Testing for Linear and Nonlinear Gaussian Processes in Nonstationary Time Series

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Surrogate data methods have been widely applied to produce synthetic data, while maintaining the same statistical properties as the original. By using such methods, one can analyze certain properties of time series. In this context, Theiler’s surrogate data methods are the most commonly considered approaches. These are based on the Fourier transform, limiting them to be applied only on stationary time series. Consequently, time series including nonstationary behavior, such as trend, produces spurious high frequencies with Theiler’s methods, resulting in inconsistent surrogates. To solve this problem, we present two new methods that combine time series decomposition techniques and surrogate data methods. These new methods initially decompose time series into a set of monocomponents and the trend. Afterwards, traditional surrogate methods are applied on those individual monocomponents and a set of surrogates is obtained. Finally, all individual surrogates plus the trend signal are combined in order to create a single surrogate series. Using this method, one can investigate linear and nonlinear Gaussian processes in time series, irrespective of the presence of nonstationary behavior.

Keywords: Surrogate data; decomposition method; Fourier transform; nonstationary time series; nonlinear time series.

1. Introduction

Surrogate data methods [Theiler et al., 1992; Maiwald et al., 2008; Schreiber & Schmitz, 2000; Small & Judd, 1998b] are traditionally applied on experimental data to test against specific null hypotheses. This is achieved by generating an ensemble of surrogate data: each surrogate dataset is expected to be similar to the original data, but also consistent with the underlying null hypotheses. Specifically, properties of this underlying null hypothesis (for example linear correlation for the null of linearly filtered noise) will produce the same statistical estimate from the original data and the surrogates. However, other properties of data,
unrelated to the null hypothesis, are randomized. In other words, only features consistent with the null are maintained so that statistical sampling from either surrogates or the original data will provide the same results. A secondary application of surrogate data is to control estimators of certain nonlinear properties such as fractal dimension or the bispectrum for supurious results. In this sense, surrogate methods have previously been used to provide bounds for the certainty of estimates of nonlinear quantities from data [Small & Judd, 1998a; Zhao et al., 2008].

The most commonly considered surrogate data methods are based on the Fourier transform, such as Fourier Transformed (FT)\(^1\) and Amplitude Adjusted Fourier Transformed (AAFT) methods [Thélier et al., 1992]. These methods basically apply the Fourier transform on the original data, producing amplitudes and phases, and substitute the original phases by uniform random phases. Afterwards, they apply the inverse Fourier transform to obtain the surrogate data. As this new data was generated using the Fourier transform, which assumes data periodicity, nonstationary characteristics are not represented in the surrogate. The main reason behind this is that by applying the Fourier transform on nonstationary time series, the difference between the first and last observations, caused by the trend, produces spurious high frequencies, as a consequence, inconsistent surrogates are produced. Other authors, such as Nakamura et al., 2006; Nakamura & Small, 2005, have also addressed this issue, but they consider parametric approaches to produce surrogate data which strongly depends on data being analyzed. On the other hand, this paper relies on the Empirical Mode Decomposition (EMD) method to automatically produce surrogate data.

Aiming at overcoming these problems, we extended these two Fourier-based methods by first decomposing the time series into a set of components plus a residue. Every component contains similar behavior and the residue corresponds to trends. Then, surrogate data is produced based on each individual component. Later on, all produced surrogates plus the original residue are added to compose the surrogate, which is indeed considered as synthetic data based on the original time series.

To separate components, we employ the Empirical Mode Decomposition (EMD) method. The last component produced by EMD (i.e. the trend) contains nonstationary features. According to our experiments, we confirmed that two new methods improve Thélier’s methods, generating synthetic data for nonstationary time series.

The remainder of this paper is organized as follows. In Sec. 2, we present an overview of the surrogate data methods and we discuss important surrogate methods. The proposed approach is presented in Sec. 3. In Sec. 4, we present an analysis of the proposed approach. Experimental results as well as a detailed discussion about the advantage of our approach are presented in Secs. 5 and 6; in Sec. 7, we present a discussion on the use of the constant phase randomization to prove that our approach allows to investigate linear and nonlinear Gaussian processes in time series; finally, in Sec. 8, we draw conclusions and discuss future work.

