Abstract

Previous work of the authors has shown that an important class of locally \((G, 2)\)-arc transitive graphs are those for which \(G\) acts faithfully and quasiprimively on each of its two orbits on vertices. In this paper we give a complete classification in the case where the two quasiprimitive actions of \(G\) are of different types. The graphs obtained have amalgams previously unknown to the authors and involve both an almost simple 2-transitive action and an affine 2-transitive action on the neighbourhoods of vertices.

Key words: locally \(s\)-arc transitive graphs, quasiprimitive

1 Introduction

An \(s\text{-}arc\) in a graph \(\Gamma\) is an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that each \(v_i\) is adjacent to \(v_{i+1}\) while \(v_i \neq v_{i+2}\). Given \(G \leq \text{Aut}(\Gamma)\) we say that \(\Gamma\) is \(locally (G, s)\)-arc transitive if for each vertex \(v\), the stabiliser \(G_v\) acts transitively on the set of \(s\)-arcs starting at \(v\). Provided that all vertices of \(\Gamma\)

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have valency at least three, a locally \((G, s)\)-arc transitive graph is also locally \((G, s - 1)\)-arc transitive. Throughout this paper we assume that this is the case for \(\Gamma\). If \(G\) is not vertex transitive, then a locally \((G, 2)\)-arc transitive graph is a bipartite graph and the two parts \(\Delta_1, \Delta_2\) of the bipartition are \(G\)-orbits.

The study of locally \((G, s)\)-arc transitive graphs goes back to Tutte \cite{15,16} who showed that if \(\Gamma\) has valency three and \(G\) is vertex transitive then \(s \leq 5\). This was extended by Weiss \cite{17} who showed that if \(\Gamma\) has valency at least three and \(G\) is vertex transitive then \(s \leq 7\). Stellmacher \cite{14} has proved that if \(G\) is vertex intransitive and all vertices have valency at least three then \(s \leq 9\). This inequality is sharp as demonstrated by the incidence graphs of the classical generalised octagons associated with the simple groups \(^2F_4(q)\).

In \cite{4}, the authors initiated a global analysis of locally \((G, s)\)-arc transitive graphs for which \(G\) has two orbits on vertices. This extended the work of the third author \cite{11} in the vertex transitive case. It was shown that the important graphs to study are those where \(G\) acts faithfully on both \(G\)-orbits and quasiprimitively on at least one. (A transitive permutation group \(G\) is quasiprimitive if every nontrivial normal subgroup is transitive.) In the case where \(G\) acts quasiprimitively on both orbits, the possible quasiprimitive types were studied and it was shown that either the two quasiprimitive actions are of the same type, or one is of Simple Diagonal type and the other is of Product Action type \cite[Theorem 1.2]{4}. (A description of these two actions will be given in Subsection 2.1.) We say that graphs in the latter case are of \(\{SD, PA\}\)-type. The first example of a locally 2-arc transitive graph of \(\{SD, PA\}\)-type was given in \cite[Example 4.1]{4}. Five infinite families were provided in \cite[Example 4.2]{6} while an infinite family of locally 5-arc transitive graphs of \(\{SD, PA\}\)-type was given in \cite{5}.

In this paper we give a general construction (Construction 3.3) of locally \((G, 2)\)-arc transitive graphs of \(\{SD, PA\}\)-type which proves the following theorem.

\textbf{Theorem 1.1} For each simple group \(T = \text{PSL}(n, q)\) \((n \geq 2)\), \(\text{PSU}(3, q)\), \(\text{Ree}(q)’\) or \(\text{Sz}(q)\), and positive integer \(d\) such that, if \(T = \text{PSL}(n, q)\) with \(n \geq 3\), then \(d = 1\), there exists a locally \((G, 2)\)-arc transitive graph \(\Gamma\) with the following properties.

\begin{enumerate}
  \item \(\text{soc}(G) = T^{q^d}\)
  \item \(G\) induces \(\text{AGL}(d, q)\) on the set of \(q^d\) simple direct factors of \(\text{soc}(G)\).
  \item \(G\) has two orbits \(\Delta_1\) and \(\Delta_2\) on \(V\) such that \(G\) acts quasiprimittively of type SD on \(\Delta_1\) and quasiprimittively of type PA on \(\Delta_2\).
  \item For \(v \in \Delta_1\), we have \(G_v^{\Gamma(v)} = \text{PGL}(n, q), \text{PGU}(3, q), \text{Aut}(\text{Ree}(q))\) or \(\text{Aut}(\text{Sz}(q))\), and \(|\Gamma(v)| = (q^n - 1)/(q - 1), q^3 + 1, q^3 + 1\) and \(q^2 + 1\) respectively.
\end{enumerate}
(5) For $w \in \Delta_2$, we have $G^\Gamma_{w} = \Gamma L(d, q)$ and $|\Gamma(w)| = q^d$.

We construct the group $G$ in Section 3.1 while $G_v$ is the group $L$ constructed in Section 3.1 and $G_w$ is the group $R$ constructed in Section 3.2. The amalgams $(G_v, G_w, G_{vw})$ were previously unknown to the authors.

In Construction 3.10 we find that certain quotients of the graphs yielded by Construction 3.3 are also of $\{\text{SD}, \text{PA}\}$-type. We prove that all locally $(G, 2)$-arc transitive graphs of $\{\text{SD}, \text{PA}\}$-type can be constructed in this way.

**Theorem 1.2** Let $\Gamma$ be a locally $(G, 2)$-arc transitive graph of $\{\text{SD}, \text{PA}\}$-type. Then $\Gamma$ is isomorphic to a graph arising from Construction 3.3 or to a normal quotient of such a graph as in Construction 3.10.

An important tool in the proof of this theorem is the classification in [3, Theorem 1.1] of all codes $C$ containing the constant code $E$ which have a weight preserving group of automorphisms $H$ which acts transitively on the nontrivial cosets of $E$ in $C$. One consequence of Theorem 1.2 is that we can bound the degree of local $s$-arc transitivity for graphs of $\{\text{SD}, \text{PA}\}$-type.

**Corollary 1.3** Let $\Gamma$ be a locally $(G, s)$-arc transitive graph of $\{\text{SD}, \text{PA}\}$-type which is not locally $(G, s + 1)$-arc transitive. Then either $s = 5$ or $s \leq 3$. Moreover, there exists a locally $(G, 5)$-arc transitive graph of $\{\text{SD}, \text{PA}\}$-type.

This paper is set out as follows. In Section 2 we give the necessary background. We outline the global analysis of locally $s$-arc transitive graphs in Subsection 2.1 and describe the two quasiprimitive types crucial to this paper. In Subsection 2.2 we give an overview of constructing graphs via cosets and in Subsection 2.3 we collate some required information about 2-transitive groups. Section 3 provides the two general constructions of locally $(G, 2)$-arc transitive graphs of $\{\text{SD}, \text{PA}\}$-type and we show that the constructions produce graphs which are at most locally $(G, 5)$-arc transitive. Finally, in Section 4 we prove that all locally 2-arc transitive graphs of $\{\text{SD}, \text{PA}\}$-type can be obtained from Construction 3.3 or Construction 3.10.

2 Preliminaries

2.1 Quotient graphs and quasiprimitive types

First we give an outline of the global analysis initiated in [4] to which the reader is referred for the details. Let $\Gamma$ be a locally $(G, 2)$-arc transitive graph such that $G$ has two orbits $\Delta_1$ and $\Delta_2$ on vertices and each vertex has valency at least three. Suppose that $G$ has a nontrivial normal subgroup $N$ which acts
Let $\Gamma_N$ be the quotient graph of $\Gamma$ whose vertex set is the set of $N$-orbits on $VT$ such that two orbits $B_1$ and $B_2$ are adjacent if there exist $v_1 \in B_1$ and $v_2 \in B_2$ such that $v_1$ and $v_2$ are adjacent in $\Gamma$. Then $\Gamma_N$ is locally $(G/N, s)$-arc transitive such that $G/N$ has two orbits on $VT_N$. Moreover, $\Gamma$ is a cover of $\Gamma_N$, that is, if $B_1$ and $B_2$ are adjacent in $\Gamma_N$ then for all $v \in B_1$ there exists a unique $u \in B_2$ which is adjacent to $v$ in $\Gamma$ (see [4, Theorem 1.1]). If $G/N$ is not faithful on one of its orbits then $\Gamma_N$ is a complete bipartite graph. Also, by the maximality of $N$, $G/N$ acts quasiprimtively on at least one of its two orbits on the vertex set of $\Gamma_N$. Hence the “basic” graphs to study are those for which $G$ acts faithfully on both orbits and quasiprimitively on at least one.

