The problem can be solved graphically as shown below:

\[ \begin{align*}
    x_1, x_2 & \geq 0 \\
    f & \leq 1x \\
    f & \geq 1x \\
    g & \leq 2x + 1x \\
    g & \geq 2x + 1x \\
    z x - 1x z - = d \\
\end{align*} \]

**Graphical Method:** Consider a simple LP problem:

This is a linear programming (LP) problem since the objective function and all the constraints are linear.

\[ \begin{align*}
    \text{subject to} & \\
    f & \leq 1x, \quad (1) \\
    g & \geq 2x + 1x, \quad (2) \\
    & \geq 2x + 1x. \quad (3) \\
\end{align*} \]

So, mathematically, we have

\[ \begin{align*}
    x_1 & \geq 0 \\
    x_2 & \geq 0 \\
    x_3 & \geq 0 \\
\end{align*} \]

We also need to impose

(4)

\[ 0 \leq x_0 + x_1 + x_2 \leq 800 \]

(5)

\[ 0 \leq x_0 + x_1 + x_2 \leq 800 \]

These can be formulated as

- Total profit per week = $20x_1 + 60x_2 + 80x_3$
- Total profit per week = \( \geq \)
- Total profit per week = \( \leq \)
- Total profit per week = \( = \)

**Mathematical Form of the Above Problem:**

1. Find the mathematical form of the above problem:

   \[ \begin{align*}
   \text{subject to:} & \\
   f & \leq 1x, \quad (1) \\
   g & \geq 2x + 1x, \quad (2) \\
   & \geq 2x + 1x. \quad (3) \\
   \end{align*} \]

2. Solve the LP problem:

   \[ \begin{align*}
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0 \\
   \end{align*} \]

   We also need to impose

   \[ \begin{align*}
   0 \leq x_0 + x_1 + x_2 \leq 800 \\
   0 \leq x_0 + x_1 + x_2 \leq 800 \\
   \end{align*} \]

   These can be formulated as

   - Total profit per week = $20x_1 + 60x_2 + 80x_3$
   - Total profit per week = \( \geq \)
   - Total profit per week = \( \leq \)
   - Total profit per week = \( = \)
The standard form:

Now, we present some methods for reducing a general LP problem to

1. We assume that \( m \geq n \). Otherwise, \( m = n \) is determinate.

2. We assume that in \( m \) columns of \( A \) are sufficient to span \( \mathbb{R}^n \).

3. We assume that the objective function is \( \sum_{i=1}^{n} a_i x_i \).

The following is called the standard form of LP problems:

\[
0 \leq \mathbf{q} \mathbf{x} \leq \mathbf{b}
\]

subject to

\[
\mathbf{A} \mathbf{x} = \mathbf{c}
\]

where

\[
\mathbf{q} = [q_1, q_2, \ldots, q_m]^T
\]

\[
\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m]
\]

\[
\mathbf{b} = [b_1, b_2, \ldots, b_m]^T
\]

\[
\mathbf{c} = [c_1, c_2, \ldots, c_n]^T
\]

\[
\mathbf{x} = [x_1, x_2, \ldots, x_n]^T
\]

\[
\mathbf{q} = [q_1, q_2, \ldots, q_n]^T
\]

\[
\mathbf{a}_i = [a_{i1}, a_{i2}, \ldots, a_{in}]^T
\]

\[
\mathbf{b}_i = [b_{i1}, b_{i2}, \ldots, b_{im}]^T
\]

Note that graphical methods can only solve problems in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) dimensions.
1. If \( x \) is a solution that satisfies \( A x = q \) and \( q \geq 0 \), then \( x \) is feasible.

2. If \( x \) is a feasible solution that satisfies \( A x = q \), then \( q \geq 0 \).

Definition. Consider the standard form of an LP problem:

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad A x = b \quad \text{and} \quad x \geq 0.
\end{align*}
\]

NB. (ii) is more popular than (i).

Then, replace all \( x \) in the cost and the constraints by this expression:

\[
\left( \sum_{i=1}^{m} a_{ij} x_j \right) / \lambda \quad \text{for all } j.
\]

Solving this gives

\[
hq = \sum_{i \in \{1, \ldots, m\} : \lambda_i \neq 0} \frac{\lambda_i}{\lambda} h_i
\]

where there is at least one \( i \in \{1, \ldots, m\} \) such that \( \lambda_i \neq 0 \). So,

\[
x = x_1, x_2, \ldots, x_n
\]

with unknowns

Moreover, the problem becomes an \((n+1)\)-dimensional problem where the \( x \) and \( h \) are replaced by \( x = \begin{bmatrix} x_1 & \ldots & x_n & h \end{bmatrix}^T \).