2. Surrogate Methods

The study of surrogate data was introduced by Thélier et al. [1992], for whom the main objective was to analyze time series to confirm whether they belong to the same generative process. In general, this evaluation is performed in two straightforward steps. First, synthetic data is produced combining part of the original data properties and another specific generation process. This step is repeatedly performed to produce a set of surrogates. In the second step, discriminating statistics are computed to compare the original time series against all surrogates. Based on the computed values, one can verify the similarity among them, and, consequently, state whether or not they were created using the same process. Discriminating statistics can be computed using different methods, such as the Grassberger–Procaccia (GP) correlation dimension [Grassberger & Procaccia, 1983], Autocorrelation Function (ACF) [Box et al., 1994], Spectral Density (SD) [Brockwell & Davis, 2002], Average Mutual Information (AMI) [Abarbanel, 1996] and Space-Time Separation Plot (STP) [Provanzale et al., 1992].

The results obtained by the discriminating statistics are then used to perform statistical

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\(^1\)In this paper, FT stands for the Thélier’s Fourier Transformed method used to produce surrogates. On the other hand, the basic Fourier transform is represented by \( F \).
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[Theiler et al., 1992] proposed two methods to generate surrogate data which have been widely studied and employed [Maiwald et al., 2008; Small & Tse, 2003; Higuchi & Aihara, 1997-2008; Small & Tse, 2003; Ikeguchi & Aihara, 1997-2008]: the Fourier Transformed (FT) surrogate; and the Amplitude Adjusted Fourier Transformed (AAFT) surrogate.

The FT method was designed to identify the nonlinear property in time series. This method defines as the null hypothesis that the analyzed time series is linear [Theiler et al., 1992; Maiwald et al., 2008]. Hence, this method produces surrogates using a linear process. Alternatively, optimal estimation leads to the opposite constraint results, i.e. this parameter can be unfairly estimated and results would be misleading, alternatively, optimal estimation leads to the opposite problem, over-fitting of the IAAFT to the data. The second surrogate data method proposed by Theiler et al. is called the Amplitude Adjusted Fourier Transformed (AAFT) method. In this case, the null hypothesis assumes that the time series is stationary, distributions are similar, but not necessarily equal, once there is an ACF, but not equal, once there is an

\[ X(f) = A(f) \cdot e^{i\phi(f)} \]  

Given the inherent linearity of the uniform distribution, this randomization creates surrogates with phases varying in a linear way. Finally, the surrogate data \( y(t) \) is obtained by applying the Inverse Fourier transform \( F^{-1} \) [Theiler et al., 1992]

\[ y(t) = F^{-1}(X(f)) \]

\[ = F^{-1}\left\{ \int_{-\infty}^{\infty} X(f) \cdot e^{i\phi(f)} df \right\}. \]

The second surrogate data method proposed by Theiler et al. is called the Amplitude Adjusted Fourier Transformed (AAFT) method. In this case, the null hypothesis assumes that the time series is linear, observations may be influenced by a nonlinear static transform [Theiler et al., 1992]. According to Theiler et al. [1992], most conventional methods used to estimate non-linearity indicate that a given time series is nonlinear, but they do not provide further information to conclude if the nonlinearity comes from the time series dynamics or from the amplitude distribution [Theiler et al., 1992]. In summary, this method generates a new series \( y(t) \) using a Gaussian process. This new time series is reordered so that the ranks agree with the ranks of \( x_i(t) \). After that, the FT method is applied on \( y(t) \), generating a new series \( y'(t) \). Finally, AAFT produces the surrogate data \( x_i(t) \) for \( x(t) \) by reordering the observations in \( x(t) \) in a way that its ranks agree with the ones of \( y(t) \) [Theiler et al., 1992].

The drawback of the AAFT method is that surrogates are produced respecting the same amplitude distribution of the original time series and presenting similar ACF, but not equal, once there is an adjustment on the amplitude [Theiler et al., 1992; Lucio et al., 2012]. Aiming at improving AAFT to produce surrogates that preserve both amplitude distribution and ACF, Schreiber and Schmitz [1996] proposed a new method called Iterative Amplitude Adjusted Fourier Transformed (IAAFT). However, this method is not considered in our comparative study because it introduces an unacknowledged but user-tunable parameter which may over- or under-constrain results, i.e. this parameter can be unfairly estimated and results would be misleading, alternatively, optimal estimation leads to the opposite problem, over-fitting of the IAAFT to the data. Besides that, in this work, we are interested in analyzing the advantages of using decomposition

2Technically, the tests fail to reject the null hypothesis.
methods to produce surrogate data regardless of the presence of nonstationary behavior. A comparative study among those methods can be found in [Maiwald et al., 2008; Schreiber & Schmitz, 1996; Lucio et al., 2012].