In [11] an O’Nan–Scott-like theorem for the structure of quasiprimitive groups was given by the third author. We follow the subdivision into 8 disjoint types and the notation given in [12]. An investigation of the possible quasiprimitive types for $G$ for a locally $(G, 2)$-arc transitive graph was undertaken in [4] and it was proved that in the case where $G$ is quasiprimitive on both orbits, either $G$ is of the same type on both orbits and only 4 of the 8 types occur, or one action is of Simple Diagonal (SD) type and the other is of Product Action (PA) type. Moreover, it is not possible to have $G$ acting with SD type on both orbits. We now give a more explicit description of these two important types.

Let $G$ be a quasiprimitive group acting on a set $\Omega$ and suppose that $G$ has a unique minimal normal subgroup $N \cong T^k$ for some finite nonabelian simple group $T$ and positive integer $k \geq 2$. Then $G$ is quasiprimitive of type SD if and only if for all $\alpha \in \Omega$, $N_{\alpha} \cong T$ and $G$ transitively permutes the $k$ simple direct factors of $N$. We now give a more explicit description of actions of SD type. Let $N = T^k$ act on the set $\Omega$ of right cosets of

$$N_{\alpha} = \{(t, t, \ldots, t) \mid t \in T\}$$

in $N$. Then $\{(t_1, t_2, \ldots, t_{k-1}, 1) \mid t_i \in T\}$ is a set of coset representatives for $N_{\alpha}$ in $N$ and so we can identify $\Omega$ with $T^{k-1}$. Each element $\tau \in \text{Aut}(T)$ acts on $\Omega$ via

$$(t_1, t_2, \ldots, t_{k-1}, 1)^\tau = (t_1^\tau, t_2^\tau, \ldots, t_{k-1}^\tau, 1)$$

and note that if $\tau \in \text{Inn}(T)$ and is conjugation by the element $t$, then the action of $\tau$ is induced by $(t, t, \ldots, t) \in N$. Each $\sigma \in S_k$ also acts on $\Omega$ via

$$(t_1, t_2, \ldots, t_{k-1}, 1)^\sigma = (t_{k\sigma-1}^{-1} t_{1\sigma-1}^{-1}, \ldots, t_{(k-1)\sigma-1}^{-1} t_{(k-1)\sigma-1}^{-1}, 1)$$

where $t_k = 1$. Let $W = \langle N, \text{Aut}(T), S_k \rangle$. Then $W \cong T^k, \langle \text{Out}(T) \times S_k \rangle$, $W$ is the normaliser of $N$ in $\text{Sym}(\Omega)$ and the stabiliser in $W$ of the coset $N_{\alpha}$ is $\text{Aut}(T) \times S_k$. Each quasiprimitive group of type SD with socle $T^k$ is equivalent to a group $G$ acting on $\Omega$ such that $N < G < W$ and $G$ acts transitively by conjugation on the $k$ simple direct factors of $N$. Such a group $G$ is primitive
of type SD if and only if \( G \) acts primitively on the \( k \) simple direct factors of \( N \).

A quasiprimitive group \( G \) with a unique minimal normal subgroup \( N = T^k \) is of type PA if and only if \( N_\alpha \neq 1 \) and is not isomorphic to \( T^l \) for any \( l \leq k \). For a quasiprimitive group of type PA on a set \( \Omega \), there exists a partition \( \mathcal{P} \) of \( \Omega \) (possibly with blocks of size 1) such that \( G \) acts faithfully on \( \mathcal{P} \) and preserves a product structure. Thus \( G \) is isomorphic to a subgroup of \( H \wr S_k \), where \( H \) is an almost simple group with socle \( T \). Furthermore, there exists \( R < T \) such that \( N_\alpha \) is a subdirect subgroup of \( R^k \), that is, \( N_\alpha \) projects onto \( R \) in each coordinate. Moreover, for the block \( B \) of \( \mathcal{P} \) containing \( \alpha \), we have \( N_B = R^k \).

2.2 Coset graphs

Construction 3.3 defines a graph in terms of the cosets of subgroups of a group \( G \). We collect a few results concerning coset graphs here. See for example [4, Lemma 3.7] for proofs. For a subgroup \( H \) of a group \( G \), we denote \([G : H] = \{Hg \mid g \in G\}\) and the coset action of \( G \) on \([G : H]\) is right multiplication. We say that \( H \) is core-free in \( G \) if \( H \) contains no nontrivial normal subgroups of \( G \). For proper subgroups \( L, R \) of a group \( G \), \( \text{Cos}(G, L, R) \) is the graph with vertex set the disjoint union of \([G : L]\) and \([G : R]\) with \( Lx, Ry \) adjacent if and only if \( xy^{-1} \in LR \). Note that the condition \( xy^{-1} \in LR \) is equivalent to \( Lx \cap Ry \neq \emptyset \).

**Lemma 2.1** For a group \( G \) with subgroups \( L, R < G \) such that \( L \cap R \) is core-free in \( G \), the graph \( \Gamma = \text{Cos}(G, L, R) \) has the following properties:

1. \( \Gamma \) is connected if and only if \( \langle L, R \rangle = G \);
2. \( G \leq \text{Aut}(\Gamma) \), \( \Gamma \) is \( G \)-edge transitive and \( G \) has two orbits \([G : L]\) and \([G : R]\) on vertices.
3. \( G \) acts faithfully on both \([G : L]\) and \([G : R]\) if and only if both \( L \) and \( R \) are core-free.
4. \( \Gamma \) is locally \((G, 2)\)-arc transitive if and only if both the \( L \)-coset action on \([L : L \cap R]\) and the \( R \)-coset action on \([R : L \cap R]\) are \( 2 \)-transitive.

Conversely, if \( \Gamma \) is a \( G \)-edge transitive but not \( G \)-vertex transitive graph, and \( v \) and \( w \) are adjacent vertices then \( \Gamma \cong \text{Cos}(G, G_v, G_w) \).

2.3 2-transitive groups

We will require the following result about 2-transitive permutation groups which follows from the classification of all 2-transitive groups, see for example
Table 1

| $T$                  | $H$                           | $|\Omega|$ | $|M|$ |
|----------------------|-------------------------------|------------|------|
| PSL$(2,q)$, for $q \geq 4$ | $[g] \rtimes C_{(q-1)/(2q-1)}$ | $q+1$      | $q$  |
| PSL$(n,q)$, for $n \geq 3$ | $[g_{n-1}] \rtimes (C_{q-1} \circ \text{SL}(n-1,q)) \cdot C_{(q-1,n-1)}$ | $q^{n-1}/q-1$ | $q^{n-1}$ |
| PSU$(3,q)$, for $q \geq 3$ | $[g^3] \rtimes C_{(q^2-1)/(3,q+1)}$ | $q^3+1$    | $q$  |
| $Ree(q)$, $q = 3^{2m+1} \geq 27$ | $[g^3] \rtimes C_{q-1}$ | $q^3+1$    | $q$  |
| $Sz(q)$, $q = 2^{2m+1} \geq 8$ | $[g^2] \rtimes C_{q-1}$ | $q^2+1$    | $q$  |

[2]. We use $[n]$ to denote a group of order $n$.

**Theorem 2.2** Let $T$ be the socle of an almost simple group $A$ which acts 2-transitively on a set $\Omega$. Suppose that $H = T - T_\alpha$ for some $\alpha \in \Omega$ and that $H$ is not almost simple. Then $T$, $H$ and $|\Omega|$ are given in Table 1. Furthermore, $H$ has a unique minimal normal subgroup $M$ and the order of $M$ is given in Table 1.

We make several observations.

**Remark 2.3**

(1) $T$ is 2-transitive except in the case where $T = PSL(2,8)$ and $|\Omega| = 28$. Here $A = PGL(2,8) \cong Ree(3)$ is 2-transitive on 28 points and $T = A'$. From now on we include this case with the Ree groups.

(2) $M$ is isomorphic to the additive group of the field $GF(q)$, except when $T = PSL(n,q)$ with $n \geq 3$. (Here we regard $PSL(2,8)$ in line 5 as $Ree(3)'$.) In this exceptional case, $M$ is isomorphic to the additive group of an $(n-1)$-dimensional vector space over $GF(q)$.

(3) $H$ induces $GF(q)$-linear automorphisms of $M$.

(4) $H$ acts transitively by conjugation on the nontrivial elements of $M$ except in the case where $T = PSL(2,q)$ for $q$ odd. In this case, if we let $A = PGL(2,q)$ then $A_\alpha$ acts transitively on $M \setminus \{1\}$.

(5) If $|M| = q^m$ and $B = N_{\text{Sym}(|\Omega|)}(T)$, then $M$ is the unique minimal normal subgroup of $B_\alpha$ and $B_\alpha$ induces $\Gamma L(m,q)$ on $M$.

We also collect the following information in the case where $T \neq PSL(n,q)$ with $n \geq 3$, that is, for the rank one groups of Lie type.

