respect to the Poincaré metric) normal field to the horosphere $\hat{S}$ in $\mathbb{H}^{n+1}$ of radius 1/2 at $w$ gives the local stable manifold $W^s_\epsilon(\tilde{z})$. Notice that $\phi_{e_1}(\gamma) \in W^s_{\epsilon_0}(\tilde{z})$ for some small $\epsilon_0 > 0$ (in fact $0 < \epsilon_1 < \text{Const } \mu$).

Let $0 < \lambda \leq \epsilon$. We will now construct $\tilde{y} \in \Lambda \cap W^s_\lambda(\tilde{x})$ which satisfies the requirements of (SNIC).

Using the decomposition of $\mathbb{H}^{n+1}$ as a direct sum of one- or two-dimensional invariant subspaces and Kronecker’s theorem (or by considering the subgroup of the orthogonal group of $\mathbb{H}^{n+1}$ generated by $A$) it follows that there exist arbitrarily large integers $m \geq 1$ such that

\begin{equation}
\|A^m - I\| < \frac{\mu}{2},
\end{equation}

where $I$ is the identity map on $\mathbb{H}^{n+1}$. Fix $m$ with (6.7) so large that

\begin{equation}
k^m \|\hat{y}\| \geq \frac{8}{\lambda}.
\end{equation}
and set $q = \tilde{y}^m(\tilde{y}')$. Since $\tilde{y}' \in L(\Gamma)$ and $L(\Gamma)$ is $\Gamma$-invariant, we have $q \in L(\Gamma)$.

We are now going to show that

$$
(6.9) \quad \|q/\|q\| - b\| \leq 6\epsilon .
$$

First, by (6.2), $\|\tilde{y}'\| \leq \|\tilde{y}' - \tilde{x}'\| + \|\tilde{x}'\| < \epsilon + 2\epsilon \|\tilde{y}'\|$, which gives $\|\tilde{y}'\| < \frac{1}{1-2\epsilon} < 1/2$. Hence $\|\varphi(\tilde{y}')\| < 1$, so

$$
\|A^m \varphi(\tilde{y}') - \varphi(\tilde{y}')\| \leq \frac{\mu}{2} \|\varphi(\tilde{y}')\| \leq \frac{\mu}{2} ,
$$

and therefore

$$
\|\varphi^{-1} A^m \varphi(\tilde{y}') - \tilde{y}'\| \leq 2 \|A^m \varphi(\tilde{y}') - \varphi(\tilde{y}')\| \leq \mu .
$$

That is, for $s = \varphi^{-1} A^m \varphi(\tilde{y}') \in \partial H^{n+1}$ we have $\|s - \tilde{y}'\| \leq \mu$. This and (6.6) imply

$$
\left\| \frac{s}{\|s\|} - \frac{\tilde{y}'}{\|\tilde{y}'\|} \right\| \leq \|s\| \left( \frac{1}{\|s\|} - \frac{1}{\|\tilde{y}'\|} \right) + \frac{\|s - \tilde{y}'\|}{\|\tilde{y}'\|} \leq 2 \frac{\|s - \tilde{y}'\|}{\|\tilde{y}'\|} \leq \frac{2\mu}{\|\tilde{y}'\|} < \epsilon .
$$

As we saw earlier, $\|\tilde{x}'\| \leq 2\epsilon \|\tilde{y}'\|$, so

$$
\left\| \frac{\tilde{y}'}{\|\tilde{y}'\|} - \frac{\tilde{y}' - \tilde{x}'}{\|\tilde{y}' - \tilde{x}'\|} \right\| \leq \frac{\|\tilde{x}'\|}{\|\tilde{y}'\|} + \frac{\|\tilde{y}' - \tilde{x}'\|}{\|\tilde{y}'\|} \frac{1}{\|\tilde{y}'\|} - \frac{1}{\|\tilde{y}' - \tilde{x}'\|} \left\| \frac{\tilde{y}'}{\|\tilde{y}'\|} \right\| \leq 2 \frac{\|\tilde{x}'\|}{\|\tilde{y}'\|} \leq 4\epsilon .
$$

Combining this with (6.2) gives

$$
\left\| \frac{\tilde{y}'}{\|\tilde{y}'\|} - b \right\| \leq 5\epsilon .
$$

Thus,

$$
\left\| \frac{q}{\|q\|} - b \right\| = \left\| \frac{s}{\|s\|} - b \right\| \leq \left\| \frac{s}{\|s\|} - \frac{\tilde{y}'}{\|\tilde{y}'\|} \right\| + \left\| \frac{\tilde{y}'}{\|\tilde{y}'\|} - b \right\| \leq 6\epsilon ,
$$

which proves (6.9).