6. \( x \) satisfies \( -\infty < x < \infty \).

\[
\max \quad c^T x \quad \text{subject to} \quad \begin{bmatrix} \lambda_1 & \cdots & \lambda_n & 1 \end{bmatrix}^T c = 0
\]

Multiplying by \(-1\) we have

\[
\min \quad c^T x
\]

The objective is maximized, i.e., \( \max \).
Definitions.

All these \( x \)'s are basic solutions with respect to the basis

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
and \( x \).

1. Columns 1 and 2 form \( H \) is

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
Since \( \det(H) \neq 0 \), \( H \) is

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
We have these combinations:

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

Example: Consider the problem

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
This is called a basic solution, \( x \). These are basic variables and

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
This for and let let \( |x|, \lambda |x|, \xi |x| \).

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
Solve where

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
\( q = Bx \)

det(\(B\)) \neq 0, \( B \) is

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
possible (to form an \( H \) matrix). Then, \( B \) is non-singular, i.e.,

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
is the only zero solution (for all \( \xi = 0 \), \( \lambda = 0 \)).

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
Linear independence, each other if

\[
\begin{pmatrix}
\xi \\
\lambda
\end{pmatrix}
\]
minimize \( G \).

A \( x \) is an optimal feasible solution if \( x \) is a feasible solution and
Proof of part (ii) is omitted here.

We can repeat this process until the remaining columns are linearly independent and so have a basic feasible solution.

Case 1: The set \( \{ x_1, \ldots, x_n \} \) is linearly dependent. Then \( z = \sum_{j=1}^n c_j x_j \) is a feasible solution.

Case 2: \( m \) of the basic variables in a basic solution is zero.

Case 3: \( m \) of the basic variables in a basic solution is nonzero.

These are the two cases.

Therefore, \( m \) of the basic variables in a basic solution is zero.

Thus, the feasible solution is a basic feasible solution with \( m \) of the basic variables in a basic solution being zero.

Proof of (ii): Assume that the solution is not a basic feasible solution. Then, there exists a variable \( x_k \) with a positive coefficient in the objective function that is not in the basis.

Let \( x_k \) be the variable to enter the basis. Then, we can replace \( x_k \) with a combination of the other variables in the basis.

Thus, the solution is a basic feasible solution.

Theorem (Fundamental Theorem of LP) Consider a linear program:

1. If a basic feasible solution is degenerate, it is called a degenerate feasible solution.
2. If \( x \) is a basic solution and \( x \) is feasible, then it is called a basic feasible solution.
3. If \( x \) is a basic feasible solution and \( x \) is feasible, then it is called a basic feasible solution.

If none of the basic variables in a basic solution is zero,
Theorem 3. Let $C$ be a convex set. Then $C$ is convex.

Proof. Let $x, y \in C$. Then, for any $t \in [0, 1]$, we have $tx + (1 - t)y = (tx + (1 - t)y) \in C$.

Examples of extreme points:

1. $x$ in a square: some boundary points can be extreme points.
2. $x$ in a circle: some boundary points are extreme points.
3. $x$ in a line: no two distinct points on the line can be extreme points.
4. $x$ in a plane: no three distinct points in the plane can be extreme points.

Proposition 1. $A \cap B$ is convex.

Proof. $A \cap B$ is a convex set.

Theorem 4. Let $C$ and $D$ be convex sets. Then $C \cup D$ is also convex.

Proof. Let $x \in C$ and $y \in D$. Then $tx + (1 - t)y \in C \cup D$.

Theorem 5. If $C$ is a convex set, then

$$
    
$$

Proof. Let $x \in C$. Then $tx \in C$ for any $t \in [0, 1]$.

Theorem 6. Let $C$ be a convex set and $a$ be a real number.

Proof. For any $t \in [0, 1]$, we have $tx + (1 - t)a \in C$.
0 \leq \beta_2 - x \quad \text{and} \quad 0 < \beta_3 + x

and \exists \beta \leq 0 \text{ such that } 0 \leq \beta_2 - x < \beta_3 + x. \quad \text{Since } x \neq 0, \text{ we can define } 0 = (\beta_2 - x) + (0 < \beta_3 + x).

Thus, for any \varepsilon > 0, we have

0 = \beta_2 + \varepsilon < \beta_3 + (0 < \beta_2 + x)

We assume the contradiction: i.e., all $w_i$ are linearly independent. Hence, there exist $z \neq 0$ such that $x = \sum z_i w_i$, and $x$ is not in the column space of $A$. By contradiction, we assume that $x$ is not in the column space of $A$, and $w_i$ are linearly independent, where $z$ contains the ith component of $x$.