**Remark 2.4** Let $A = PGL(2,q)$, $Ree(q)$, $Sz(q)$ or $PGU(3,q)$ and $\alpha \in \Omega$.

(1) Then $A_\alpha = O_p(A_\alpha) \rtimes P$, where $q = p^e$, $O_p(A_\alpha)$ is the largest normal
$p$-subgroup of $A_\alpha$ and $P = A_{\alpha, \alpha_2}$ for some $\alpha_2 \in \Omega \setminus \{\alpha\}$. Furthermore, the subgroup $M$ of Theorem 2.2 is the centre of $O_p(A_\alpha)$ and $O_p(A_\alpha)$ acts regularly on $\Omega \setminus \{\alpha\}$.

(2) In each case $|M| = q$ and $P$ induces $GF(q)$-multiplication on $M$. Moreover, $|P| = q - 1$, except for $A = PGU(3, q)$, in which case $|P| = q^2 - 1$ and $P$ has a subgroup of order $q + 1$ which acts trivially on $M$. In the PGU(3, $q$) case, $P$ is isomorphic to the multiplicative group of $GF(q^2)$ and the action of $P$ on $M$ is given by $\lambda : x \mapsto \lambda^q x$, for all $x \in M$. See for example [10, Lemma 1.11(ii)].

(3) Let $A \neq PGU(3, q)$ and $B = N_{\text{Sym}(\Omega)}(T)$. Let $\phi$ be the Frobenius automorphism of $GF(q)$, that is, $\phi$ raises each field element to its $p^{\text{th}}$ power. Then $\phi$ defines an automorphism of $A$, and $B = A \rtimes \langle \phi \rangle$. Moreover, since $A$ is 2-transitive on $\Omega$ we can choose $\alpha, \alpha_2 \in \Omega$ such that $B_{\alpha, \alpha_2} = P \rtimes \langle \phi \rangle$. The group automorphism $\phi$ induces the field automorphism $\phi$ on $M$.

(4) Let $A = PGU(3, q)$ and $B = N_{\text{Sym}(\Omega)}(T)$. Let $\varphi$ be the Frobenius automorphism of $GF(q^2)$, that is $\varphi$ raises each field element to its $p^{\text{th}}$ power. (Note that $GU(3, q)$ consists of matrices whose elements lie in $GF(q^2)$.) Then $\varphi$ defines an automorphism of $A$, and $B = A \rtimes \langle \varphi \rangle$. Moreover, since $A$ is 2-transitive on $\Omega$ we can choose $\alpha, \alpha_2 \in \Omega$ such that $B_{\alpha, \alpha_2} = P \rtimes \langle \varphi \rangle$. Note that we still have $|M| = GF(q)$. Looking at the matrix representation for $M$ given in [10] we see that $\varphi^e$, where $q = p^e$, induces multiplication by $-1$ on $M$.

We also have the following lemma concerning subgroups of $AGL(1, q)$.

Lemma 2.5 Let $K$ be a subgroup of $AGL(1, q)$ whose order is divisible by $q(q - 1)$. Then $K$ acts transitively on $GF(q)$.

PROOF. Let $q = p^e$ where $p$ is prime and let $N$ be the unique minimal normal subgroup of $AGL(1, q)$. Then $N \cong C_p^e$ and note that $|AGL(1, q)| = q(q - 1)e$. If $e = 1$ then $K = AGL(1, q)$ and so the result holds. Thus we assume that $e \geq 2$.

Suppose that $(p, e) \neq (2, 6)$ and that if $e = 2$ then $p$ is not a Mersenne prime. Let $r$ be a primitive prime divisor of $p^e - 1$. Such an $r$ exists by [18]. Then $K$ has an element $g$ of order $r$, and as $r$ is coprime to $e$ it follows that $g \in AGL(1, q)$. Then as $q$ is coprime to $r$ (conjugating $K$ by an element of $AGL(d, q)$ if necessary) we may assume that $g \in GL(1, q)$. Such a $g$ normalises no proper nontrivial subgroup of $N$, and so $K \cap N = 1$ or $N \leq K$. The first is not possible as $e < q$ and $q$ divides $|K|$ and so $N \leq K$. As $N$ acts transitively on $GF(q)$ it follows that so does $K$.

Suppose now that $e = 2$ and $p$ is a Mersenne prime. Then $q$ is odd and $N$ is the unique subgroup of $AGL(1, q)$ of order $q$. Hence $N \leq K$ and again we
have that $K$ is transitive. This leaves us to consider the case where $p = 2$ and $e = 6$. As $2^6$ divides $|K|$ it follows that $|K \cap N| \geq 2^5$. However, $K$ contains an element $g$ of order 7 which we may assume belongs to $\text{GL}(1, 2^6)$. Such an element does not normalise an index 2 subgroup of $N$ and so $N \leq K$ and we are done.

3 Construction

In this section we give a general method for constructing locally $(G, 2)$-arc transitive graphs of type \{SD, PA\}. We start with a general construction where $\text{soc}(G) = T^q^d$ for $T = \text{PSL}(n, q)$, $\text{PGU}(3, q)$, $\text{Ree}(q)'$ or $Sz(q)$, and $d$ a positive integer such that if $T = \text{PGL}(n, q)$ for $n \geq 3$ we have $d = 1$. The group induced by $G$ by conjugation on the set of simple direct factors of $\text{soc}(G)$ is $\text{AGL}(d, q)$. Moreover, if $g \in G$ induces an element of $\text{AGL}(d, q)$ on coordinates whose associated field automorphism is nontrivial, then it also induces a corresponding field automorphism of $T$ on each direct factor. We construct two subgroups $L$ and $R$ of $G$ with $L$ corresponding to constant functions $f : GF(q)^d \to T$ and $R$ related to certain affine functions with domain $GF(q)^d$. Then for every subgroup $K \leq \text{AGL}(d, q)$ whose projection onto $\Gamma \text{L}(d, q)$ is transitive on the set of 1-spaces of $GF(q)^d$, we find a subgroup $G_K$ of $G$ which induces $K$ on the simple direct factors of $\text{soc}(G)$. If $K$ is not transitive on $GF(q)^d$ then $G_K$ has a normal subgroup which is intransitive on each part of the bipartition and the quotient graph with respect to this normal subgroup is also of \{SD, PA\} type. We now proceed with the details so that we can define $G$, $L$ and $R$ precisely.

3.1 The groups $G$ and $L$

Let $A$ be one of $\text{PGL}(n, q)$, $\text{PGU}(3, q)$, $\text{Ree}(q)$ or $Sz(q)$ so that $A$ is an almost simple group with socle $T$. For each choice of $A$, there exists a set $\Omega$ such that $A$ acts 2-transitively on $\Omega$ and for $\alpha \in \Omega$, $A_\alpha$ is not almost simple. Up to permutational isomorphism, the choice of $\Omega$ is unique. Note that there are two possible $A$ with the same socle $T = \text{PSL}(2, 8)$, one being $\text{PGL}(2, 8)$ and the other $\text{Ree}(3) \cong \text{PGL}(2, 8)$.

Let $d$ be a positive integer with the added restriction that if $A = \text{PGL}(n, q)$ for $n \geq 3$, then $d = 1$. We define the group

$$F = \{ f : GF(q)^d \to A \}$$

with multiplication defined pointwise, that is, $(fg)(a) = f(a)g(a)$. Then $F \cong A^{q^d}$. We take $GF(q)^d$ to consist of column vectors and sometimes regard each
Let \( f \in F \) as the \( q^d \)-tuple given by the evaluation of \( f \) at each element \( a \in \text{GF}(q)^d \). Note that \( N := \text{soc}(F) = \{ f : \text{GF}(q)^d \to T \} \), where \( T = \text{soc}(A) \). For each \( h \in A \) we let \( f_h \in F \) denote the constant function with value \( h \).

Each \( g \in AGL(d, q) \) defines an automorphism \( \sigma_g \) of \( F \) via the action

\[
f_{\sigma_g}(a) = f(a^{q^{-1}})
\]

for all \( a \in \text{GF}(q)^d \). Let \( \phi \) be the Frobenius automorphism of \( \text{GF}(q) \). Then \( \phi \) defines a semilinear map on \( \text{GF}(q)^d \) and there is an associated automorphism \( \sigma_{\phi} \) of \( F \).

When \( A = \text{PGL}(d, q) \), \( \text{Ree}(q) \) or \( \text{Sz}(q) \), the Frobenius automorphism \( \phi \) defines an automorphism of \( A \), which we will also denote by \( \phi \). Note that our notation does not distinguish between the field automorphism \( \phi \), the related automorphism of \( A \) and the semilinear map induced on \( \text{GF}(q)^d \). The meaning should be clear from the context.