The main problem faced by Theiler’s FT and AAMF methods is related to the stationarity restriction imposed by the Fourier transform. Theiler’s FT and AAMF methods cannot create surrogate data sufficiently similar to time series characterized by nonstationary behavior. In such situations, the surrogates produced by those methods are affected by the amplitude variation in the Fourier transform, resulting in surrogates completely different from the original time series [Nakamura et al., 2006; Nakamura & Small, 2005].

Another surrogate method called the Small Shufﬂe Surrogate (SSS) was proposed by Nakamura and Small [2005, 2006]. To generate surrogate data, this method performs the following steps: (i) the original time series \( x(t) \) is analyzed and the indices of its observations are stored in a list \( i(t) \); (ii) a new index list is created by considering equation \( i'(t) = i(t) + A \cdot g(t) \), in which \( A \) represents an amplitude and \( g(t) \) is a sequence of Gaussian random numbers. In this equation, the amplitude \( A \) is responsible for deﬁning the scale of changes in the index list \( i(t) \); (iii) list \( i'(t) \) is sorted and stored in a new list \( i(t) \); (iv) ﬁnally, a surrogate \( s(t) \) is obtained by selecting values of the original time series \( x(t) \) according to new indexes \( i(t) \), i.e. \( s(t) = x(i(t)) \).

According to the authors [Nakamura & Small, 2005], this method can be used to investigate irregular fluctuations in time series, once it destroys local structures or correlations and keeps the global behavior, such as trend. Hence, the null hypothesis addressed by this new method is that the time series consists of a de-trended component, producing a set of monocomponent surrogates. Those individual surrogates are combined to produce a single surrogate data, which is ﬁnally re-trended by adding the residue obtained in the ﬁrst step. These detrending and re-trending steps allow}

### 3. Improving Surrogate Methods by Decomposing Time Series

By applying these Fourier-based methods on time series with trends, the observations are inﬂuenced by spurious high frequencies, which affect the general behavior of surrogates. In order to overcome this drawback, we initially decompose time series into a set of monocomponents plus a residue, which represents the time series trend. Afterwards, we apply traditional surrogate methods on every monocomponent, producing a set of monocomponent surrogates. Those individual surrogates are combined to produce a single surrogate data, which is ﬁnally re-trended by adding the residue obtained in the ﬁrst step. These detrending and re-trending steps allow
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The decomposition step in our method permits detrending the time series, by the residue extraction, before applying any surrogate method. Hence, as next step, our method executes the Theiler’s FT method on all decomposed monocomponents \( \{h_m(t)\} \), except on residue \( r_M(t) \), producing a set of monocomponent surrogates.

In the last step, all monocomponent surrogates are summed to form a single surrogate data. Finally, the global trend of the original time series is combined to this single surrogate by adding the residue. In summary, this new method, called EMD-FT is defined by (8) in which \( y(t) \) is the surrogate data, \( X_m, k(f) \) represents the coefficients obtained applying the Discrete Fourier transform on the \( m \)th monocomponent, and \( \varphi(f) \) represents the values obtained with the phase randomization

\[
y(t) = \sum_{m=1}^{M-1} \left( \sum_{k=1}^{N} X_m, k(f) \cdot e^{i\varphi(f)} \right) + r_M(t).
\]

On the other hand, the adapted EMD-AAFT method was created using exactly the same steps previously presented, with the AAFT method adopted to produce surrogate data for every monocomponent instead.

One of the most important contributions of the proposed methods is the possibility of removing nonstationary influences during the decomposition of the original time series. After decomposition, every monocomponent contains simpler behavior which is better represented using sinusoidal functions. Thus, when we apply Theiler’s surrogate data methods on every monocomponent to produce surrogates, except the residue which represents the time series trend. Consequently, by using FT and AAFT in our methods, one can test the linearity in stationary or nonstationary time series, which is
not possible using Theiler’s methods directly on the original time series.

In order to evaluate the proposed methods, in the next section, we analytically demonstrate that surrogates produced by the original FT and AAFT methods are similar to the proposed EMD-FT and EMD-AAFT methods, even when there is a trend embedded in the time series.