Let $x$ be an extreme point of $Y$. We assume that the vector $x$ is an extreme point of $Y$. Let $z \neq 0$ such that $x = \sum z_i w_i$. Since all $w_i$ are linearly independent, it follows that

$0 = \sum z_i (w_i - w_i) + \sum (\beta_i - \beta_i) x_i + \|x - 1\| x_i$

Subtracting this from $8$, $q = \sum z_i w_i + \sum \beta_i x_i + 0 \varepsilon$

Recall that $\varepsilon$ is chosen such that $\varepsilon \leq \|Y - x\|$. Thus,

$\sum (0 \leq \beta_i - \beta_i + \varepsilon) x_i = 0$

and

This is because $x_i = 0$ for all $i \neq 1$.

Let $x \neq 0$, $0 \neq \beta_i$. Then, for any $x \in \mathbb{F}$, the number of $w_i$ satisfies

$\sum (0 \leq \beta_i - \beta_i + \varepsilon) x_i = 0$

and

Finally, we have

$A^T(0 \leq \beta_i - \beta_i + \varepsilon) x_i = 0$

This is because $x_i = 0$ for all $i \neq 1$.
**Theorem 8.** If K is bounded, then at least one optimal feasible solution is an extreme point of K.

Proof. Omitted.

Example: Consider the following constraints:

\[
\begin{align*}
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
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\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\
\ell & \leq \ell x + \ell y \\end{align*}
\]

The number of these feasible solutions is finite and is equal to the number of extreme points. Hence, there are no unbounded convex sets in \( \mathbb{R}^n \).

Definition (Polyhedron): A set \( \mathcal{C} \subseteq \mathbb{R}^n \) is said to be a convex set if \( \mathcal{C} \) is the convex hull of its extreme points.

**Example 5.** If the constraint set is non-empty, there is at least one extreme point.

Proof. Since \( \mathcal{C} \) is non-empty, there is a basic feasible solution. Thus, \( \mathcal{C} \) is bounded. Hence, \( \mathcal{C} \) is compact. Therefore, there is a bounded minimizer, say \( (x_1, x_2, \ldots, x_n) \). Since \( \mathcal{C} \) is bounded, there is a finite number of vertices. Hence, \( \mathcal{C} \) is a polyhedron.

**Corollary 2.** Consider the LP problem in the fundamental Theorem 8.

**Corollary 3.** There are only a finite number of extreme points in the set \( \mathcal{C} \).

Therefore, the set of feasible solutions is finite. Hence, there is a finite number of extreme points. The set of extreme points is the set of vertices of the convex polyhedron \( \mathcal{C} \).

**Proof.** Since \( \mathcal{C} \) is non-empty, there is a basic feasible solution. Thus, \( \mathcal{C} \) is bounded. Hence, \( \mathcal{C} \) is compact. Therefore, there is a bounded minimizer, say \( (x_1, x_2, \ldots, x_n) \). Since \( \mathcal{C} \) is bounded, there is a finite number of vertices. Hence, \( \mathcal{C} \) is a polyhedron.

**Proof.** Omitted.
\[
\begin{align*}
\left[ f_0 - \sum_{k=1}^{m} f_k H_k \right] x & \leq \sum_{k=1}^{m} - q_k H_k \quad = \\
& \left( \begin{array}{c}
\begin{array}{c}
(\Pi) \\
(\Pi)
\end{array}
\end{array} \right) \left( \begin{array}{c}
\begin{array}{c}
(\Pi) \\
(\Pi)
\end{array}
\end{array} \right)
\end{align*}
\]

Substituting this into (10), we have:

\[
N x N_{k-1} - q_{k-1} = H x
\]

From (II), we have:

\[
\sum_{k=1}^{m} f_k x \leq N x N_c
\]

and

\[
\sum_{k=1}^{m} f_k x \leq N x N
\]

Rule that:

\[
0 \leq N x \cdot H x
\]

subject to

\[
N x N_c + N x H = 0
\]

subject to

\[
N x N_c + N x H = 0
\]

Similarly, we let \(c' \leq \frac{1}{\lambda} \sum_{k=1}^{m} N_k H_k \).

\[
\left( \begin{array}{c}
N x \\
H x
\end{array} \right) \left( \begin{array}{c}
N x \\
H x
\end{array} \right) = v
\]

\[
\text{subject to}
\]

\[
0 \leq x
\]

Consider the following:

1. If there is no basic feasible solution, then we may find a basic feasible solution using the simplex algorithm.

2. If there is no basic feasible solution, then we may define the \(z\) to be the cost of the new basic feasible solution.

3. If there is no basic feasible solution, then we may define the \(z\) to be the cost of the new basic feasible solution.

