Partial Solutions to Practice Sheet 2

1.

2.

3. (a) Consider one analyser. Let \( X_t \) be the state of the analyser (1 = ‘operating’ or 2 = ‘offline’) at time \( t \geq 0 \). The description given implies that \( X_t \) is a continuous time Markov Chain with transition rate matrix

\[
Q = \begin{pmatrix}
-\beta & \beta \\
\alpha & -\alpha
\end{pmatrix}
\]

where the first row and column correspond to the state where the analyser is in operation. This chain is clearly irreducible, so it has a unique limiting distribution \( \pi \) which is the unique solution of

\[
\pi Q = 0
\]

together with \( \pi_1 + \pi_2 = 1 \). This states that

\[
-\beta \pi_1 + \alpha \pi_2 = 0 \\
\beta \pi_1 - \alpha \pi_2 = 0
\]

so we have \( \pi_1 = (\alpha/\beta) \pi_2 \), yielding

\[
\pi_1 = \frac{\alpha}{\alpha + \beta}, \quad \pi_2 = \frac{\beta}{\alpha + \beta}.
\]

That is, \( \lim_{t \to \infty} \mathbb{P}\{X_t = 1\} = \pi_1 = \alpha/(\alpha + \beta) \).

Now consider the second analyser, and let \( Y_t \) be its state at time \( t \). Similarly we have \( \lim_{t \to \infty} \mathbb{P}\{Y_t = 1\} = \pi_1 = \alpha/(\alpha + \beta) \).

Since the two analysers are independent, we have

\[
\mathbb{P}\{\text{both analysers operating at time } t\} = \mathbb{P}\{X_t = 1, Y_t = 1\} = \mathbb{P}\{X_t = 1\} \mathbb{P}\{Y_t = 1\}
\]

It follows that

\[
\lim_{t \to \infty} \mathbb{P}\{\text{both analysers operating at time } t\} = \pi_1^2 = \frac{\alpha^2}{(\alpha + \beta)^2}
\]
(b) Keeping the same notation, let $W_t = (X_t, Y_t)$ be the pair of numbers giving the states of the two analysers. For example, $W_t = (1, 2)$ means that the first analyser is operating and the second is offline.

It is easy to see that $W_t$ is a continuous time Markov Chain on the state space \{(1, 1), (1, 2), (2, 1), (2, 2)\}. If $W_0 = (1, 2)$ then the first analyser will break down after a random time $T \sim \text{exponential}(\beta)$ while the second analyser will be repaired after a random time $S \sim \text{exponential}(\alpha)$, and $S$ and $T$ are independent. By question 3 on Assignment 3, the time to the first transition $U = \min\{S, T\}$ is exponentially distributed with rate $\alpha + \beta$. Similar arguments hold for the other states.

Thus $W_t$ is a four-state Markov Chain in continuous time, with rate matrix

\[
Q = \begin{pmatrix}
(1, 1) & (1, 2) & (2, 1) & (2, 2) \\
(1, 1) & -2\beta & \beta & \beta & 0 \\
(1, 2) & \alpha & - (\alpha + \beta) & 0 & \beta \\
(2, 1) & \alpha & 0 & -(\alpha + \beta) & \beta \\
(2, 2) & 0 & \alpha & \alpha & -2\beta
\end{pmatrix}
\]

This chain is irreducible and so has a unique limiting distribution, which is the solution of $\pi Q = 0$ or

\[
\begin{align*}
-2\beta \pi_{(1,1)} + \alpha \pi_{(1,2)} + \alpha \pi_{(2,1)} &= 0 \\
\beta \pi_{(1,1)} - (\alpha + \beta) \pi_{(1,2)} + \alpha \pi_{(2,1)} &= 0 \\
\beta \pi_{(1,1)} - (\alpha + \beta) \pi_{(2,1)} + \alpha \pi_{(2,2)} &= 0 \\
\beta \pi_{(1,2)} + \beta \pi_{(2,1)} - 2\alpha \pi_{(2,2)} &= 0
\end{align*}
\]