4. Analyzing the Proposed Surrogate Methods

In order to investigate the efficiency of the proposed methods, we analyzed the surrogate data produced by FT and EMD-FT methods. The main objective of this analysis is to ensure that the adapted method produces surrogates with the same behavior as the traditional one and, consequently, the same null hypothesis can be used by both methods, even when there is nonstationary behavior in the data.

All analysis presented in this section is based on constant phase randomization, i.e. the same phase randomization is applied on every IMF, in order to simplify the stated theorem and proof. However, this is not mandatory, because one can employ our methods to apply different phase randomizations to IMFs (discussed in Sec. 7).

It is important to highlight that we used the EMD-FT method because it is simpler and more intuitive, but the same results can be extended to the EMD-AAFT method. In order to proceed with this analysis, we first present a theorem which states that the surrogate data produced by both methods are exactly the same, even when there is a trend embedded in the time series. By proving this theorem, we confirm the same hypothesis test used by FT and EMD-FT methods. The main objective of this analysis is to ensure that the adapted method produces surrogates with the same behavior as the traditional one and, consequently, the same null hypothesis can be used by both methods, even when there is nonstationary behavior in the data.

The second part of our proof is, initially, obtained by applying the EMD method on the same nonlinear time series $x(t)$, which returns a set of monocomponents $\{h_m(t)\}$ and a residue $r_M(t)$. By definition, residue $r_M(t)$ represents the trend component of the time series. The result is equivalent to the trend extracted by the EMD method.

Considering the definition of Fourier transform [Eq. (1)], we can rewrite the previous equation as

$$y(t)^{FT} = \text{FT}(z(t)) + r_M(t).$$

Theorem 1. If the nonstationary trend can be linearly decoupled from a nonlinear time series, then Theiler’s FT and the proposed EMD-FT method produce the same surrogate data.

Proof. To establish this theorem, we need to prove that the surrogate data produced by both methods are exactly the same to linear and nonlinear time series. However, we first assume that the trend can somehow be detached from the time series. Therefore, considering this assumption, we can rewrite a nonlinear time series as $z(t) = x(t) + r_M(t)$, in which $z(t)$ represents the time series observations and $r_M(t)$ is the trend. We use $r_M(t)$ to represent the trend just to keep the same pattern used to describe the Empirical Mode Decomposition (EMD) method. Hence, in the first part of our proof, after applying Theiler’s FT surrogate method on the observations of $z(t)$, i.e. the detrended series, we obtain surrogate $y(t)^{FT}$

$$y(t)^{FT} = \text{FT}(z(t)) + r_M(t).$$

The second part of our proof is, initially, obtained by applying the EMD method on the same nonlinear time series $x(t)$, which returns a set of monocomponents $\{h_m(t)\}$ and a residue $r_M(t)$. By definition, residue $r_M(t)$ represents the trend component of the time series. The result is equivalent to the trend extracted by the EMD method.

$$y(t)^{EMD} = \text{EMD}(x(t)) = h_1(t) + \cdots + h_n(t) + r_M(t)$$

$$y(t)^{EMD-FT} = \text{FT}(h_1(t)) + \cdots + \text{FT}(h_n(t)) + r_M(t)$$

$$= \left( \frac{1}{N} \sum_{k=1}^{N} X_{1,k}(f) \cdot e^{i\varphi(f)} \right)$$

$$+ \cdots + \left( \frac{1}{N} \sum_{k=1}^{N} X_{n,k}(f) \cdot e^{i\varphi(f)} \right)$$

$$+ r_M(t).$$

An important step of our proof is stated by assuming the phase randomization is performed only once for all monocomponents $h_m(t)$, i.e. all

$$X_k$$ and $\varphi$ are the Fourier domain decompositions of the time series and are hence obtained from time dependent signals — they are expressed here in the frequency domain form as functions of frequency only.
monocomponents were randomized considering the same sequence of values. Hence, surrogate 
\( y(t)^{\text{EMD-FT}} \) can be rewritten to emphasize the randomized phase according to
\[
y(t)^{\text{EMD-FT}} = \left[ \frac{1}{N} \sum_{k=1}^{N} X_{1,k}(t) \right] + \cdots + \left[ \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right] 
= \sum_{m=1}^{M-1} \left( \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right) \cdot e^{r \varphi(t)} + r_m(t),
\] (12)
\[
y(t)^{\text{FT}} = \sum_{m=1}^{M-1} \left( \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right) \cdot e^{r \varphi(t)} + r_m(t).
\]