4. Repeat steps 2 and 3 until no further improvement is possible.

5. Exchange a basic variable with an old basic variable so that the cost is decreased.

The method in Tableau form:

\[
\text{subject to}
\]

\[
\text{subject to}
\]

\[
\text{subject to}
\]

\[
\text{subject to}
\]

The Simplex Algorithm:

The method in Tableau form:

\[
\text{subject to}
\]

\[
\text{subject to}
\]

\[
\text{subject to}
\]

\[
\text{subject to}
\]

The Simplex Algorithm:
\[
\begin{bmatrix}
0.1 \\
0.1
\end{bmatrix} = \begin{bmatrix}
4 \ x \\
4
\end{bmatrix} = 4p
\]

so that
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = (4p \ 4p) = d
\]

We choose the initial basis \( B = \begin{bmatrix} 4p \ 4p \end{bmatrix} \) and \( x \) is chosen so that
\[
\begin{bmatrix}
0.1 \\
0.1
\end{bmatrix} = \begin{bmatrix}
4 \ z \\
4
\end{bmatrix} = r
\]

Obviously
\[
0 < x
\]
\[
0 < x
\]
\[
0 < x
\]
\[
0 = x
\]
\[
0 = x
\]
\[
0 = x
\]
\[
0 = x
\]

Solution. Introducing slack variables \( s_x x z \), we have
\[
\begin{align*}
0 & \leq x + s_x x
\\ 0 & \leq x + s_z z
\\ \sum_{i=1}^{n} x_i - s_x x &= z
\end{align*}
\]

Example 1:
\[
\frac{d_{2p}}{d_{2p}} = \frac{f_{2p}}{f_{2p}}
\]

\[
d \neq 1 \Rightarrow \frac{d_{2p}}{d_{2p}} - \frac{f_{2p}}{f_{2p}}
\]

coefficient, \( d \), of the new tableau is given by

Rule III. Let the coefficient of the old tableau be denoted by \( a \); then the

\[
\begin{align*}
\frac{1 + m_j - f_{1j} B_{1j}}{m_j} &= x
\\ \frac{1 + m_j - f_{1j} B_{1j}}{m_j} &= x
\end{align*}
\]

become non-basic if

Let \( x \) be the only component of \( B \). Then, \( B \) is chosen to

Rule II. Choose the basic variable \( x \) to become non-basic.

Rule I. The optimal tableau will then be unique.

In the tableau, \( x \) is the only variable, and \( f \) is the set of indices of non-basic variables.

\[
\begin{align*}
\{ f \} &= \begin{bmatrix} f_j \end{bmatrix}
\\ \{ f \} &= \begin{bmatrix} f_j \end{bmatrix}
\end{align*}
\]

The simplex (tableau) method:

\[
\begin{aligned}
\text{Rule I.} & \quad \text{Choose \( x \) \text{ as \text{ chosen \ according \ the \ following \ method:} \}} \\
\text{Rule II.} & \quad \text{Choose the \text{ basic \ variable \( x \) \text{ to become \non-basic.} \}
\\ \text{Rule II.} & \quad \text{Choose the \text{ basic \ variable \( x \) \text{ to become \non-basic.} \}
\\ \text{Rule I.} & \quad \text{Choose the \text{ optimal \ tableau \ will \ then \ be \ unique.} \}
\end{aligned}
\]

\[
\begin{aligned}
\text{Rule I.} & \quad \text{Choose \( x \) \text{ as \text{ chosen \ according \ the \ following \ method:} \}} \\
\text{Rule II.} & \quad \text{Choose the \text{ basic \ variable \( x \) \text{ to become \non-basic.} \}
\\ \text{Rule II.} & \quad \text{Choose the \text{ basic \ variable \( x \) \text{ to become \non-basic.} \}
\\ \text{Rule I.} & \quad \text{Choose the \text{ optimal \ tableau \ will \ then \ be \ unique.} \}
\end{aligned}
\]

Therefore:

\[
\begin{align*}
(f_j - f) x & \sum_{u=1}^{m_j} - q \ x
\\ (f_j - f) x & \sum_{u=1}^{m_j} - q \ x
\end{align*}
\]

\[
\begin{align*}
(f_j - f) x & \sum_{u=1}^{m_j} - q \ x
\\ (f_j - f) x & \sum_{u=1}^{m_j} - q \ x
\end{align*}
\]

Set \( x \) from the above.

We have from the above.
\[
\begin{bmatrix}
1 & u & w & 1 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Suppose \( x \) is of the form \( x = \sum_{i=1}^{n} a_i x_i \) with
\[
0 \leq x
\]

such that
\[
x_d = z
\]

Consider the standard form of the above LP problem:

Proof of Rule III

Since all \( z_i - c_i \) we have obtained the optimal feasible solution.

