Rate of Convergence: Suppose \( \{\beta_n\}^\infty_1 \) is a sequence converging to zero, and \( \{\alpha_n\}^\infty_1 \) converges to \( \alpha \). If \( \exists K > 0 \) such that
\[
|\alpha_n - \alpha| \leq K|\beta_n|
\]
for large \( n \), then we say \( \{\alpha_n\} \) converges to \( \alpha \) with rate of convergence \( O(\beta_n) \). This is normally denoted as \( \alpha_n = \alpha + O(\beta_n) \).

Example. Find the rate of convergence of
\[
\lim_{n \to \infty} [\ln(n+1) - \ln n] = 0.
\]
Solution I. Note that \( \ln(n+1) - \ln n = \ln(1 + 1/n) \). Let \( x = 1/n \). Find \( p > 0 \) such that
\[
\lim_{x \to 0} \left| \frac{\ln(1 + x)}{x^p} \right| = \text{const} \neq 0.
\]
Note that
\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x^p} = \lim_{x \to 0} \frac{1/(1 + x)}{px^{p-1}}.
\]
So, when \( p = 1 \), we have
\[
\lim_{x \to 0} \frac{1/(1 + x)}{px^{p-1}} = 1.
\]
This implies that
\[
\ln(1 + 1/n) = O\left(\frac{1}{n}\right),
\]
or \( \ln(n+1) - \ln n \) converges to zero with the rate \( O(1/n) \).
Solution II. Using the mean value theorem we have that \( \exists \xi \in (0, 1/n) \) such that
\[
|\ln(n+1) - \ln n| = |\ln(1 + 1/n)| = \left| \frac{1}{n} - \frac{1}{1 + \xi/n^2} \right| \leq \frac{1}{n}.
\]
Therefore, \( \ln(n+1) - \ln n \to 0 \) with the rate \( O(1/n) \).

Example. The sequence \( \{F_n\} \) described by \( F_0 = 1, F_1 = 1, \) and \( F_{n+2} = F_n + F_{n+1} \) for \( n \geq 0 \), is called a Fibonacci sequence. Consider \( \{x_n\} \) where \( x_{n+1} = F_{n+1}/F_n \). Show that \( \{x_n\} \) is convergent, and find its limit.
Solution. Using the definition we have
\[
x_{n+1} = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{x_n}
\]
for \( n = 1, 2, \ldots \). Let us consider
\[
g(x) = 1 + \frac{1}{x}.
\]
Clearly, \( g(x) > 1 \) for all \( x > 0 \). Also for any \( b > 2 \), \( g(b) = 1 + 1/b < 2 < b \). Therefore
\[
1 + \delta < g(x) < b, \quad \forall x \in [1 + \delta, b],
\]
where \( \delta \) is any small positive number and \( b > 2 \). This implies that \( g(x) \) has a fixed point in \([1 + \delta, b]\). Differentiating w.r.t. \( x \) gives
\[
g'(x) = -\frac{1}{x^2} \Rightarrow |g(x)| \leq \frac{1}{x^2} \leq \frac{1}{(1 + \delta)^2} < 1.
\]
Therefore, \( g \) satisfies the fixed point theorem in \([1 + \delta, b]\), and thus \( x_{n+1} = g(x_n) \) converges. Solving \( x = 1 + 1/x \) gives \( x = \frac{1}{2}(1 \pm \sqrt{5}) \). So,

\[
\lim_{n \to \infty} x_n = \frac{1 + \sqrt{5}}{2}.
\]

**Example.** Show that the sequence

\[
x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad n \geq 1
\]

converges to \( \sqrt{2} \) whenever \( x_0 > \sqrt{2} \).

Solution. Let \( g(x) = x/2 + 1/x \). Then,

\[
g'(x) = \frac{1}{2} - \frac{1}{x^2} \geq 0, \quad x \geq \sqrt{2}.
\]

So, \( g \) is increasing. Since \( g(\sqrt{2}) = \sqrt{2} \) we have that \( g(x) \geq \sqrt{2} \) for all \( x \geq \sqrt{2} \). Also, for any \( b > 2 \),

\[
g(b) = \frac{1}{2}b + \frac{1}{b} < b.
\]

Therefore, \( g(x) \in [\sqrt{2}, b] \). Differentiating gives

\[
|g'(x)| = \left| \frac{1}{2} - \frac{1}{x^2} \right| < \frac{1}{2}
\]

for \( x \geq \sqrt{2} \). Therefore, by the fixed point theorem, the sequence converges to the unique fixed point of \( g \), i.e., to \( \sqrt{2} \).

**Example.** Let \( A > 0 \) be a constant and \( g(x) = 2x - Ax^2 \). (a) Show that if the fixed point iteration converges to a nonzero limit, then the limit is \( x = 1/A \). (b) Find an interval about \( 1/A \) for which the fixed point iteration converges, provided that \( x_0 \) is in the interval.