For each integer \( i \), we define the constant map

\[
f_{\phi^i} : \text{GF}(q)^d \to \text{Aut}(A)
\]

\[
a \mapsto \phi^i.
\]

Then \( (f_{\phi})^{-1} = f_{\phi^{-1}} \) and \( f_{\phi} \) defines an automorphism of \( F \) by

\[
f_{f_{\phi}}(a) = (f(a))^\phi.
\]

Let \( \rho = f_{\phi}\sigma_{\phi} \), where multiplication is composition in \( \text{Aut}(F) \). Then for all \( f \in F \), \( g \in AGL(d, q) \), and \( a \in \text{GF}(q)^d \),

\[
f_{\rho^{-1}\sigma_{\phi}}(a) = (f_{\rho^{-1}\sigma_{\phi}}(a^{\phi^{-1}}))_{\phi}
\]

\[
= (f_{\rho^{-1}}((a^{\phi^{-1}})^{g^{-1}}))_{\phi}
\]

\[
= f(a^{\phi^{-1}g^{-1}})_{\phi}
\]

\[
= f_{\rho^{-1}g_{\phi}}(a)
\]

and so \( \rho^{-1}\sigma_{\phi}g = \sigma_{\phi^{-1}g_{\phi}} \). Let

\[
K_0 = \langle \rho, \sigma_g \mid g \in AGL(d, q) \rangle \leq \text{Aut}(F). \tag{3.2}
\]

Then

\[
\Phi : K_0 \to AGL(d, q)
\]

\[
\sigma_g \mapsto g
\]

\[
\rho \mapsto \phi
\]

is an isomorphism from \( K_0 \) to \( AGL(d, q) \). Note that \( K_0 \) induces the group \( AGL(d, q) \) on the \( q^d \) simple direct factors of \( N \).
Field automorphisms of PGU(3, q) arise from field automorphisms of GF(q^2) and so we need to treat this case slightly differently. Let \( \varphi \) be the Frobenius automorphism of GF(q^2). Then \( \varphi \) defines a field automorphism of \( A = \text{PGU}(3, q) \) which we also denote by \( \varphi \). For each integer \( i \), let \( f_\varphi^i \) be the constant function

\[
    f_\varphi^i : \text{GF}(q)^d \to \text{Aut}(A)
    \quad a \mapsto \varphi^i.
\]

Then \((f_\varphi)^{-1} = f_{\varphi^{-1}}\) and \( f_\varphi \) defines an automorphism of \( F \) by

\[
    f^{f_\varphi}(a) = (f(a))^\varphi.
\]

Furthermore, \( \varphi \) fixes setwise the subfield \( \text{GF}(q) \) of \( \text{GF}(q^2) \) and induces on \( \text{GF}(q) \) the field automorphism \( \phi \). Let \( \rho_u = f_\varphi \sigma_\phi \), where multiplication is composition in \( \text{Aut}(F) \). Then for all \( f \in F \), \( g \in \text{AGL}(d, q) \), and \( a \in \text{GF}(q)^d \),

\[
    f^{\rho_u^{-1} \sigma g \rho_u}(a) = (f^{\rho_u^{-1} \sigma_g (a^{\phi^{-1}})})^\varphi
    = (f^{\rho_u^{-1} ((a^{\phi^{-1}})g^{-1})})^\varphi
    = f(a^{\phi^{-1}g^{-1}\phi})
    = f^{\sigma_{\phi^{-1}g\phi}}(a)
\]

and so \( \rho_u^{-1} \sigma \rho_u = \sigma_{\phi^{-1}g\phi} \). Let

\[
    K_u = \langle \rho_u, \sigma_g \mid g \in \text{AGL}(d, q) \rangle \leq \text{Aut}(F) \tag{3.4}
\]

Then

\[
    \Phi_u : K_u \to \text{AGL}(d, q)
    \quad \sigma_g \mapsto g \tag{3.5}
    \quad \rho_u \mapsto \phi
\]

is a homomorphism from \( K_u \) onto \( \text{AGL}(d, q) \) with kernel \( \langle \rho_u^e \rangle \), where \( q = p^e \). Note that \( \rho_u^e = f_\varphi^e \) and so \( K_u \cong 2, \text{AGL}(d, q) \) where the extension is split if and only if \( e \) is odd. Moreover, \( K_u \) induces \( \text{AGL}(d, q) \) on the \( q^d \) simple direct factors of \( N \).

We are now in a position to define \( G \) and \( L \). Note that when \( A \neq \text{PGU}(3, q) \), the group \( K_0 \) of automorphism of \( F \) normalises the subgroups \( N \) and \( \{ f_h \mid h \in A \} \), while when \( A = \text{PGU}(3, q) \), the subgroups \( N \) and \( \{ f_h \mid h \in A \} \) are normalised by \( K_u \). Thus when \( A \neq \text{PGU}(3, q) \) we let

\[
    G = \langle N, f_h \mid h \in A \rangle \rtimes K_0 \tag{3.6}
\]

and

\[
    L = \{ f_h \mid h \in A \} \rtimes K_0. \tag{3.7}
\]
Note that $G \cong (T^{q^d}.(A/T)) \rtimes \text{AGL}(d, q)$ while $L \cong A \rtimes \text{AGL}(d, q)$. When $A = \text{PGU}(3, q)$ we let

$$G = \langle N, f_h \mid h \in A \rangle \rtimes K_u$$

(3.8)

and

$$L = \{f_h \mid h \in A \} \rtimes K_u.$$  

(3.9)

Then $G \cong (T^{q^d}.(A/T)) \rtimes (2 \cdot \text{AGL}(d, q))$ and $L \cong A \rtimes (2 \cdot \text{AGL}(d, q))$.

### 3.2 The group $R$

When $A = \text{PGL}(d, q), \text{Sz}(q)$ or $\text{Ree}(q)$, there is a point $\alpha \in \Omega$ such that the group automorphism $\phi$ normalises $A_\alpha$. Hence $\langle A, \phi \rangle$ acts 2-transitively on $\Omega$ with point stabiliser $\langle A_\alpha, \phi \rangle$. Similarly, when $A = \text{PGU}(3, q)$ there exists $\alpha \in \Omega$ such that $\langle A, \varphi \rangle$ acts 2-transitively with point stabiliser $\langle A_\alpha, \varphi \rangle$. Now for all $A$, the point stabiliser $A_\alpha$ has a unique minimal normal subgroup $M$ given in Table 1. In each case $M$ is isomorphic to the additive group of an $m$-dimensional vector space over $\text{GF}(q)$ and the group of automorphisms induced by conjugation by $A_\alpha$ on $M$ is $\text{GL}(m, q)$. See Remark 2.3. Note that $m = 1$ except when $A = \text{PGL}(n, q)$, in which case $m = n - 1$. Each $\lambda \in \text{GF}(q)^*$ induces an automorphism of $M$, this being scalar multiplication, and we denote the image of each $l \in M$ under this automorphism by $l^\lambda$. We also define $l^0 = 1_M$ for all $l \in M$. Then for $\lambda, \mu \in \text{GF}(q)$, the distributivity of scalar multiplication implies that $l^{\lambda + \mu} = l^\lambda l^\mu$. Each $h \in A_\alpha$ also induces an automorphism of $M$. Note that $A_\alpha$ acts transitively on the nontrivial elements of $M$ and induces $\text{GF}(q)$-linear automorphisms, that is, for all $h \in A_\alpha$, $\lambda \in \text{GF}(q)^*$ and $l \in M$,

$$(l^\lambda)^h = (l^h)^\lambda.$$  

(3.10)

Furthermore when $A \neq \text{PGU}(3, q)$,

$$(l^\lambda)^\phi = (l^\phi)^{\lambda^e}$$

(3.11)

while when $A = \text{PGU}(3, q)$

$$(l^\lambda)^\varphi = (l^\varphi)^{\lambda^e}.$$  

(3.12)

Recall from Remark 2.4(4) that if $q = p^e$ then $\varphi^e$ induces multiplication by $-1$ on $M$.

As $M$ is a $\text{GF}(q)$-vector space we can study linear functions $f : \text{GF}(q)^d \rightarrow M$. Since composition in $M$ is written multiplicatively, the linearity conditions become: for all $a, b \in \text{GF}(q)^d$ and $\lambda \in \text{GF}(q)$, we have $f(a + b) = f(a)f(b)$ and $f(\lambda a) = f(a)^\lambda$. We have the following lemma.
Lemma 3.1 Let

\[ Y = \langle f : \text{GF}(q)^d \to M \mid f \text{ constant} \rangle \]  

(3.13)

and

\[ X = \langle f : \text{GF}(q)^d \to M \mid f \text{ constant or linear} \rangle. \]  

(3.14)

Then \( Y \cong M \) and \( X/Y \cong M^d \). Moreover, when \( A \neq \text{PGU}(3, q) \) both \( X \) and \( Y \) are normalised by \( K_0 \), while when \( A = \text{PGU}(3, q) \), both \( X \) and \( Y \) are normalised by \( K_u \) (as defined in (3.2) and (3.4)).

**Proof.** The first assertion is trivial while the second assertion follows from the fact that the set of all linear functions \( f : \text{GF}(q)^d \to M \) is a set of coset representatives for \( Y \) in \( X \). When \( A \neq \text{PGU}(3, q) \) the group automorphism \( \phi \) normalises \( M \) and so \( K_0 \) is normalised by \( Y \). Furthermore, it follows from [3, Lemma 2.3] that \( X \) is normalised by \( K_0 \) (since \( \rho \) acts on \( M^k \) in the same way that \( \tau_0 \sigma_0 \) does there). Similar calculations show that when \( A = \text{PGU}(3, q) \) the group \( K_u \) normalises \( X \) and \( Y \).