<table>
<thead>
<tr>
<th>( \bar{e} )</th>
<th>( \bar{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5( e )</td>
<td>0</td>
</tr>
<tr>
<td>5( e )</td>
<td>1</td>
</tr>
<tr>
<td>3( e )</td>
<td>1</td>
</tr>
<tr>
<td>0 ( e )</td>
<td>0</td>
</tr>
</tbody>
</table>

From this we have

\[
\begin{bmatrix}
\bar{e} \\
\bar{z}
\end{bmatrix}
\]

Now, \( \bar{e} = 1 \) and \( \bar{z} = 1 \). Note that for this case \( \bar{e} = 1 \).

<table>
<thead>
<tr>
<th>( \bar{e} )</th>
<th>( \bar{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3 ( e )</td>
<td></td>
</tr>
<tr>
<td>4 ( e )</td>
<td></td>
</tr>
</tbody>
</table>

Rule II: We calculate the new tableau entries.
Consider a fully worked example. This generates the coefficients of the new tableau.

\[
\begin{align*}
0 & \geq x \\
1 & \leq bx + cx + dx + 1x \\
\tau & \leq bx + cx + dx + 1x \\
0 & \geq bx + cx + dx + 1x
\end{align*}
\]

subject to

with

\[
\begin{align*}
\phi_p^m = \sum_{\phi_p^m} + \phi_p^m \phi_p^m := \\
\phi_p^m \left( \phi_p^m \phi_p^m - \phi_p^m \phi_p^m \right) \sum_{\phi_p^m} + \phi_p^m \phi_p^m = \\
\phi_p^m \sum_{\phi_p^m} + \phi_p^m \phi_p^m - \phi_p^m \phi_p^m = \phi_p^m
\end{align*}
\]

Substituting (18) into (19) we have

\[
\phi_p^m \sum_{\phi_p^m} + \phi_p^m \phi_p^m = \phi_p^m
\]

From (17) we have

\[
\phi_p^m \sum_{\phi_p^m} \phi_p^m = \phi_p^m
\]
Evaluation of the optimality condition:

\[ \mathbf{x} = z > 0 \quad \text{and} \quad \mathbf{z} \text{ is given in the tableau. From this we have} \]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

\[
\text{and} \quad \mathbf{z} \text{ is given in the tableau. From this we have}
\]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

Similarly, we can evaluate the new that for \( z \).

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

\[
\text{Similarly, we can evaluate the new that for } \mathbf{z}. \]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

\[
\text{Similarly, we can evaluate the new that for } \mathbf{z}. \]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

\[
\text{Similarly, we can evaluate the new that for } \mathbf{z}. \]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]

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\text{Similarly, we can evaluate the new that for } \mathbf{z}. \]

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\text{Similarly, we can evaluate the new that for } \mathbf{z}. \]

\[
\begin{bmatrix}
\mathbf{z}/1 & 0 \\
\mathbf{z}/\mathbf{x} & 1
\end{bmatrix} = H
\]
Example: Solve down immediately, no initial feasible solution allowed. The basic feasible solution can not be written. In the case where the constraint is not redundant, we can choose \( \mathbf{y} = 0 \). In that case, a basic feasible solution is easy to find.

\[
\begin{bmatrix}
0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

subject to

\[
0 \leq 1 \mathbf{x}
\]

**The problem becomes**

\[
\begin{bmatrix}
\mathbf{q} \\
\mathbf{x}
\end{bmatrix} \leq \begin{bmatrix}
1 & 2
\end{bmatrix}
\]

Introducing slack variables, such that

\[
0 \leq \mathbf{x}
\]

\[
\begin{bmatrix}
\mathbf{q} \\
\mathbf{x}
\end{bmatrix} \leq \begin{bmatrix}
1 & 2
\end{bmatrix}
\]

subject to

\[
\begin{bmatrix}
\mathbf{q} \\
\mathbf{x}
\end{bmatrix} = \begin{bmatrix}
1 & 2
\end{bmatrix}
\]

Consider the LP problem:

A method for finding a basic feasible solution

**Two-Phase Method**

\[
\mathbf{y} \rightarrow \mathbf{y}^* = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

and

\[
\mathbf{y}^* \rightarrow \mathbf{y} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

so the optimal solution obtained:

\[
\mathbf{y}^* = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

**Table:**
Table 1: Initially, we choose \( H \) to be the \( 3 \times 3 \) identity matrix.

\[
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

So

\[
\begin{align*}
x + y + z &= x
\end{align*}
\]

with

\[
\begin{align*}
x + y + z &= x
\end{align*}
\]

Phase I: Consider the artificial LP problem:

\[
\begin{align*}
0 \leq x
\end{align*}
\]

subject to

\[
\begin{align*}
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
x + y + z &= 0
\end{align*}
\]

Example. Solve the following examples to demonstrate this algorithm.

| Artificial variable | From the initial solution/phase I by dropping the
|                    | original LP problem by the simplex method. This can start
|                    | phase II: Starting from the basic feasible solution found in phase I, solve
|                    | the original LP problem by the simplex method. To find the basis, we might consider
|                    | solving the following equations. However, solving gives that \( x = 0 \).
|                    | Here, the initial basis is not a feasible solution.

