A nonlinear dynamical systems modelling approach unveils chaotic dynamics in simulations of large strain behaviour of a granular material under biaxial compression.

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Chaos in sand

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What can we learn from the data?
Embed

\[ x_t \]

\[ \downarrow \]

\[ z_t = (x_t, x_{t-\tau}, \ldots, x_{t-(d-1)\tau}) \]

\( d \) embedding dimension
\( \tau \) embedding lag
Model

\[
v_t = (x_t, x_{t-1}, x_{t-2}, x_{t-3}, \ldots x_{t-(d_e-1)})
\]

\[
F(v_t) = x_{t+1}
\]

\[
= \lambda_0 + \sum_{i=1}^{b_1} \lambda_i v_{t-\ell_i} + \ldots + \sum_{i=1}^{b_2} \lambda_{i+b_1} \left( \frac{\|v_t - c_i\|}{r_i} \right)
\]

\[
\phi(x; s) = \begin{cases} 
  e^{-\frac{x^2}{2}}, \\
  x^3, \\
  \tanh \frac{x}{s}, \text{or} \\
  \cos(2\pi \frac{x}{s}) \exp \left(-2\left(\frac{\pi}{5}\right)^2 \left(\frac{x}{s}\right)^2\right) - \ldots \\
  \exp \left(-\frac{5^2}{2} - 2\left(\frac{1}{5}\right)^2 \left(\frac{x}{s}\right)^2\right)
\end{cases}
\]
Compare

• Linear Statistics:
  • Skewness
  • Kurtosis

• Nonlinear measures
  • Correlation Dimension
  • Entropy
  • Noise

• False neighbours
  • Lyapunov exponents

• Complex network
  • Motif distribution
  • Mean path-length
  • Assortativity
  • Clustering
Motifs: flows

Chaos

Hyper-Chaos

Periodic+Noise

Periodic
Data

- DEM simulation
- Bulk measurements of the stress ratio
- Initially dense assembly of grains under biaxial compression with constant confining pressure
- Densely packed granular assembly – observed during the large strain or so-called critical state regime in the presence of a fully developed shear band.
Results: Linear noise? No.

FIG. 4. Linearly filtered noise Comparison of nonlinear statistics computed with the Gaussian Kernel algorithm (correlation dimension, entropy and noise level) and higher order linear distribution statistics (skewness and kurtosis) for the original data in Fig. 2 and linear surrogates (monotonic nonlinear transformations of linearly filtered noise). As expected the linear statistics (lower right plots) show no difference. However, the nonlinear measures indicate that the linear model does not adequately described the dynamics (upper panels and lower left). For the nonlinear measures, the solid blue lines (no error bars) indicates the statistic values computed for the data (as a function of the embedding dimension — a parameter of the statistic). The tight error bars (red) are the mean and standard deviation from 100 simulations, the larger error bars (green) are the full range (minimum to maximum). For the higher order linear statistics, a distribution of values is plotted, the value for the data is indicated as a solid line.
Stable Node? Maybe...

GKA – correlation dimension

GKA – entropy

GKA – noise

GKA – kurtosis

GKA – skewness
Stable Focus? No.
Transient Chaos?
Probably.
... & complex network measures?

<table>
<thead>
<tr>
<th>model</th>
<th>N</th>
<th>diameter</th>
<th>clustering</th>
<th>assortativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>13.1</td>
<td>0.353</td>
<td>-0.122</td>
<td></td>
</tr>
<tr>
<td>Stable node</td>
<td>58</td>
<td>16.0 ± 2.7</td>
<td>0.360 ± 0.0251</td>
<td>-0.079 ± 0.060</td>
</tr>
<tr>
<td>Stable focus</td>
<td>100</td>
<td>12.1 ± 0.88</td>
<td>0.388 ± 0.0146</td>
<td>-0.0084 ± 0.053</td>
</tr>
<tr>
<td>Transient chaos</td>
<td>77</td>
<td>13.1 ± 0.84</td>
<td>0.358 ± 0.0172</td>
<td>-0.053 ± 0.058</td>
</tr>
</tbody>
</table>

FIG. 8. Motif frequency figure For each of the three model dynamics we computed the motif super-family which occurred most frequently — shown here as a percentage. Motif family ADBCEF is the corresponding family for the data (the right-most batch of columns). In models exhibiting a stable focus, for example Motif ABDCEF occurred for all but one of the model simulations, that exception was ADBCEF. According to this measure, stable node and transient chaos models exhibited behavior most similar to the data. Inset: the structure of the six four node motifs.
Dynamics
Features from data

Shear direction

T(1) < T(2) < T(3) < T(4)

[Graphs showing force/mass over time index]
Network measures

Community membership from epsilon networks

Node closeness centrality

Network measures
Thanks.

See also: Poster, this afternoon.