We have the following lemma.

**Lemma 3.2** Let \( F \) be as in (3.1). Then \( F \) has a subgroup

\[ F_L = \langle X, f_h \mid h \in A_\alpha \rangle \cong M^{d+1}.(A_\alpha/M). \]

If \( A \neq \text{PGU}(3, q) \) then \( F_L \) is normalised by \( K_0 \) while if \( A = \text{PGU}(3, q) \) then \( F_L \) is normalised by \( K_u \).

**Proof.** Let \( h \in A_\alpha \). If \( f \in X \) is a constant function then as \( h \) normalises \( M \), \( f^h \) is also a constant function in \( X \). If \( f \in X \) is linear then for all \( a, b \in \text{GF}(q)^d \),

\[
\begin{align*}
    f_h^{-1} ff_h(a + b) &= h^{-1}f(a + b)h \\
    &= h^{-1}f(a)f(b)h \\
    &= f(a)^h f(b)^h \\
    &= f_h^{-1} ff_h(a)f_h^{-1} ff_h(b)
\end{align*}
\]

and for all \( \lambda \in \text{GF}(q) \),

\[
\begin{align*}
    f_h^{-1} ff_h(\lambda a) &= h^{-1}f(\lambda a)h \\
    &= h^{-1}f(a)^\lambda h \\
    &= (f(a)^\lambda)^h \\
    &= (f_h^{-1} ff_h(a))^\lambda.
\end{align*}
\]
Hence $f_h^{-1}f_h$ is a linear function and so $f_h$ normalises $X$. For $h \in M$, $f_h \in X$ and so $\langle X, f_h \mid h \in A_\alpha \rangle \cong M^{d+1}$. If $A \neq \text{PGU}(3, q)$ then Lemma 3.1 implies that, $K_0$ normalises $X$ and since $K_0$ normalises the subgroup $\{f_h \mid h \in A_\alpha\}$ it follows that $K_0$ normalises $F_L$. Similarly, when $A = \text{PGU}(3, q)$, $K_u$ normalises $X$, $\{f_h \mid h \in A_\alpha\}$ and $F_L$.

We now define the subgroup $R$. When $A = \text{PGL}(d, q)$, $\text{Ree}(q)$ or $\text{Sz}(q)$ let

$$R = \langle X, f_h \mid h \in A_\alpha \rangle \rtimes K_0$$  \hspace{1cm} (3.15)

Note that $R \cong (M^{d+1}.(A_\alpha/M)) \rtimes \text{AGL}(d, q)$. When $A = \text{PGU}(3, q)$ then let

$$R = \langle X, f_h \mid h \in A_\alpha \rangle \rtimes K_u$$  \hspace{1cm} (3.16)

In this case we have that $R \cong (M^{d+1}.(A_\alpha/M)) \rtimes (2. \text{AGL}(d, q))$.

### 3.3 The main construction

We can now give our general construction.

**Construction 3.3** We begin with the following:

- a 2-transitive almost simple group $A$ on a set $\Omega$, such that $A = \text{PGL}(n, q)$, $\text{PGU}(3, q)$, $\text{Ree}(q)$ or $\text{Sz}(q)$, and for $\alpha \in \Omega$, $A_\alpha$ has a unique minimal normal subgroup $M$ which is elementary abelian,
- a positive integer $d$, such that if $A = \text{PGL}(n, q)$ with $n \geq 3$, then $d = 1$,

and we construct a bipartite graph $\Gamma(A, d)$.

Recall the definition of $F$ from (3.1), and let $N = \text{soc}(F) \cong T_q^d$, where $T = \text{soc}(A)$. Let $X$ be the group generated by the set of constant or linear functions $f : \text{GF}(q)^d \rightarrow M$. Let $f_h \in F$ be the constant function with value $h$. If $A \neq \text{PGU}(3, q)$ recall $K_0$ from (3.2) and define

$$G = \langle N, f_h \mid h \in A \rangle \rtimes K_0,$$

$$L = \langle f_h \mid h \in A \rangle \rtimes K_0,$$

and

$$R = \langle X, f_h \mid h \in A_\alpha \rangle \rtimes K_0,$$
while when $A = \text{PSU}(3,q)$, recall $K_u$ from (3.4) and define

$$G = \langle N, f_h \mid h \in A \rangle \rtimes K_u,$$

$$L = \langle f_h \mid h \in A \rangle \rtimes K_u, \text{ and}$$

$$R = \langle X, f_h \mid h \in A_\alpha \rangle \rtimes K_u.$$

We can then construct the bipartite graph

$$\Gamma(A,d) := \text{Cos}(G,L,R)$$

as defined in Subsection 2.2.

If $d = 1$ and $A = \text{PGL}(2,q)$ with $q = p^e$, then the graph $\Gamma(A,1)$ is the graph $G(p,e)$ constructed and studied in [5].

Let $\Gamma = \Gamma(A,d)$ as yielded by Construction 3.3. Then $G$ has two orbits on $V\Gamma$, these being $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$. Now

$$|\Delta_1| = |G : L| = T^{q^d-1}$$

and $G$ acts quasiprimatively of type SD on $\Delta_1$. Also,

$$|\Delta_2| = |G : R| = \frac{|\Omega||T|^{q^d-1}}{|M|^d}.$$

Furthermore, $G$ acts quasiprimatively of type PA on $\Delta_2$.

Let $v$ be the vertex of $\Gamma$ given by the coset $L$. Then

$$|\Gamma(v)| = |L : L \cap R| = |A : A_\alpha| = |\Omega|,$$

and $G_v^{\Gamma(v)} = \text{PGL}(n,q)$, $\text{PGU}(3,q)$, $\text{Aut}(\text{Ree}(q))$ or $\text{Aut}(\text{Sz}(q))$. Let $w$ be the vertex of $\Gamma$ given by the coset $R$. Then $w \in \Gamma(v)$ and

$$|\Gamma(w)| = |R : L \cap R| = |M|^d.$$

Moreover, $G_w^{\Gamma(w)} = \text{AGL}(d,q)$. Thus $\Gamma$ has valency $\{ |\Omega|, |M|^d \}$. Also $G_v = L$, $G_w = R$ and if $A \neq \text{PGU}(3,q)$ then

$$G_{vw} = L \cap R = \{ f_h \mid h \in A_\alpha \} \rtimes K_0,$$

while if $A = \text{PGU}(3,q)$ we have

$$G_{vw} = L \cap R = \{ f_h \mid h \in A_\alpha \} \rtimes K_u.$$

Moreover, $G = NG_{vw}$ as $A = TA_\alpha$. Thus $N$ acts transitively on the set of edges of $\Gamma$ and so by Lemma 2.1, $\Gamma \cong \text{Cos}(N,N_v,N_w)$.
Collecting this information together we have the following lemma.

**Lemma 3.4** Each graph $\Gamma(A, d)$ yielded by Construction 3.3 is bipartite of valency $\{|\Omega|, |M|^d\}$ and $G$ acts quasiprimivitely of type SD on $\Delta_1$ and quasiprimitively of type PA on $\Delta_2$. Moreover, $G = N(L \cap R)$, that is, $N$ is transitive on edges.

Our next lemma shows that $\Gamma$ is connected. A subgroup $D$ of $N = T^k$ is **subdirect** if $D$ projects onto $T$ in each of the $k$ simple direct factors of $N$, and is a **full diagonal subgroup** if it is subdirect and isomorphic to $T$.

**Lemma 3.5** Each graph $\Gamma(A, d)$ yielded by Construction 3.3 is connected.

**PROOF.** Let $\Gamma = \Gamma(A, d)$ and $L, R, N$ and $G$ be as obtained from Construction 3.3. Let $D = \langle L, R \rangle \cap N$. By Lemma 3.4, it follows that $L \cap N$ is a full diagonal subgroup of $N = T^d$ and hence $D$ is a subdirect subgroup of $N$. Thus, it follows from a well known lemma, (see for example [13, p 328]), that there is a partition $I$ of $GF(q^d)$ with $D = \prod_{I \in I} T_I$, where each $T_I$ is a group of functions $f : GF(q^d) \to T$ for which $f(a) = 1$ for all $a \notin I$, the image of each $a \in I$ under $T_I$ is equal to $T$ and $T_I \cong T$. Choose $I \in I$ and let $I = \{b_1, \ldots, b_s\}$. Now for each $i \leq s$ there exists $\tau_i \in \text{Aut}(T)$, with $\tau_1 = 1$, such that for each $t \in T$ there exists a unique $f \in T_I$ with $f(b_i) = t^{\tau_i}$ for all $b_i \in I$.