Finally, in order to prove the theorem stated in this section, we need to evaluate the relation 
\( y(t)^{\text{FT}} = y(t)^{\text{EMD-FT}} \), i.e. observations generated by both methods are equal:
\[
y(t)^{\text{FT}} = y(t)^{\text{EMD-FT}},
\]
\[
\left( \frac{1}{N} \sum_{k=1}^{N} X_k(f) \cdot e^{r \varphi(f)} \right) + r_m(t) = \sum_{m=1}^{M-1} \left( \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right) \cdot e^{r \varphi(f)} + r_m(t).
\] (13)

By subtracting \( r_m(t) \) from both sides of Eq. (12), we obtain the equality
\[
\left( \frac{1}{N} \sum_{k=1}^{N} X_k(f) \cdot e^{r \varphi(f)} \right) = \sum_{m=1}^{M-1} \left( \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right) \cdot e^{r \varphi(f)}. \] (14)

Finally, we divide both sides of the equation by \( e^{r \varphi(f)} \):
\[
\left( \frac{1}{N} \sum_{k=1}^{N} X_k(f) \right) = \sum_{m=1}^{M-1} \left( \frac{1}{N} \sum_{k=1}^{N} X_{m,k}(t) \right).
\] (15)

This equality proves the sum of amplitudes, obtained by applying Theiler’s FT on EMD decomposed monocomponents, is equal to the amplitude obtained using Theiler’s FT surrogate method directly on the time series.

Therefore, we confirm the new method supports the same null hypothesis as Theiler’s FT and AAFT, but without any interference of the nonstationary behavior, once the nonlinear EMD method permits treating the trend as a separated component.

In the following section, we present the experimental setup to evaluate the proposed methods.

5. Experimental Setup
In order to evaluate the proposed methods, we analyzed two sets of time series. The first one was composed of three synthetic time series created by adding a trend to a sine function [Fig. 1(a)], a white noise process [Fig. 1(b)], and an autoregressive process [Fig. 1(c)]. The autoregressive (AR) process used in these experiments was generated considering a first-order model and a Normal distribution \( N(0, \sigma^2) \). This Normal distribution was also used to create the white noise time series presented in Fig. 1(b). Besides these time series, we also analyzed a nonlinear time series [Fig. 1(d)] created from a Lorenz system [Alligood et al., 1997].

The second set was composed of three real-world time series. The first series is illustrated in Fig. 1(e), which corresponds to a collection of yearly average global temperatures [Shumway & Stoffer, 2006]. Figure 1(f) presents the second time series that corresponds to atmospheric concentrations of CO2 [Cleveland, 1993]. Finally, Fig. 1(g) corresponds to the Dow Jones Utilities Index commonly considered in stock market analysis [Brockwell & Davis, 2002].

The evaluation process was performed by analyzing every time series and producing 99 surrogates using the methods FT, AAFT, EMD-FT, and EMD-AAFT. Then, the original and surrogate time series were evaluated considering three types of analyses: (i) visual inspection of time series plots; (ii) visual inspection of plots produced by the
Fig. 1. Time series used to evaluate the proposed methods: (a) A sine function combined with a trend, (b) a white noise process combined with a trend, (c) autoregressive (AR) process combined with a trend, (d) nonlinear time series produced by a Lorenz system with chaotic behavior, (e) global temperature [Shumway & Stoffer, 2006], (f) atmospheric concentrations of CO$_2$ [Cleveland, 1993] and (g) Dow Jones Utilities Index [Brockwell & Davis, 2002].

The Autocorrelation Function (ACF) [Provanzale et al., 1992] and the Average Mutual Information (AMI) [Abarbanel, 1996; Nakamura & Small, 2005; Nakamura et al., 2006]; (iii) hypothesis test considering the discriminating statistics obtained using AMI.

The Autocorrelation Function allows to identify temporal correlations present in series at different lags in time, depicting the difference among the methods. Formally, the Autocorrelation function $\rho(h)$ of a time series $x(t)$ is obtained by computing the autocovariances of $x(t)$ and its time-shifted version $x(t+h)$ as defined in Eq. (16), in which $E[\cdot]$ is the expected value of the expression, and $\mu$ and $\sigma^2$ are the variance and mean of $x(t)$, respectively.

$$\rho(h) = \frac{E[(x(t) - \mu)(x(t+h) - \mu)]}{\sigma^2}. \quad (16)$$

In summary, ACF allows to analyze the similarity among time series observations. By considering it as discriminating function, we can evaluate whether the similarity among the original time series observations agrees with the similarities of surrogates or not.