Next, denote by $S_0$ the horosphere at $q$ externally tangent to $\hat{S}$. Let $R$ be the radius of $S_0$ and $u$ be the tangent point of $\hat{S}$ and $S_0$. Then $\tilde{y} = (u, \xi) \in W^2_{c_0}(\hat{z})$ for some vector $\xi$ (assuming that $\|q\|$, and therefore $R$ is chosen sufficiently large), and $W^2_{c_0}(\tilde{y})$ coincides locally with the outward unit normal field to $S_0$. Notice that $\lim_{t \to 0} \phi_t(\tilde{y}) = w \in L(\Gamma)$ and $\lim_{t \to -\infty} \phi_t(\tilde{y}) = q \in L(\Gamma)$, so the definition of $\Lambda$ implies $\tilde{y} \in \Lambda$.

Since $(R + 1/2)^2 - (R - 1/2)^2 = \|q - w\|^2$, we have $2R = \|q - w\|^2$, so if $\alpha$ is the angle between the radii of $\hat{S}$ through $\hat{x}$ and $u$, then by elementary geometry,

$$
\|u - \tilde{x}'\| = \sqrt{1 - \cos \alpha} = \sqrt{1 - \frac{R - 1/2}{R + 1/2}} \leq \frac{1}{\sqrt{2R} + 1} < \frac{1}{\sqrt{2R}} = \frac{1}{\|q - w\|} < \frac{2}{\|q\|} \leq \frac{\lambda}{2} ,
$$

since $\|q\| > 4/\lambda$ by (6.8). Similarly, $\|\xi + e_1\| < \lambda/2$ assuming $m$ is chosen sufficiently large.

Set $\epsilon' = \mu$ and consider an arbitrary $z = (x, -e_1) \in W^2_{c_0}(\hat{z})$; then $\|x' - \tilde{x}'\| < \mu$. Since $\|\tilde{x}'\| = \|w\| = \frac{1}{\mu} - d = \frac{\mu}{1 + \sqrt{1 - \mu^2}} < \mu$, it follows that $\|x'\| < 2\mu$. Let $a \in \partial H^{n+1}$ and $h \in \mathbb{R}$ be such that $|a| = 1$, $|\langle a, b \rangle| \geq \theta_0$ and $|h| < \mu$. We will now show that (2.1) holds with $\delta = \frac{\theta_0}{4\|q\|}$.

Let $S$ be the horosphere of radius $1/2$ with diameter $[x, x']$; then locally $W^1_{c_0}(z)$ coincides with the inward unit normal field to $S$. So, for

$$
\sigma = \pi_\tilde{y}(z) = [z, \tilde{y}] = W^1_{c_0}(z) \cap \phi_{[-\epsilon, \epsilon]}(W^u_{c_0}(\tilde{y}))
$$
we have $\sigma = (v, \eta)$ for some $v \in S$ and $\phi_t(\sigma) \in W^u_{t_0}(\tilde{y})$. Thus, if $S_1$ is the horosphere at $x'$ tangent to $S_0$ (necessarily at the foot point of $\phi_t(\sigma)$) and $r_1$ is the radius of $S_1$, then $t_1 = \ln(2r_1)$. On the other hand, by elementary geometry, $(R + r_1)^2 = \|q - x''\|^2 + (R - r_1)^2$ so $r_1 = \|q - x''\|^2/(4R)$ and therefore $\Delta(\tilde{z}, \tilde{y}) = t_1 = \ln \frac{\|q - x''\|^2}{2R}$.

In the same way for $\exp_z(ha) = (x + ha, -e_1)$ one obtains $\Delta(\exp_z(ha), \tilde{y}) = \ln \frac{\|q - x' - ha\|^2}{\|q - x''\|^2}$.