Suppose that $s \geq 2$ and let $f_h \in L$ be a constant function whose value is $h$, for some $h \in T$. Then for all $i = 1, \ldots, s$, we have $f_h^I(b_i) = (t^{\tau_i})^h$. Since $L$ normalises $D$, we have $f_h^I \in T_I$ and so it follows that $(t^{\tau_i})^h = f_h^I(b_i) = (f_h^I(b_1))^{\tau_i} = (t^h)^{\tau_i}$. Hence for all $h, t \in T$ we have $t^{\tau_i} = t^{hr_i}$. Thus each $\tau_i$ centralises $\text{Inn}(T)$ and so each $\tau_i = 1$, that is, all functions in $T_I$ are constant on $I$ and trivial elsewhere. Furthermore, each function in $D$ is constant on $I$. However, there exists a linear function $f \in X \leq D$ which is not constant on $I$, a contradiction. Thus $s = 1$ and $N = D \leq \langle L, R \rangle$. As $G = NL$ it follows that $\langle L, R \rangle = G$ and so by Lemma 2.1, $\Gamma$ is connected.

### 3.4 Local s-arc transitivity

Recall the homomorphisms $\Phi$ and $\Phi_u$ from $K_0$ and $K_u$ respectively, onto $\Gamma\Gamma L(d, q)$. For any $K \leq \Gamma\Gamma L(d, q)$ we have $\Phi^{-1}(K) \leq K_0$ and $\Phi_u^{-1}(K) \leq K_u$. Thus when $A \neq \text{PGU}(3, q)$, $G$ has a subgroup

$$G_K = \langle N, f_h \mid h \in A_u \rangle \rtimes \Phi^{-1}(K),$$

(3.17)
while when $A = \text{PGU}(3, q)$, $G$ has a subgroup

$$G_K = \langle N, f_h \mid h \in A_\alpha \rangle \rtimes \Phi_u^{-1}(K). \quad (3.18)$$

Note that in both cases, $G = G_{\text{AGL}(d,q)}$. We wish to determine the largest $s$ such that $\Gamma(A,d)$ is locally $(G_K,s)$-arc transitive.

For each $g \in \text{AGL}(d,q)$ we can uniquely write $g = g_1g_2$ for some translation $g_1$ and some $g_2 \in \Gamma L(d,q)$. Since the subgroup of all translations is normal in $\text{AGL}(d,q)$ we can define the projection map $\pi: \text{AGL}(d,q) \rightarrow \Gamma L(d,q)$ which takes $g = g_1g_2$ to $g_2$. Moreover, there exists a field automorphism $\phi^i$ such that for all $a \in \text{GF}(q)^d$ and $\lambda \in \text{GF}(q)$, we have $(\lambda a)^{g_2} = \lambda^{\phi^i}a^{g_2}$. We call $\phi^i$ the field automorphism associated with $g_2$ and $g$.

We will require the following lemma.

**Lemma 3.6** Let $A = \text{PGL}(2,q)$, $\text{PGU}(3,q)$, $\text{Ree}(q)$ or $\text{Sz}(q)$, act 2-transitively on a set $\Omega$ of size $q+1$, $q^3+1$, $q^3+1$ or $q^2+1$ respectively. Let $P = A_{\alpha,\alpha_2}$ for $\alpha,\alpha_2 \in \Omega$ with $\alpha \neq \alpha_2$, and let $M$ be the centre of $O_p(A_\alpha)$. Let $K \leq \text{AGL}(d,q)$ for $d \geq 2$ such that $\pi(K)$ acts transitively on the set of 1-spaces of $\text{GF}(q)^d$. If $A \neq \text{PGU}(3,q)$ then $\{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$ acts transitively on the set of nontrivial elements of $X/Y$. If $A = \text{PGU}(3,q)$ then $\{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K)$ acts transitively on the set of nontrivial elements of $X/Y$.

**PROOF.** Let $f_1, f_2 : \text{GF}(q)^d \rightarrow M$ be linear functions, and recall that $M$ is a 1-dimensional vector space over $\text{GF}(q)$. Then $W_1 = \ker(f_1)$ and $W_2 = \ker(f_2)$ both have dimension $d-1$. Since $\pi(K)$ acts transitively on the set of 1-spaces, by Block’s Lemma [1], it also acts transitively on the set of hyperplanes of $\text{GF}(q)^d$. Hence there exists $g = g_1g_2 \in K$ such that $g_1$ is a translation and $g_2 \in \Gamma L(d,q)$, and $(W_1)^{g_2^{-1}} = W_2$.

Suppose first that $A \neq \text{PGU}(3,q)$. Hence $(Yf_1)^{\Phi^{-1}(\lambda)} = Y(\lambda f_2)$ for some $\lambda \in \text{GF}(q)$. Now there exists $h \in P$ such that $h$ induces the automorphism of $M$ corresponding to scalar multiplication by $\lambda^{-1}$ (Remark 2.4) and hence $(Yf_1)^{\Phi^{-1}(\lambda)}h = Yf_2$. Thus $\{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$ acts transitively by conjugation on the set of nontrivial elements of $X/Y$.

Suppose now that $A = \text{PGU}(3,q)$. Then $\Phi_u$ is not an isomorphism so $\Phi^{-1}_u(g)$ is not a unique element. Let $\phi^i$ be the field automorphism of $\text{GF}(q)$ associated with $g_2 \in \Gamma L(d,q)$. Then $\Phi_u(f_\phi \sigma_g) = g$ and $(Yf_1)^{f_\phi \sigma_g} = Y(\lambda f_2)$ for some $\lambda \in \text{GF}(q)$. Then taking the image under a suitable element $f_h$, $h \in P$, it follows that $\{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K)$ acts transitively by conjugation on the set of nontrivial elements of $X/Y$. 

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We can now prove the following lemma which completes the proof of Theorem 1.1.

**Lemma 3.7** Each graph \( \Gamma(A, d) \) yielded by Construction 3.3 is locally \((G, 2)\)-arc transitive. Furthermore, let \( K \leq A\Gamma L(d, q) \) such that \( \pi(K) \) is transitive on the set of 1-spaces of \( GF(q)^d \). Then \( \Gamma(A, d) \) is locally \((G_K, 2)\)-arc transitive.

**PROOF.** Let \( \Gamma = \Gamma(A, d) \), and let \( v \) be the vertex corresponding to the coset \( L \) and \( w \) be the vertex corresponding to the coset \( R \). When \( A \neq \text{PGU}(3, q) \), the action of \((G_K)_v\) on \( \Gamma(v) \) is equivalent to the action of \( \langle A, \phi \rangle \) on \( \Omega \) and so in this case \((G_K)_v\) acts 2-transitively on \( \Gamma(v) \). When \( A = \text{PGU}(3, q) \), the action of \((G_K)_v\) on \( \Gamma(v) \) is equivalent to the action of \( \langle A, \varphi \rangle = \text{PGL}(3, q) \) on \( \Omega \) and so in all cases \((G_K)_v\) acts 2-transitively on \( \Gamma(v) \).

Now \((G_K)_w = X(G_K)_{vw} \) and so \( X \) acts transitively on \( \Gamma(w) \). Furthermore, \( X \) is abelian and \( X_v = Y \). Hence as \( X \lhd (G_K)_w \) we can identify \( \Gamma(w) \) with \( X/Y \) such that \( X \) acts by right multiplication and \((G_K)_{vw}\) acts by conjugation. We may take the set of linear functions as a set of coset representatives for \( Y \) in \( X \). When \( A \neq \text{PGU}(3, q) \), we have \( (G_K)_{vw} = \{f_h \mid h \in A_\alpha \} \rtimes \Phi^{-1}(K) \) while when \( A = \text{PGU}(3, q) \) we have \( (G_K)_{vw} = \{f_h \mid h \in A_\alpha \} \rtimes \Phi^{-1}_w(K) \).

Suppose first that \( d = 1 \) and let \( a \in GF(q) \setminus \{0\} \). Then each linear function \( f : GF(q) \to M \) is determined by \( f(a) \). Furthermore, for each \( h \in A_\alpha \), let \( f_h \) be the constant function with value \( h \). Then \( f_h(a) = f(a)^h \) and as \( A_\alpha \) acts transitively by conjugation on the nontrivial elements of \( M \) it follows that \((G_K)_{vw}\) acts transitively on the set of nontrivial elements of \( X/Y \) and hence acts transitively on \( \Gamma(w) \setminus \{v\} \).

Suppose now that \( d \geq 2 \). Note that \( A \neq \text{PGL}(n, q) \) with \( n \geq 3 \) in this case. Then by Lemma 3.6, when \( A \neq \text{PGU}(3, q) \) the subgroup \( \{f_h \mid h \in P\} \rtimes \Phi^{-1}(K) \) of \((G_K)_{vw}\) acts transitively on the set of nontrivial elements of \( X/Y \) while when \( A = \text{PGU}(3, q) \) the subgroup \( \{f_h \mid h \in P\} \rtimes \Phi^{-1}_w(K) \) acts transitively on the set of nontrivial cosets of \( Y \) in \( X \). Hence in all cases \((G_K)_{vw}\) acts transitively on \( \Gamma(w) \setminus \{v\} \) and so for all values of \( d \), \((G_K)_w\) acts 2-transitively on \( \Gamma(w) \). Thus \( \Gamma \) is locally \((G_K, 2)\)-arc transitive. Moreover, since \( G = G_{\Delta L(d, q)} \) and \( \Gamma L(d, q) \) acts transitively on the set of 1-spaces of \( GF(q)^d \), it follows that \( \Gamma(A, d) \) is locally \((G, 2)\)-arc transitive.