Phase II: Solve the artificial LP problem (22) by the simplex method.
Therefore, we have obtained a basic feasible solution to the

\[
\begin{array}{cccccc|c}
0 & 0 & 0 & 0 & 0 & f_0-f_x \\
\frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{4} & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\
\end{array}
\]

The cost and \( f_0 - f_x \) column can by the usual means of artificial variables and then removing this column. The above can be done by replacing the row vector and the corresponding column of the artificial LP problem. We then state the last tableau as the final tableau.
There is no feasible solution to the original problem, if \( \alpha \neq d \neq 0 \), 50.

<table>
<thead>
<tr>
<th>( r )</th>
<th>0</th>
<th>1</th>
<th>-</th>
<th>0</th>
<th>0</th>
<th>( f_2 - f_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/13</td>
<td>1</td>
<td>1-</td>
<td>5/5</td>
<td>-</td>
<td>5/5</td>
<td>0</td>
</tr>
<tr>
<td>3/5</td>
<td>0</td>
<td>0</td>
<td>5/5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5/11</td>
<td>0</td>
<td>0</td>
<td>5/5</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3}{10} )</td>
<td>( f_2 - f_x )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

So \( \pi = 4 \) and \( \pi = 1 \).

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>0</th>
<th>1</th>
<th>-</th>
<th>0</th>
<th>0</th>
<th>( f_2 - f_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>1-</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(d) ( \pi )</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Initially, we choose \( f = (0, 0, 0, 0) \), which is the \( 3 \times \) X 3 identity matrix. The tableau is

\[
\begin{bmatrix}
0 & 1 & -1 & 0 & 1 & 1 \\
\varepsilon & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]
subject to \( 0 \leq x \), with

To find an initial basic feasible solution, we need to introduce three artificial variables. The problem can be written as the following standard form:

\[
\begin{bmatrix}
0 & 1 & -1 & 0 & 1 & 1 \\
\varepsilon & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]
subject to \( 0 \leq x \), with

Solution, introducing slack variables \( x_2 \) and surplus variable \( x_3 \).
$$\begin{align*}
\text{(39)} & \quad \mathbf{x}^* \geq V_a \mathbf{Y} \\
\text{subject to} & \quad q \mathbf{x}^* \\
\text{max} & \quad \max
\end{align*}$$

Let $A = a - a$ we further write the above as

$$\begin{align*}
\mathbf{x}^* \geq V_a \mathbf{Y} & \quad \text{subject to} \\
q \mathbf{x}^* & \quad \text{max}
\end{align*}$$

Rewriting in the $\mathbf{a} \cdot \mathbf{x}$ form we have

$$(a_1, \ldots, a_n) = a^* \text{ and } (a_1, \ldots, a_n) = a^* \text{ with } (a) = \theta \mathbf{y}^*$$

$$\begin{align*}
0 & \geq \theta \\
\mathbf{x}^* & \geq \begin{bmatrix} V^- \mathbf{Y} \theta \end{bmatrix} \\
\text{subject to} & \quad q \mathbf{x}^* \\
\text{max} & \quad \max
\end{align*}$$

The corresponding dual problem is

$$\begin{align*}
q^* \geq x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} & \quad \text{subject to} \\
q & \geq x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} \\
\text{max} & \quad \max
\end{align*}$$

or in matrix form

$$\begin{align*}
0 & \geq x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} \\
q & \geq x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} \quad \text{subject to} \\
q & = x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} \\
\text{max} & \quad \max
\end{align*}$$

We have the dual of (39) can be written as

$$\begin{align*}
q & \geq x \mathbf{Y} \\
q \geq x \mathbf{Y} & \quad \text{subject to} \\
\text{max} & \quad \max
\end{align*}$$

Since $A \mathbf{Y} = q$ can be expressed as

$$0 \geq x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} \quad \text{subject to} \\
\text{max} & \quad \max$$

Consider the standard form of an LP problem:

Diagonal form of matrix

$$\begin{align*}
\text{Definition: } \text{The pair of problems (25) and (26) is called the } \\
\text{symmetric form of duality.} \\
\text{The dual problem is sometimes referred to as the primal problem. The } \\
\text{primal problem is the following:} \\
\text{subject to} \\
\text{max} & \quad \max
\end{align*}$$

$$\begin{align*}
q = x \begin{bmatrix} V^- \mathbf{Y} \end{bmatrix} & \quad \text{subject to} \\
\text{max} & \quad \max
\end{align*}$$

Consider the following LP problem:

Dual Theory
Therefore, we have proved the theorem.

