1. **(Deferred Exam 2002)**

An "ion channel" is an important feature of many biological cells. Ion channels have only a few possible 'states' of activity (e.g. On/Off or Off/Low/High) and are observed to switch spontaneously between these states. A simple probabilistic model of the ion channel assumes that, if $X_n$ is the state of the channel at time $n$ milliseconds ($n = 0, 1, 2, \ldots$) then $X_0, X_1, X_2, \ldots$ is a Markov Chain.

(a) Define the term "Markov chain". [2 marks]

(b) Explain briefly in non-technical language what it means to assume that the ion channel can be described by a Markov chain. [1 mark]

(c) Suppose that $(X_n)$ is a time-homogeneous Markov chain with transition probability matrix $P$. Let

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$$

and let $P^{(n)}$ be the matrix with entries $P_{ij}^{(n)}$ for all $i, j$. Prove that $P^{(n+1)} = P^{(n)}P$ and hence derive an expression for $P^{(n)}$ in terms of $P$. Each step in the derivation should be justified explicitly. [8 marks]

(d) Suppose that the ion channel is a time-homogeneous Markov chain $(X_n)$ with state space $\{1, 2, 3, 4\}$, transition probability matrix

$$P = \begin{pmatrix}
0 & 0 & 1 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}$$

and initial state $X_0 = 1$. Compute the probability distribution of $X_4$. [3 marks]

(e) For the Markov chain described in part (d), decide whether the distribution of $X_n$ converges to a limit as $n \to \infty$, and if so, compute this limit. [5 marks]
2. (Deferred exam 2002)

(a) Suppose \((X_n, n = 0, 1, 2, \ldots)\) is a time-homogeneous Markov chain with a finite state space. Assume all states are either absorbing or transient. Let \(P\) be the transition probability matrix, and let \(T\) be the sub-matrix of \(P\) corresponding to transitions between the transient states only.

Let \(V_j\) be the total number of visits to state \(j\) in the entire history of the chain, and

\[ u_{i,j} = \mathbb{E}[V_j | X_0 = i] \]

the expected number of visits to state \(j\) if the chain starts in state \(i\). Let \(U\) be the matrix with entries \(u_{i,j}\) for all transient states \(i, j\). Prove that

\[ U = I + TU \]

where \(I\) is the identity matrix. [10 marks]

(b) A game of tennis has just reached Deuce. Assume that the remainder of the game is a Markov chain with the following transition diagram:

Using the result of (a), calculate the expected number of times during the game from now on (including the present) that the score is Deuce.

[Hint: It may be useful to know that

\[
\begin{pmatrix}
1 & -p & 0 \\
-q & 1 & -p \\
0 & -q & 1
\end{pmatrix}
\begin{pmatrix}
1 - pq & p & p^2 \\
q & 1 & p \\
q & q & 1 - pq
\end{pmatrix}
= (1 - 2pq)I
\]

[5 marks]
3. The air in the crew compartment of a spacecraft is continuously monitored using an oxygen analyser. The analyser is based on a fuel cell which eventually burns out and must be replaced. The analyser operates online for an exponentially distributed time, with failure rate $\beta$, then burns out. The time needed to replace the burnt-out fuel cell is exponentially distributed, with rate $\alpha$. Successive failure times and repair times are independent.

To increase reliability, there are actually two of these analysers, which are (probabilistically) independent. Find the long-run probability that \textit{at least one} of the analysers is operating, by

(a) treating each analyser separately as a two-state Markov Chain in continuous time, and using the fact that the two analysers are independent;

(b) treating the pair of analysers as a four-state Markov Chain in continuous time;

(c) letting $Z_t =$ the number of analysers that are operating at time $t$, and arguing that $Z_t$ is a continuous-time Markov Chain on the state space $\{0, 1, 2\}$.

4. A single-celled animal lives for an exponentially distributed time with parameter $\beta$, then divides into two. Each offspring animal also lives for an exponential ($\beta$) time then divides into two. The lives of the offspring are independent of each other and of their parent.

Let $X_t$ be the number of animals alive at time $t$. Show that $X_t$ is a linear birth process.

5. A system for protecting welders against electric shock consists of two insulating components which will eventually fail. If both components are working, then the time until one of the components fails, is exponentially distributed with rate $\alpha$. After one component has failed, and one is still working, the time to failure of the remaining component is exponentially distributed with rate $3\alpha$. When both components have failed, the system is replaced with a new one containing two working components. Calculate the fraction of time (in a long run average) during which the safety system has only one working component.