We split the rest of our analysis into case where \( A \) is a rank one Lie group and the case where the rank of \( A \) is greater than one.

**Lemma 3.8** Let \( A = \text{PGL}(n, q) \) with \( n \geq 3 \). Then each graph \( \Gamma(A, 1) \) yielded by Construction 3.3 is not locally \((G, 3)\)-arc transitive.

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PROOF. Let $\Gamma = \Gamma(A,d)$, let $v$ be the vertex corresponding to the coset $L$ and $w$ be the vertex corresponding to the coset $R$. Then there exists $u \in \Gamma(v)\backslash\{w\}$ and $\alpha_2 \in \Omega\backslash\{\alpha\}$ such that

$$G_{uvw} = \{f_h \mid h \in A_{\alpha,\alpha_2}\} \rtimes \Phi^{-1}(K).$$

Now $A_{\alpha,\alpha_2}$ normalises a unique index $q$ subgroup of $M = O_p(A_{\alpha})$, where $q = p^t$, and so $\{f_h \mid h \in A_{\alpha,\alpha_2}\} \rtimes \Phi^{-1}(K)$ does not act transitively by conjugation on the set of nontrivial elements of $X/Y$. Hence $G_{uvw}$ does not act transitively on $\Gamma(w)\backslash\{v\}$ and so $\Gamma$ is not locally $(G,3)$-arc transitive.

We have the following proposition.

**Proposition 3.9**  
(1) Each graph $\Gamma(A,d)$, where $A = \text{PGL}(2,q)$, $\text{PGU}(3,q)$, $S_2(q)$ or $\text{Ree}(q)$, is locally $(G_K,3)$-arc transitive.

(2) The graph $\Gamma(A,d)$ is locally $(G,4)$-arc transitive if and only if $A = \text{PGL}(2,q)$, $d = 1$ and $q$ is even. Furthermore, when $q$ is even, the graph $\Gamma(\text{PGL}(2,q),1)$ is locally 5-arc transitive but not locally 6-arc transitive.

(3) If $\Gamma(\text{PGL}(2,2^e),1)$ is locally $(G_K,4)$-arc transitive then $K$ acts transitively on $\text{GF}(q)$.

**PROOF.** Let $\Gamma = \Gamma(A,d)$. Choose $u \in \Gamma(v)\backslash\{w\}$ such that when $A \neq \text{PGU}(3,q)$,

$$(G_K)_{uvw} = \{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$$

and when $A = \text{PGU}(3,q)$,

$$(G_K)_{uvw} = \{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K).$$

Note that $u$ corresponds to the point $\alpha_2 \in \Omega\backslash\{\alpha\}$ such that $A_{\alpha,\alpha_2} = P$. If $d = 1$, then $\{f_h \mid h \in P\}$ acts transitively by conjugation on the set of nontrivial elements of $X/Y$ and so $(G_K)_{uvw}$ acts transitively on $\Gamma(w)\backslash\{v\}$.

When $d > 1$, we have from Lemma 3.6 that in all cases, the group $(G_K)_{uvw}$ acts transitively on the set of nontrivial elements of $X/Y$. Hence for all possibilities for $A$, $(G_K)_{uvw}$ acts transitively on $\Gamma(w)\backslash\{v\}$. Thus $(G_K)_u$ acts transitively on the set of 3-arcs starting at $u$.

Let $\{e_1, \ldots, e_d\}$ be a basis for $\text{GF}(q)^d$ such that the semilinear map $\phi$ fixes each $e_i$. Fix $l_1 \in M\backslash\{1\}$ such that $l_1$ is centralised by the group automorphism $\phi$ of $A$ ($\phi$ when $A = \text{PGU}(3,q)$) and let $f_1 \in X$ be the linear function which maps $e_1$ to $l_1$ and sends each $e_i$, $i \geq 2$, to $1_M$. Let $x = Lf_1 \in \Gamma(w)$. Since $O_p(A_{\alpha})$ centralises $M$ (Remark 2.4(1)), we have

$$\{f_h \mid h \in O_p(A_{\alpha})\} \leq (G_K)_{uvw}.$$
Then as $O_p(A_u)$ acts transitively on $\Omega \setminus \{\alpha\}$ (Remark 2.4), it follows that $(G_K)_{xuv}$ acts transitively on $\Gamma(v) \setminus \{w\}$. Hence $(G_K)_x$ acts transitively on the set of 3-arcs starting at $x$ and so $\Gamma$ is locally $(G_K,3)$-arc transitive. This completes the proof of part 1.

Next we investigate 4-arcs. Now $G_{uvw} \leq G_{uvw}$. Let $f_h \sigma \in G_{uvw}$ where $\sigma \in K_0 = \Phi^{-1}(\GammaGL(d,q))$ if $A \neq \PGU(3,q)$ while $\sigma \in K_u = \Phi^{-1}_u(\GammaGL(d,q))$ if $A = \PGU(3,q)$. Then

$$Lf_1 = Lf_1f_h \sigma = L(\lambda f_1)\sigma$$

for some $\lambda \in GF(q)$.

Then as $K_0$ and $K_u$ induce semilinear maps on $X$ it follows that $\sigma$ fixes setwise

$$\{L(\mu f_1) \mid \mu \in GF(q)\}.$$

Since $A$ acts 2-transitively on $\Omega$ there exists $t \in A$ such that $\alpha t = \alpha_2$ and $\alpha_2^2 = \alpha$. Furthermore, we can choose $t$ such that $t$ is centralised by the group automorphism $\phi$ (alternatively by $\varphi$ when $A = \PGU(3,q)$). This can be done by choosing $t \in \PGL(2,p), \PGU(3,p), Ree(p)$, or $Sz(p)$ appropriately. Let $f_t : GF(q)^d \to A$ be the constant function with value $t$. Then $f_t \in G_v$ and interchanges $u$ and $w$. Thus $f_t$ maps $\Gamma(w)$ to $\Gamma(u)$. Then

$$\Gamma(u) = \Gamma(w)f_t = \{Lff_t \mid f \text{ linear}\}.$$

Also note that $t$ normalises $P = A_{a,\alpha_2}$. Let $f_h \sigma \in G_{uvwx}$, where $\sigma \in K_0$ if $A \neq \PGU(3,q)$ while $\sigma \in K_u$ if $A = \PGU(3,q)$. Then $Lf_1f_t \in \Gamma(u)$ and

$$Lf_1f_t f_h \sigma = Lf_1f_t f_h \sigma f_t^{-1}f_t = Lf_1f_t \sigma f_t$$

where $\sigma$ centralises $f_t$

$$= L(\xi f_1)\sigma f_t$$

for some $\xi \in GF(q)$, since $h^t \in P$.

Now as $f_h \sigma \in G_{uvwx}$, we have seen that $\sigma$ fixes the set

$$\{L(\mu f_1) \mid \mu \in GF(q)\}$$

setwise and so each $f_h \sigma \in G_{uvwx}$ fixes

$$\{L(\mu f_1)f_t \mid \mu \in GF(q)\} \subseteq \Gamma(u)$$

setwise. Hence if $d > 1$, then $G_{uvwx}$ is not transitive on $\Gamma(u) \setminus \{v\}$. Thus $\Gamma(A,d)$ is not locally $(G,4)$-arc transitive for $d \geq 2$.

Suppose now that $d = 1$. Then $G_{uvw}$ acts transitively on the set $\Gamma(w) \setminus \{v\}$ of size $q - 1$. If $A \neq \PGU(3,q)$ it follows that $|G_{uvwx}| = |P|q(q-1)e/(q-1) = ge|P|$. On the other hand if $A = \PGU(3,q)$ then $|G_{uvwx}| = |P|2q(q-1)e/(q-1)| = 2ge|P|$.
If \( A = \text{Ree}(q) \) or \( S\text{z}(q) \), then \(|P| = q-1\) and \(|\Omega| = q^3+1\) or \(q^2+1\) respectively. Thus in all cases \(|\Gamma(x)\setminus\{w\}| = |\Omega| - 1\) does not divide \(|G_{uvwx}|\), so \(G_{uvwx}\) is not transitive on \(\Gamma(x)\setminus\{w\}\). Hence \(\Gamma(A,1)\) is not locally \((G,4)\)-arc transitive for either of these choices for \(A\).