This is exactly the same problem as (d) with \( x \) replaced by \( v \).

or it can be written as

\[
0 \geq a \\
\sum_{v} d_{v} \geq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

The dual of the above dual is

\[
0 \geq c \\
\sum_{v} d_{v} \leq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

But

\[
0 \leq a \\
\sum_{v} d_{v} = q_{a} \\
\text{max}
\]

where here the dual can be written as

\[
0 \leq a \\
\sum_{v} d_{v} \leq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

Dual problem (d):

PROOF. Consider the pair primal problem (d):

\[
0 \geq x \\
\sum_{v} d_{v} \geq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

The dual of the dual is equivalent to the primal.

Some Basic Theorems

Theorem 10. The dual of the dual is equivalent to the primal.

\[
0 \geq x \\
\sum_{v} d_{v} \geq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

Therefore, the dual problem is

\[
0 \leq x \\
\sum_{v} d_{v} \leq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

In matrix form it is

\[
0 \leq x \\
\sum_{v} d_{v} \leq \left( \sum_{v} a_{v} \right) d_{v} \\
\text{max}
\]

Example: The pair of problems (d) (and (28)) is called the

unconstraint form dual.
We have the following unsymmetrical dual problems:

\[ \min \{ y' \mathbf{c} : \mathbf{A}_y \mathbf{y} \leq \mathbf{b}_y, \mathbf{y} \geq 0 \} \]

\[ \max \{ x' \mathbf{b} : \mathbf{A}_x \mathbf{x} \geq \mathbf{c}, \mathbf{x} \geq 0 \} \]

The second fundamental theorem is

Thus the proof is complete.

The converse statement can be proved similarly.

\[ x = q_x(y) \]

denoted by \( x \). Then, if there exists a solution to the constraints of (d), i.e., this solution be

\[ W = \sum x_k y_k > q_x(y) \]

Theorem 1. If the feasible function value in either of and (d) is optimal, then the dual is optimal.

We now have two fundamental theorems in duality theory.

Thus \( x \) and \( y \) are optimal to (d) and (q) respectively.

\[ q_x(y) = q_y(x) = 0 \] 

This implies that

\[ x = q_x(y) \]

Combining (33) and (q) we have

where \( a \) and \( b \) are feasible solutions to (d) and (q) respectively.

We now recall the assumption that for all feasible solutions \( x \) and \( y \) to their respective problems, we have

\[ q_x(y) = 0 \] 

Therefore (33) holds for all feasible solutions \( x \) and \( y \) to their respective problems. In particular we have

We now know \( q_x(y) = 0 \) if and only if \( x = y \).

Theorem 2. If the feasible function value in either of

\[ \min \{ y' \mathbf{c} : \mathbf{A}_y \mathbf{y} \leq \mathbf{b}_y, \mathbf{y} \geq 0 \} \]

\[ \max \{ x' \mathbf{b} : \mathbf{A}_x \mathbf{x} \geq \mathbf{c}, \mathbf{x} \geq 0 \} \]

We consider the following unsymmetrical dual problem:

\[ \min \{ y' \mathbf{c} : \mathbf{A}_y \mathbf{y} \leq \mathbf{b}_y, \mathbf{y} \geq 0 \} \]

\[ \max \{ x' \mathbf{b} : \mathbf{A}_x \mathbf{x} \geq \mathbf{c}, \mathbf{x} \geq 0 \} \]
The last tableau for solving this problem is

\[
\begin{bmatrix}
  0 & 0 & 2 & 1 \\
  0 & 1 & 1 & 2 \\
\end{bmatrix}
\]

subject to

\[
x[0 \leq 3 - y - z - 1 -]
\]

The standard form is

\[
0 \leq x \\
y \leq 2x + 2z + 1x \\
z \leq 3x + 2x + 1x - 1
\]

subject to

\[
x \geq 0 \\
y \geq 0 \\
z \geq 0
\]

Example. Consider the following LP problem:

\[
x \geq 0 \\
y \geq 0 \\
z \geq 0
\]

subject to

\[
x + y + z \leq 1
\]

By the contrarily in duality we have that \( z \) is optimal. Therefore, \( z \) is a feasible optimal solution. Furthermore, \( z \) is a feasible solution. Therefore,

\[
q_{1-H}^x = q_{1-H}^y = q_{1-H}^z = q_{1-H}^x
\]

So, \( x \) is a feasible solution. Therefore,

\[
f = \sum_{j \in A} \sum_{i \in N} H_{ji} y_i
\]

subject to

\[
q_{1-H}^x \geq 0 \\
q_{1-H}^y \geq 0 \\
q_{1-H}^z \geq 0
\]

Therefore, \( x \) is an optimal feasible solution to \( q_{1-H}^x \).