Hence
\[ \tilde{t} = \Delta(\exp_z(ha), \pi_\gamma(z)) = \Delta(\exp_z(ha), \tilde{y}) - \Delta(z, \tilde{y}) = \ln \frac{\|q - x' - ha\|^2}{\|q - x''\|^2}. \]

Using the fact that $|\ln(1 + x)| \geq |x|/2$ for $|x| < 1$, one gets
\[ |\tilde{t}| = \ln \left[ 1 - \frac{2h}{\|q - x''\|} \left( \frac{q - x'}{\|q - x''\|} ; a \right) + \frac{h^2}{\|q - x''\|^2} \right] \geq \frac{|h|}{2\|q - x''\|} - \frac{h}{\|q - x''\|} \cdot \|
\]

Now (6.9) and $\|x''\| < 2\mu < \epsilon/8$ imply
\[ \left\| \frac{q - x'}{\|q - x''\|} - b \right\| \leq 6\epsilon + \left\| \frac{q - x'}{\|q - x''\|} - q \right\| \leq 6\epsilon + 2\|x''\|/\|q\| < 6\epsilon + 4\mu < 7\epsilon, \]

and using $|\langle a, b \rangle| \geq \theta_0$ we get
\[ 2\left| \left\langle \frac{q - x'}{\|q - x''\|} ; a \right\rangle \right| \geq 2|\langle b, a \rangle| - 2\left| \left\langle \frac{q - x'}{\|q - x''\|} - b, a \right\rangle \right| \geq 2\theta_0 - 14\epsilon. \]

Moreover, $\|q - x''\| \geq 1 - 2\mu > 1/2$, so $|h|/\|q - x''\| < 2\mu \leq \epsilon$ which combined with the above, $\|q - x''\| \leq \|q\| + 2\mu < 2\|q\|$ and (6.1) gives
\[ |\tilde{t}| \geq \frac{|h|}{4\|q\|} (2\theta_0 - 15\epsilon) \geq \frac{\theta_0}{4\|q\|} |h| = \delta |h| \]

for all $h$ with $|h| < \epsilon$. This proves (2.1), so the condition (SNIC) is fulfilled, thus completing the proof of Theorem 1.3.

7 Appendix – Proofs of some technical lemmas

Proof of Lemma 5.3. (a) Let $u, u' \in U_i$ for some $i = 1, \ldots, k$. Since $M^m_i$ is continuous on $U_i$, it is enough to consider the case $u, u' \in \Int_\Lambda(U_i)$. Given $v \in U$ with $\sigma^m(v) = u$, let $C[i] = C[i_0, \ldots, i_m]$ be the cylinder of length $m$ containing $v$ (see the beginning of Sect. 3). Since the sequence $i = [i_0, \ldots, i_m]$ is admissible, the Markov property implies $i_m = i$ and $\sigma^m(C[i]) = U_i$. Moreover, $\sigma^m_i : C[i] \longrightarrow U_i$ is a homeomorphism, so there exists a unique $v' = v'(v) \in C[i]$ such that $\sigma^m(v') = u'$. By (2.2), $d(\sigma^j(v), \sigma^j(v'(v))) \leq \frac{1}{c_0 \gamma^{m-j}} d(u, u')$ for all $j = 0, 1, \ldots, m - 1$. Consequently, using (2.3),
\[ |f^{(a)}_m(v) - f^{(a)}_m(v')| \leq \sum_{j=0}^{m-1} |f^{(a)}(\sigma^j(v)) - f^{(a)}(\sigma^j(v'))| \leq \sum_{j=0}^{m-1} \frac{1}{c_0 \gamma^{m-j}} d(u, u') \leq \frac{T}{c_0 (\gamma - 1)} d(u, u'). \]

Using this, the definition of $M_a$, and the fact that $M_a 1 = 1$ (hence $M^m_a 1 = 1$), we get
\[ \frac{|(M^m_a H)(u) - (M^m_a H)(u')|}{M^m_a H(u')} = \sum_{\sigma^m v = u} e^{f^{(a)}_m(v)} H(v) - \sum_{\sigma^m v = u} e^{f^{(a)}_m(v'(v))} H(v'(v)) \frac{1}{M^m_a H(u')} \]

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\[
\begin{align*}
&\leq \left| \sum_{\sigma^m v = u} e^{f_m(v)} (H(v) - H(v')) \right| + \left| \sum_{\sigma^m v = u} e^{f_m(v)} - e^{f_m(v')} \right| H(v') \\
&\leq \sum_{\sigma^m v = u} e^{f_m(v)} B H(v') d(v, v') + \sum_{\sigma^m v = u} e^{f_m(v)} H(v') \\
&\leq B d(u, u') \sum_{\sigma^m v = u} e^{f_m(v)} - e^{f_m(v')} e^{f_m(v')} H(v') \\
&\leq B d(u, u') \sum_{\sigma^m v = u} e^{f_m(v)} - e^{f_m(v')} e^{f_m(v')} H(v') \\
&\leq B d(u, u') \sum_{\sigma^m v = u} e^{f_m(v)} - e^{f_m(v')} e^{f_m(v')} H(v') \\
&\leq \frac{T |U|}{c_0 \gamma^m} \frac{T}{c_0 (\gamma - 1)} d(u, u') \leq A_0 \left[ \frac{B}{c_0} + \frac{T}{\gamma - 1} \right] d(u, u') ,
\end{align*}
\]
provided \( A_0 \geq \frac{T |U|}{c_0 \gamma^m} \), \( \gamma \geq 1 \).