In general a 4-by-4 system of equations could be troublesome to solve, but we can exploit the symmetry. The equations are symmetric in $\pi_{(1,2)}$ and $\pi_{(2,1)}$, that is, these two variables can be exchanged without affecting the equations. Since the solution is unique, we must have $\pi_{(1,2)} = \pi_{(2,1)}$. Then the first equation simplifies to

\[
\pi_{(1,1)} = \frac{\alpha}{\beta} \pi_{(1,2)}
\]

and the fourth equation becomes

\[
\pi_{(2,2)} = \frac{\beta}{\alpha} \pi_{(1,2)}
\]

so we have (abbreviating $\pi_{(1,2)}$ to $p$, say)

\[
\pi_{(1,1)} = \frac{\alpha}{\beta} p; \quad \pi_{(1,2)} = p; \quad \pi_{(2,1)} = p; \quad \pi_{(2,2)} = \frac{\beta}{\alpha} p
\]

Since the probabilities must sum to 1, we must have

\[
\frac{\alpha}{\beta} p + p + p + \frac{\beta}{\alpha} p = 1
\]
yielding
\[
p = \frac{1}{2 + \alpha/\beta + \beta/\alpha} = \frac{\alpha\beta}{2\alpha\beta + \alpha^2 + \beta^2} = \frac{\alpha\beta}{(\alpha + \beta)^2}
\]
and so
\[
\pi_{(1,1)} = \frac{\alpha^2}{(\alpha + \beta)^2}; \quad \pi_{(1,2)} = \frac{\alpha\beta}{(\alpha + \beta)^2}; \quad \pi_{(2,1)} = \frac{\alpha\beta}{(\alpha + \beta)^2}; \quad \pi_{(2,2)} = \frac{\beta^2}{(\alpha + \beta)^2}.
\]

The long-run probability that both analysers are operating is \(\pi_{(1,1)} = \frac{\alpha^2}{(\alpha + \beta)^2}\).

(c) let \(Z_t\) = the number of analysers that are operating at time \(t\). Then by the same arguments as above, \(Z_t\) is a continuous-time Markov Chain on the state space \(\{0, 1, 2\}\) with rate matrix
\[
Q = \begin{pmatrix}
-2\alpha & 2\alpha & 0 \\
\beta & -(\alpha + \beta) & \alpha \\
0 & 2\beta & -2\beta
\end{pmatrix}
\]
This chain is irreducible so it has a limit distribution \(\pi\) which is the solution of \(\pi Q = 0\). By the same methods the solution is
\[
\pi_0 = \frac{\beta^2}{(\alpha + \beta)^2}; \quad \pi_1 = \frac{2\alpha\beta}{(\alpha + \beta)^2}; \quad \pi_2 = \frac{\alpha^2}{(\alpha + \beta)^2}.
\]
Once again the limiting probability that both analysers are operating is \(\pi_2 = \frac{\alpha^2}{(\alpha + \beta)^2}\).
4. Each animal which is alive at time $t$ has a remaining lifetime $T_i$ which has an exponential ($\beta$) distribution. (Note: by the memoryless property of the exponential distribution it does not matter how long the animal has already been alive; given that it is still alive at time $t$, its remaining lifetime is exponential).

If there are currently $k$ animals alive, then the first new animal will be created at time $T = \min\{T_1, \ldots, T_k\}$ where $T_1, \ldots, T_k$ are the remaining lifetimes of the $k$ existing animals. By the same methods as in Question 3 of Assignment 3, $T$ must have an exponential distribution with rate $k\beta$. At time $T$ the number of animals increases to $k + 1$.

Thus $X_t$ is a continuous time Markov Chain on the positive integers with transition rates $q_{k,k+1} = k\beta$ for $k = 1, 2, \ldots$. This is what we called the Yule process or linear birth process.

5. Let $X_t$ be the number of components that are working. Then $X_t$ is a continuous time Markov Chain with state space $\{1, 2\}$ and transition rate matrix

$$Q = \begin{pmatrix} -3\alpha & 3\alpha \\ \alpha & -\alpha \end{pmatrix}$$

This chain is clearly irreducible and so has a limiting distribution $\pi$ which is the solution of $\pi Q = 0$. The solution is

$$\pi_1 = \frac{1}{4}; \quad \pi_2 = \frac{3}{4}$$

that is, the fraction of time (in a long run average) during which the safety system has only one working component is one quarter.