Theorem 12 (Strong Duality Theorem) If any one of the primal or
\[ 0 = x^T P^2 - V_d(x) \]

we have

Since \( q^T x = 0 \), we can reverse the above process so that eventually

\[ x^T P^2 = q^T (x) \]

\[ \text{optimal, } q \text{ of } \text{the previous complementary } \text{dual } \]

By part (ii) of the previous complementary theorem, we should be

\[ x^T P^2 = q^T (x) \]

But \( V(x) \) is lower bound.

So

\[ 0 = x^T P^2 - V_d(x) \]

or

\[ 0 = x^T P^2 - V_d(x) \]

Combining these two, we have

Combine this to get the complementary slackness

Suppose (i) and (ii) are not satisfied, then we have other

where \( n \) is the number of columns of \( P \).

\[ \text{Lemma (iv)} \]

\[ 0 \leq x \Rightarrow 0 \leq x \]

\[ 0 > x \Rightarrow 0 > x \]

Theorem 2.1: Let \( x \) be a feasible solution to the primal

\[ \text{complementary slackness theorem. In this form, and} \]

In the following theorem, we provide the relationship between (and

\[ \text{Problem (d)} \]

\[ 0 \leq x \]

\[ \text{subject to} \]

\[ x \in \mathbb{R}^n \]

\[ \text{max} \]

\[ \text{Problem (e)} \]

Consider the asymmetric form of duality

\[ \text{Complementary slackness theorem} \]

is an optimal solution to the dual problem.

\[ \left[ \begin{array}{c}
1 - 1 \\
1 - 1
\end{array} \right] = \left[ \begin{array}{c}
1 - 1 \\
1 - 1
\end{array} \right] = \text{dual of } \text{problem} \]

By the above theorem,

\[ \left[ \begin{array}{c}
\varepsilon_1 - 1 \\
\varepsilon_2 - 1
\end{array} \right] = q \text{ and } \left[ \begin{array}{c}
1 - 1 \\
1 - 1
\end{array} \right] = 1 - 1 \]

Now, \[ \left[ \begin{array}{c}
\varepsilon_1 - 1 \\
\varepsilon_2 - 1
\end{array} \right] = q \}

with \[ \left[ \begin{array}{c}
1 \\
1
\end{array} \right] = 1 - 1 \]

So, the optimal solution is \( x = 0 \) and \( \varepsilon = -1 \). Note that

\[ \begin{array}{cccccc}
-1 & 1 & - & 0 & 1 & 0 \\
2 & \varepsilon - 1 & 0 & 1 & 0 & \varepsilon \\
1 & 1 & 0 & 1 & \varepsilon & \varepsilon \\
H & 0 & 0 & -3 & - & 1 \end{array} \]
□

**Proof.** Omitted.

Row vector \( a \) and \( b \) denote respectively the column vector and the coordinate vector of an arbitrary point \( x \) in \( \mathbb{R}^n \). Let \( \mathbf{y} \) be an arbitrary point of \( \mathbb{R}^n \). Then, the vector \( \mathbf{y} \) is a \( n \times n \) matrix. The following properties hold:

1. If \( \mathbf{y} \) is a \( n \times n \) matrix, then \( \mathbf{y} \) is also a \( n \times n \) matrix.
2. If \( \mathbf{y} \) is a \( n \times n \) matrix, then \( \mathbf{y} \) is also a \( n \times n \) matrix.
3. If \( \mathbf{y} \) is a \( n \times n \) matrix, then \( \mathbf{y} \) is also a \( n \times n \) matrix.

**Theorem 16.** Let \( x \) be a \( n \times n \) matrix, and \( \mathbf{y} \) be an arbitrary point of \( \mathbb{R}^n \). Then, the following properties hold:

1. If \( x \) is a \( n \times n \) matrix, then \( x \) is also a \( n \times n \) matrix.
2. If \( x \) is a \( n \times n \) matrix, then \( x \) is also a \( n \times n \) matrix.

**Proof.** Omitted.

For the symmetric form of duality we have the following:

- \( x \) is a \( n \times n \) matrix, and \( \mathbf{y} \) is an arbitrary point of \( \mathbb{R}^n \). Then \( x \) is a \( n \times n \) matrix.
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Note that the above implies that

\[
0 = \left[ x(1 - a_1 \lambda, \mathbf{y}) \right] \sum_{\alpha=0}^n
\]