Solution. (a). Solving \( g(x) = 2x - Ax^2 = x \) gives \( x = 0 \) or \( 1/A \). Therefore, the nonlinear solution is \( 1/A \). (b) Consider the derivative of \( g \) given by \( g'(x) = 2(1 - Ax) \). Solving \( |g'(x)| = 2|1 - Ax| \leq K < 1 \) we have

\[
\frac{1}{A}(1 - \frac{K}{2}) \leq x \leq \frac{1}{A}(1 + \frac{K}{2}).
\]

We rewrite \( g \) as \( g(x) = -A(x-1/A)^2 + 1/A \). From this we see that

\[
\max_{\frac{1}{4}(1-\frac{K}{2}) \leq x \leq \frac{1}{2}(1+\frac{K}{2})} \frac{1}{A} - \frac{K^2}{4A} = \frac{1}{A} \left( 1 - \frac{K^2}{4} \right) > \frac{1}{A} \left( 1 - \frac{K}{2} \right)
\]

It is easy to see that

\[
\frac{1}{A} - \frac{K^2}{4A} = \frac{1}{A} \left( 1 - \frac{K^2}{4} \right) > \frac{1}{A} \left( 1 - \frac{K}{2} \right)
\]

since \( |K| \leq 1 \). Therefore, \( g \in [\frac{1}{4}(1 - \frac{K}{2}), \frac{1}{2}(1 + \frac{K}{2})] \) and \( g'(x) \leq 1 < 1 \) for all \( x \in [\frac{1}{4}(1 - \frac{K}{2}), \frac{1}{2}(1 + \frac{K}{2})] \)

So, \( x_{n+1} = g(x_n) \) converges for any \( x_0 \in [\frac{1}{4}(1 - \frac{K}{2}), \frac{1}{2}(1 + \frac{K}{2})] \).

**Example.** Show that for any positive integer \( k \), \( x_k = 1/n^k \) converges to zero linearly.

Solution. Let \( \alpha \) be a positive number and consider

\[
\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = \lim_{n \to \infty} \frac{1/(n+1)^k}{1/(n^k)^\alpha} = \left( \frac{n^\alpha}{n+1} \right)^k.
\]

When \( \alpha = 1 \), we have \( \lim_{n \to \infty} (n/(n+1))^k = 1 \). So, it converges to zero linearly with asymptotic error constant 1.
**Example.** Show that \( x_n = 10^{-2^n} \) converges to zero quadratically.

**Solution.** Consider the following limit
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{10^{-2^{n+1}}}{10^{-2^n}} = \lim_{n \to \infty} 10^{-2^{n-2^n+1}}.
\]
Therefore, when \( \alpha = 2 \), we have \( \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 1 \). This implies that \( x_n \) converges to zero quadratically with the asymptotic constant 1.

**Worksheet 1, Q.6**

**Solution:** We write \( x = [3(x^2 - 1)]^{1/4} \) and let \( g(x) = [3(x^2 - 1)]^{1/4} = 3^{1/4}(x^2 + 1)^{1/4} \). Then,
\[
g'(x) = 3^{1/4} \cdot \frac{1}{4} (x^2 + 1)^{-3/4} \cdot 2x \geq 0, \quad \forall x \geq 1.
\]
So, \( g \) is increasing. But \( g(1) = 6^{1/4} > 1 \) and \( g(2) = 15^{1/5} < 2 \), so \( g(x) \in [1, 2] \) when \( x \in [1, 2] \). Also, \( |g'(x)| < \frac{3^{1/4}}{2^{3/4}} < 1 \) for all \( x \in [1, 2] \). Therefore, the fixed point iteration \( x_{n+1} = g(x_n) \), with \( x_0 = 1 \) converges to a solution of \( x^4 - 3x^3 - 3 = 0 \).

**Worksheet 1, Q.7(a)**

**Solution:** \( x = \sqrt{e^x/\sqrt{3}} =: g(x) \). So,
\[
g'(x) = \frac{1}{\sqrt{3}} \cdot e^{x/2} \cdot \frac{1}{2} > 0.
\]
Thus, \( g \) is increasing in \([0, \infty)\). Also, we have
\[
|g'(x)| \leq e^{1/2} / (2 \cdot 3^{1/2}) < 1, \quad \forall x \in [0, 1].
\]
Furthermore, \( g(0) = 1/\sqrt{3} > 0 \) and \( g(1) = \sqrt{e/3} < 1 \). Therefore, \( x_{n+1} = g(x_n) \) converges to the fixed point of \( g \) in \([0, 1]\) if \( x_0 \in [0, 1] \). We choose \( x_0 = 0 \). Then \( x_1 = 0.77035, x_2 = 0.84872, ..., x_8 = 0.90948, ... \).

**Worksheet 1, Q.7(b)**

**Solution:** Consider \([0, \pi/3]\) and choose \( x = \cos x =: g(x) \). Then, \( g' = -\sin x \leq 0 \) for \( x \in [0, \pi/2] \), or \( g \) is decreasing. \( g(0) = 1 \) and \( g(\pi/3) = 0 \). So, \( g \in [0, \pi/3] \). Furthermore, it is easy to check that \( |g'| \leq \sin(\pi/3) < 1 \). Therefore, the fixed point iteration converges to the unique fixed point in \([0, \pi/3]\), which is about 0.739085.