The Average Mutual Information (AMI) can be seen as a nonlinear version of ACF, which helps to determine the dependence between past and future observations [Abarbanel, 1996; Nakamura & Small, 2005; Nakamura et al., 2006]. Equation (17) defines AMI, in which $p(x(t))$ represents the marginal probability distribution function of $x(t)$ and $p(x(t), x(t+h))$ is the joint probability distribution function of $x(t)$ and $x(t+h)$, having $h$ as the time lag.

In all experiments, the time lag was varied within the interval [1, 20] and 16 bins were used to discretize data and estimate probabilities as follows:

$$I(h) = \sum_x p(x(t), x(t+h)) \times \log\left(\frac{p(x(t), x(t+h))}{p(x(t))p(x(t+h))}\right). \quad (17)$$
6. Experiments

This section presents the experimental results in two subsections. First, we analyzed the synthetic time series, then the real ones.

6.1. Synthetic time series

In the first synthetic experiment, we analyzed a time series created by the combination of a sine function and a linear sequence of observations.

![Fig. 2. Surrogates generated from the synthetic time series created by the combination of trend and sine functions. On the left side, the original time series (red continuous line) and its surrogates (dashed lines) are presented. In the middle and right side, the ACF and AMI plots are shown.](image-url)
added to simulate a trend behavior. This time series (red continuous line) and the surrogates (dashed lines) generated by FT, AAFT, EMD-FT, and EMD-AAFT are illustrated in Figs. 2(a), 2(d), 2(g) and 2(j), respectively. By visually inspecting these plots, we observe EMD-FT produced surrogates whose behavior is very similar to the original series. Although the surrogates produced by the EMD-AAFT method are also close to the original time series, we notice the presence of small noise changing the expected behavior.

These conclusions were also drawn by analyzing the discriminating statistics. According to ACF [Figs. 2(b), 2(e), 2(h) and 2(k)] and AMI [Figs. 2(c), 2(f), 2(i) and 2(l)] plots, the original data (red continuous line) only falls within the surrogate distribution (dashed lines) produced by the EMD-FT method.

In the second synthetic experiment, we analyzed a time series [Fig. 3(a)] created by the combination of a white noise process and a linear sequence of observations used to simulate a trend behavior. By analyzing the plots in Fig. 3, we confirm the proposed methods produce surrogates, whose behavior is closer to the original time series than the Theiler’s FT and AAFT methods. In case of EMD-FT and EMD-AAFT, we observe no significant differences.

The third synthetic time series was created combining an autoregressive process and a linear sequence of observations. Observing Fig. 4, the proposed methods generated surrogates more consistently with the original time series. We highlight there is no significant difference between EMD-FT and EMD-AAFT when visually inspecting time series plots.

The last synthetic time series was created considering the outputs produced by a Lorenz system [Eq. (18)] with parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$.

$$\begin{align*}
\frac{dx}{dt} &= \sigma(y-x), \\
\frac{dy}{dt} &= x(\rho-z) - y, \\
\frac{dz}{dt} &= xy - \beta z. 
\end{align*}$$

These values produced a nonlinear time series with chaotic behavior [Alligood et al., 1997]. Therefore, by applying the Theiler’s methods on this nonlinear time series, the null hypothesis may be rejected, once the produced surrogates are linear. Similarly, hypothesis tests on the surrogates produced by the methods proposed in this manuscript also showed the null hypothesis was rejected, as presented in Fig. 5.

In the next section, we present the results obtained when considering real-world time series.

### 6.2. Real-world time series

The first time series analyzed in this section is presented in Fig. 6(a). This time series was studied in [Shumway & Stoffer, 2006] and it is composed of yearly average values of global temperatures.

By analyzing this series, we realized observations follow a trend, i.e. the mean temperature increases over time, characterizing the time series as nonstationary. Nevertheless, the surrogates created by Theiler’s FT and AAFT cannot replicate such a trend. On the other hand, the EMD-FT and EMD-AAFT methods produced surrogates duplicating the nonstationarity of the original time series. Analyzing only the AMI plots, we conclude all methods produced surrogates which are compatible with the original time series, once they are within the surrogate distribution. However, in this situation, the ACF plot clearly indicates Theiler’s FT and AAFT surrogates are different from the original time series (the left side of Fig. 6).

