Worksheet 2, Q.4(a)
Solution: \( x_0 = 0, x_1 = 0.6, x_2 = 0.9 \). From these,

\[
L_{2,0} = \frac{(x - 0.6)(x - 0.9)}{0.54}, \quad L_{2,1} = \frac{(x - 0)(x - 0.9)}{-0.18}, \quad L_{2,2} = \frac{(x - 0)(x - 0.6)}{0.27}.
\]

Therefore,

\[
P_2(x) = \cos 0L_{2,0}(x) + \cos 0.6L_{2,1}(x) + \cos 0.9L_{2,2}(x).
\]

The error is \( |P_2(0.45) - \cos 0.45| = 0.00235 \). For the case of 1st order approximation we have

\[
P_1(x) = \cos 0\frac{x - 0.6}{-0.6} + \cos 0.6\frac{x}{0.6}.
\]

So, \( |P_1(0.45) - \cos 0.45| = 0.03145 \).

Worksheet 2, Q.6
Solution: Let \( P_3(x) = a + bx + cx^2 + dx^3 \). Then,

\[
P_3(0) = a = 0
\]
\[
P_3(0.5) = 0.5b + 0.5^2c + 0.5^3d = y
\]
\[
P_3(1) = b + c + d = 3
\]
\[
P_3(2) = 2b + 4c + 8d = 2.
\]

Now, from \( d = 6 \) we have

\[
\begin{pmatrix}
0.5 & 0.5^2 & -1 \\
1 & 1 & 0 \\
2 & 4 & 0
\end{pmatrix}
\begin{pmatrix}
b \\
c \\
y
\end{pmatrix}
= \begin{pmatrix}
-0.5^3 \cdot 6 \\
3 - 6 \\
2 - 8 \cdot 6
\end{pmatrix}
\]

Solving this gives \( b = 17, c = -20 \) and \( y = 4.25 \).

Worksheet 2, Q.16
Solution: There are 3 parameters to determine. We force the formula to be exact for \( f = 1, x \) and \( x^2 \), giving

\[
c_1 + c_2 = 1
\]
\[
c_1 \cdot 0 + c_2 \cdot 1 = \frac{1}{2}
\]
\[
c_1 + c_2x_1^2 = \frac{1}{3}.
\]
Solving this system gives
\[ x_1 = \frac{\sqrt{13} - 2}{3}, \quad c_2 = \frac{1}{2x_1}, \quad c_1 = 1 - c_2. \]

(Note there are two solutions for \( x_1 \), but we choose the positive one.)

**Worksheet 2, Q.18**

**Solution:** The Simpson’s rule on \([x_0, x_2]\) has the estimate
\[
\int_{x_0}^{x_2} f(x)dx - S(f, x_0, x_2) = -\frac{h^5}{90} f^{(4)}(\xi)
\]
for a \( \xi \in (x_0, x_2) \), where \( S(f, x_0, x_2) = \frac{b}{3}[f_0 + 4f_1 + f_2]. \) If \([a, b]\) is divided into \( n \) suintervals, where \( n \) is an even integer, then,
\[
\int_{x_0}^{x_n} f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx
\]
\[
= \sum_{j=1}^{n/2} \left[ S(f, x_{2j-2}, x_{2j}) - \frac{h^5}{90} f^{(4)}(\xi_j) \right]
\]
\[
= S_{comp}(f, a, b) - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)
\]
\[
= S_{comp}(f, a, b) - \frac{h^4}{180} \sum_{j=1}^{n/2} (2h) f^{(4)}(\xi_j)
\]
\[
\approx S_{comp}(f, a, b) - \frac{h^4}{180} \int_a^b f^{(4)}(x)dx
\]
\[
= S_{comp}(f, a, b) - \frac{h^4}{180} \left[ f^{(3)}(b) - f^{(3)}(a) \right].
\]

Therefore, the error is the last term in the above.