(b) Let \( m \geq 1 \) be an integer and \( u, u' \in U_i \) for some \( i = 1, \ldots, k \). As above, it is enough to consider the case \( u, u' \in \text{Int}_A(U_i) \). Using the notation \( v' = v'(v) \) from part (a) above, we get

\[
|L^{m}_{ab} h(u) - L^{m}_{ab} h(u')| = \left| \sum_{\sigma^m v = u} \left( e^{f_m(v)} - e^{f_m(v')} \right) h(v) - e^{f_m(v')} h(v') \right|
\]

\[
\leq \sum_{\sigma^m v = u} e^{f_m(v)} \left| h(v) - h(v') \right| + \sum_{\sigma^m v = u} \left( e^{f_m(v)} - e^{f_m(v')} \right) h(v') \]

\[
\leq \sum_{\sigma^m v = u} e^{f_m(v)} \left| h(v) - h(v') \right| + \sum_{\sigma^m v = u} \left( e^{f_m(v)} - e^{f_m(v')} \right) h(v') \]

Using elementary inequalities one checks that \( |e^{x+iy} - 1| \leq 2e^{y} (|x| + |y|) \) for real \( x \) and \( y \). This and (2.3) give \( \left| e^{f_m(v)} - e^{f_m(v')} \right| \leq \frac{T |U|}{c_0 \gamma^m} \frac{T}{\gamma - 1} \) \( |b| d(u, u') \). Assuming \( A_0 \geq \frac{T |U|}{c_0 \gamma^m} \), we now get

\[
\sum_{\sigma^m v = u} e^{f_m(v)} \left| h(v) - h(v') \right| \leq \sum_{\sigma^m v = u} \left( e^{f_m(v)} - e^{f_m(v')} \right) e^{f_m(v')} B H(v') d(v, v') \]

\[
\leq B e^{f_m(v')} \frac{T |U|}{c_0 \gamma^m} d(u, u') \left( \mathcal{M}^m_{ab} H(u') \right) ,
\]

which immediately leads to the desired inequality. \( \blacksquare \)

**Proof of Lemma 4.2.** Choosing an appropriate subsequence of \( \{u_m\} \), we may assume that \( \|u_m\| \leq \frac{1}{m} \|u_{m-1}\| \) for all \( m \geq 1 \), \( a = \sum_{m=1}^{\infty} \epsilon_m < \infty \), where \( \epsilon_m = \|u_{m+1} - u_m\| \), and that there exists \( v = \lim_{m \to \infty} \frac{u_m}{\|u_m\|} \in S^{k-1} \). It then follows that \( v = -\lim_{m \to \infty} \frac{u_m - u_{m-1}}{\|u_m - u_{m-1}\|} \).
Let \( v_m = (u_{m-1} - u_m)/\|u_{m-1} - u_m\| \) and \( a_m = \sum_{i=m}^{\infty} \epsilon_i \) for \( m \geq 1 \). Then \( a_m \to 0 \) as \( m \to \infty \), \( a = a_1 > a_2 > \ldots > a_m > \ldots \), and \( \epsilon_m = a_m - a_{m+1} \). Define the curve \( z(t) \), \( 0 \leq t \leq a \), by \( z(0) = 0 \) and

\[
z(t) = \left[ 1 - \frac{t - a_{m+1}}{\epsilon_m} \right] u_{m+1} + \frac{t - a_{m+1}}{\epsilon_m} u_m + \left( u_m - u_{m+1} \right) \left( \frac{t - a_{m+1}}{\epsilon_m} \right)^2 \left( 1 - \frac{t - a_{m+1}}{\epsilon_m} \right)
\]

for \( a_{m+1} \leq t \leq a_m \). One checks easily that \( z(t) \) is \( C^1 \) for \( 0 \leq t \leq a \). □

Acknowledgements. A preliminary version of this paper dealing with open billiard flows was written during my visit to the Erwin Schrödinger Institute for Mathematical Physics in Vienna for the Program on Scattering Theory in 2001. Thanks are due to the organizers of this activity Vesselin Petkov, Andras Vasy and Maciej Zworski and to the staff of ESI for their hospitality and support.

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