The second real-world time series considered in this study is composed of atmospheric concentrations of CO$_2$ (Fig. 7) and has a similar behavior to the synthetic series presented in Fig. 2. This series is characterized by some trend and cyclical behavior. In this scenario, the best surrogates were generated using EMD-FT and EMD-AAFT, as expected due to the presence of a trend. This is also confirmed by the ACF and AMI plots. In this situation, there is no significant difference between the EMD-FT and EMD-AAFT surrogates.

Finally, the last experiment was performed on the Dow Jones Utilities Index, which was recorded from 28 August to 18 December 1972 [Brockwell & Davis, 2002]. This time series (Fig. 8) has

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4 Although there are many current observations for this dataset, we used this period due to its adoption in several papers and textbooks on time series analysis.
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Fig. 3. Surrogates generated from the synthetic time series created by the combination of trend and white noise processes. On the left side, the original time series (red continuous line) and its surrogates (dashed lines) are presented. In the middle and right side, the ACF and AMI plots are shown.

- a trend behavior as well, benefiting the EMD-FT and EMD-AAFT methods. This conclusion is also evident in the ACF plots. However, according to the AMI plots, the only ineffective method was Theiler’s AAFT. In such situation, EMD-FT and EMD-AAFT surrogates have similar behavior.

Finally, we also applied a hypothesis test on the discriminating statistics produced by the Average...
Mutual Information. Thus, we applied hypothesis tests to compare the original time series against every surrogate produced by all methods. Then, we computed the average $p$-value $\mu_{p\text{-value}}$ for every method. At last, the following hypothesis test was applied to compare the methods, which considers a significance level of 0.01 for the one tailed test

$$
\begin{align*}
H_0 : & \quad p\text{-value} \geq 0.01 \\
H_a : & \quad p\text{-value} < 0.01 .
\end{align*}
$$

(19)
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Fig. 5. Surrogates generated from the chaotic time series created by the Lorenz system. The left side presents the original time series (red continuous line) and its surrogates (dashed lines). In the middle and right side, the ACF and AMI plots are shown.
Fig. 6. Surrogates generated from the real-world time series composed of average values, collected yearly, of global temperatures. On the left side, the original time series (red continuous line) and its surrogates (dashed lines) are presented. In the middle and right side, the ACF and AMI plots are shown.
Fig. 7. Surrogates generated from the real-world time series composed of atmospheric concentrations of CO$_2$. On the left side, the original time series (red continuous line) and its surrogates (dashed lines) are presented. In the middle and right side, the ACF and AMI plots are shown.
This test accepts the null hypothesis when the average p-value is greater than 0.01, otherwise, we accept the alternative hypothesis, which allows us to infer the surrogate and original time series were not produced using the same generation process. The obtained results were summarized in Table 1, in which letters A and R mean the null hypothesis was accepted (the tests failed to reject it) or rejected, respectively.

According to Table 1 the surrogates generated by Theiler’s FT and AAFT were significantly different from the original time series in most situations,
We were able to prove that our methods produce similar surrogates to Theiler’s FT and AAF methods, but without any stationary influence.

This assumption is not mandatory, we could produce the final surrogate by combining IMFs at different phase randomizations. Using this process, we can use our methods to: (i) filter only determinist IMFs (Rios & Mello, 2013) and apply phase randomization to produce more representative surrogates, once the stochastic behavior may be out of scope for some application domains, such as signal and image processing; (ii) filter time series trends out and produce surrogates only considering the relevant behavior which is represented by IMFs. Finally, we only add trends to compose the final surrogate, maintaining the nonstationary characteristic of the original time series (as approached in this work) that is not fulfilled by Theiler’s FT and AAF methods; (iii) filter IMFs according to amplitudes to produce surrogates at different randomization levels. For example, consider the time series shown in Fig. 1(e), which corresponds to a collection of yearly average global temperatures. By applying the EMD method on this time series, a set of IMFs is obtained as shown in Fig. 9. When noticing the EMD method on this time series, a set of IMFs is obtained as shown in Fig. 9. When noticing the relevant behavior which is represented by IMFs. Finally, we only add trends to compose the final surrogate, maintaining the nonstationary characteristic of the original time series (as approached in this work) that is not fulfilled by Theiler’s FT and AAF methods; (iii) filter IMFs according to amplitudes to produce surrogates at different randomization levels. For example, consider the time series shown in Fig. 1(e), which corresponds to a collection of yearly average global temperatures. By applying the EMD method on this time series, a set of IMFs is obtained as shown in Fig. 9. We notice the

7. Discussion on Phase Randomization

In our proof, we applied the same phase randomization for all monocomponents, i.e. after extracting a set of IMFs for a nonlinear time series, the same variable $\phi$ [Eq. (3)] used to rotate the phase of the first IMF is again used for the remaining IMFs. This assumption was used to simplify the analysis of our methods. By using a constant value for the phase, we were able to prove that our methods produce similar surrogates to Theiler’s FT and AAF methods, but without any stationary influence.

By analyzing the results obtained with the synthetic and real-world time series, we conclude that (i) our proposed EMD-FT and EMD-AAFT methods provided greater $p$-values, showing their surrogates are more similar to the original time series.

Table 1. Hypothesis test using Average Mutual Information (AMI): FT and AAFT rejected the null hypothesis in 9 out of 12 scenarios, confirming the surrogate series produced are not representative enough; EMD-FT and EMD-AAFT accepted the null hypothesis in all scenarios, confirming they produce more significant surrogate data.

<table>
<thead>
<tr>
<th>Time Series</th>
<th>FT</th>
<th>AAFM</th>
<th>EMD-FT</th>
<th>EMD-AAFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine + Trend</td>
<td>$0 (R)$</td>
<td>$0 (R)$</td>
<td>$0.753 (A)$</td>
<td>$0.013 (A)$</td>
</tr>
<tr>
<td>White noise + Trend</td>
<td>$4.105 \times 10^{-5} (R)$</td>
<td>$1.515 \times 10^{-5} (R)$</td>
<td>$0.601 (A)$</td>
<td>$0.508 (A)$</td>
</tr>
<tr>
<td>AR(1) + Trend</td>
<td>$7.882 \times 10^{-7} (R)$</td>
<td>$1.010 \times 10^{-10} (R)$</td>
<td>$0.565 (A)$</td>
<td>$0.571 (A)$</td>
</tr>
<tr>
<td>Lorenz system</td>
<td>$9.596 \times 10^{-5} (R)$</td>
<td>$3.111 \times 10^{-5} (R)$</td>
<td>$1.181 \times 10^{-5} (R)$</td>
<td>$3.862 \times 10^{-5} (R)$</td>
</tr>
<tr>
<td>Global temperature</td>
<td>$0.727 (A)$</td>
<td>$0.942 (A)$</td>
<td>$0.829 (A)$</td>
<td>$0.886 (A)$</td>
</tr>
<tr>
<td>CO2</td>
<td>$5.253 \times 10^{-3} (R)$</td>
<td>$0 (R)$</td>
<td>$0.483 (A)$</td>
<td>$0.412 (A)$</td>
</tr>
<tr>
<td>Dow Jones</td>
<td>$0.073 (A)$</td>
<td>$0.003 (R)$</td>
<td>$0.344 (A)$</td>
<td>$0.321 (A)$</td>
</tr>
</tbody>
</table>

Fig. 9. The plots from (a) to (f) show the IMFs and the residue extracted from the time series presented in Fig. 1(e).
...amplitudes of IMF's vary significantly. Hence, the phase randomization of a low-amplitude IMF adds no significant information to the final surrogate.

Finally, even using constant phase randomization, the analysis on the null hypothesis for our methods (see Sec. 4) remains consistent.

8. Concluding Remarks

In this paper, we discussed the problem faced by Theiler's FT and AAFT methods when time series present nonstationary behavior. In such situations, surrogates produced by these methods are very different from the original data. By applying statistical methods or even performing a visual inspection on the original and surrogate time series, we cannot state whether they were created from the same process or not.

In order to address this drawback, we proposed two new methods based on Theiler's techniques. The new methods initially decompose the time series into monocomponents that are, in a second step, transformed by either Theiler's FT or AAFT method. As a result, a set of monocomponent surrogates is produced, which are combined with the original time series trend to create the surrogate time series.

Experimental results on synthetic and real-world time series confirmed the proposed methods produced surrogates in accordance to the original data in the presence of nonstationarity. This is due to the extraction of the series trend, which would otherwise add spurious high frequencies. As a consequence, the proposed methods support the linear/nonlinear test for stationary and nonstationary time series, that is not possible when directly using Theiler’s methods.

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References