If \( A = \text{PGU}(3,q) \) then \(|P| = q^2-1\) and \(|\Omega| = q^3+1\). Thus \(|\Gamma(x)\setminus\{w\}| = |\Omega| - 1 = q^3\) does not divide \(|G_{uvwx}|\), so \(G_{uvwx}\) is not transitive on \(\Gamma(x)\setminus\{w\}\). Hence \(\Gamma(\text{PSU}(3,q),1)\) is not locally \((G,4)\)-arc transitive.

This leaves us to investigate the case where \( A = \text{PGL}(2,q) \). Then \(|(G_K)_{uvw}| = |P||K|\) and since \((G_K)_{uvw}\) acts transitively on the set \(\Gamma(w)\setminus\{v\}\) of size \(q-1\) it follows that \(|(G_K)_{uvw}| = |P||K|/(q-1)\). Here \(|P| = q-1\) and \(|\Omega| = q+1\). Thus \(|(G_K)_{uvw}| = |K|\) divides \(q(q-1)e\), and \(|\Gamma(x)\setminus\{w\}| = q\). Also \(|\Gamma(u)\setminus\{v\}| = q-1\) and so for \((G_K)_{uvw}\) to be transitive on both \(\Gamma(x)\setminus\{w\}\) and \(\Gamma(u)\setminus\{v\}\) we require that \(q(q-1)\) divides \(|(G_K)_{uvw}|\). Hence \(K\) is a subgroup of \(\text{ATL}(1,q)\) whose order is divisible by \(q(q-1)\). By Lemma 2.5, \(K\) acts transitively on \(\text{GF}(q)\). Hence if \(\Gamma(\text{PGL}(2,q),1)\) is locally \((G_K,4)\)-arc transitive then \(K\) is transitive on \(\text{GF}(q)\). This completes the proof of part (3). Furthermore, the graph \(\Gamma(\text{PGL}(2,q),1)\) where \(q = p^e\), is the graph \(\mathcal{G}(p,e)\) constructed in [5]. By [5, Theorem 1.1] this is locally \((G,4)\)-arc transitive if and only if \(p = 2\). Furthermore, for \(q\) even, the graph \(\Gamma(\text{PGL}(2,q),1,\text{GF}(q))\) is locally 5-arc transitive but not locally 6-arc transitive. This completes the proof of part (2).

### 3.5 Quotients

Given \( K \leq \text{ATL}(d,q) \) recall \(G_K\) defined in (3.17) and (3.18), and that the group induced by \(G_K\) on the \(q^d\) simple direct factors of \(N = \text{soc}(G) \leq G_K\) is \(K\). Hence if \(K\) is not transitive on the set of \(q^d\) vectors of \(\text{GF}(q)^d\), our group \(G_K\) is not quasiprimitive on \(\Delta_1\) or on \(\Delta_2\). However, by Lemma 3.7 if \(\pi(K)\), the projection of \(K\) onto \(\GammaL(d,q)\), is transitive on the set of 1-spaces of \(\text{GF}(q)^d\), then \(\Gamma(A,d)\) is locally \((G_K,2)\)-arc transitive. This allows us to give the following construction of more locally 2-arc transitive graphs of \(\{\text{SD,PA}\}\) type.

### Construction 3.10

We begin with

- a 2-transitive almost simple group \(A\) on a set \(\Omega\), such that \(A = \text{PGL}(n,q)\), \(\text{PGU}(3,q)\), \(\text{Ree}(q)\) or \(\text{Sz}(q)\),
- a positive integer \(d\), such that if \(A = \text{PGL}(n,q)\) with \(n \geq 3\), then \(d = 1\),
- the locally \((G,2)\)-arc transitive graph \(\Gamma(A,d)\) obtained from Construction 3.3,
- \(K \leq \text{ATL}(d,q)\) such that \(\pi(K)\) acts transitively on the set of 1-spaces of \(\text{GF}(q)^d\) but \(K\) is intransitive on \(\text{GF}(q)^d\).
• an orbit $S$ of $K$ on $\text{GF}(q)^d$ of length $k > 1$.

Now $N = \text{soc}(G) \cong T^n$ contains a subgroup

$$N_S = \{ f : \text{GF}(q)^d \to T \mid f(a) = 1 \text{ for all } a \in S \}.$$ 

Then $N_S \cong T^{n-k}$ and $N_S \trianglelefteq G_K$. Moreover, $N_S$ acts intransitively on both $\Delta_1$ and $\Delta_2$. We construct the graph $\Gamma(A, d, S)$ to be the quotient graph of $\Gamma(A, d)$ with respect to the orbits of $N_S$.

We collect the following remarks about the choice of $K$.

**Remark 3.11**

(1) If $S$ is an orbit of two groups $K_1, K_2 \leq \text{AGL}(d, q)$ on $\text{GF}(q)^d$ then Construction 3.10 yields the same graph using either $K_1$ or $K_2$.

(2) If $d = 1$ then we can take $K$ to be any intransitive subgroup of $\text{AGL}(1, q)$ and $S$ any nontrivial orbit.

(3) Suppose now that $d \geq 2$. The group of all translations in $\text{AGL}(d, q)$ is normalised by $K$. Then since $\pi(K)$ acts transitively on the set of 1-spaces of $\text{GF}(q)^d$, it follows that $K$ does not contain any nontrivial translations. Hence $K \cong \pi(K) \leq \text{GL}(d, q)$. One possibility for $K$ is $\pi(K)$ and $S = \text{GF}(q)^d \setminus \{0\}$. If $\pi(K) < \text{GL}(d, q)$ and $Z$ is the group of scalars in $\text{GL}(d, q)$ then $\pi(K)Z$ acts transitively on $\text{GF}(q)^d \setminus \{0\}$ and so is the stabiliser of a point in an affine 2-transitive group. All such groups are known, see [7–9].

For each $f \in F$ let $f|_S$ denote the restriction of $f$ to $S$. Then when $A \neq \text{PGU}(3, q)$

$$G_{K,S} := G_K/N_S \cong \langle f|_S, (f|_S)h|_S \mid f \in N, h \in A \rangle \rtimes \Phi^{-1}(K)$$

while when $A = \text{PGU}(3, q)$

$$G_{K,S} := G_K/N_S \cong \langle f|_S, (f|_S)h|_S \mid f \in N, h \in A \rangle \rtimes \Phi^{-1}_\sigma(K).$$

Note that $G_{K,S}$ is isomorphic to either $T^k.(A/T) \rtimes K$ or $T^k.(A/T) \rtimes (2.K)$ depending on $A$.

Since $K$ fixes $S$ setwise, $\Phi^{-1}(K)$ and $\Phi^{-1}_\sigma(K)$ induce automorphisms of

$$F_S = \langle f|_S, (f|_S)h|_S \mid f \in N, h \in A \rangle.$$ 

As $\pi(K)$ acts transitively on the set of 1-spaces of $\text{GF}(q)^d$, it follows that $S$ spans $\text{GF}(q)^d$. Hence if $g \in K$ acts trivially on $S$ then $g \notin \text{AGL}(d, q)$ and so has a nontrivial associated field automorphism $\phi^i$ for some $i = 1, \ldots, e - 1$, where $q = p^e$. If $A \neq \text{PGU}(3, q)$ then $\Phi^{-1}(g) = f|_S \sigma g$. Hence $\Phi^{-1}(g)$ induces
Suppose that $\Delta$ is a cover of $\Gamma$. Then $(\Delta',v)$ is a cover of $\Gamma$. Hence $\Delta'$ is a cover of $\Gamma$.

**PROOF.** By Lemma 3.5, $\Gamma(A,d)$ is connected and so $\Gamma(A,d,S)$ is connected. By [4, Lemma 5.1], $\Gamma(A,d,S)$ is locally $(G_{K}/N_{S},s)$-arc transitive, $\Gamma(A,d)$ is a cover of $\Gamma(A,d,S)$ and $G_{K,S} = G_{K}/N_{S} \leq \text{Aut}(\Gamma(A,d,S))$. Moreover, $\Gamma(A,d,S)$ is a bipartite graph with bipartition $\{\Delta_{1}',\Delta_{2}'\}$, where $\Delta_{1}'$ is the set of $N_{S}$-orbits on $\Delta_{1}$ and $\Delta_{2}'$ is the set of $N_{S}$-orbits on $\Delta_{2}$. Let $W = \text{soc}(G_{K,S}) \cong T^{k}$. Then as $K$ acts transitively on $S$ it follows that $G_{K,S}$ acts transitively on the $k$ simple direct factors of $W$. Given an $N_{S}$-orbit $B \in \Delta_{1}'$ with $v \in B$ we have that $W_{B} = N_{v}N_{S}/N_{S} \cong N_{v}/(N_{v} \cap N_{S}) \cong N_{v} \cong T$. Hence $G_{K,S}$ acts quasiprimitively of type SD on $\Delta_{1}$. Since $S$ spans $\text{GF}(q)^{d}$ we have $N_{w} \cap N_{S} = 1$ and so, given an $N_{S}$-orbit $C \in \Delta_{2}'$ with $w \in C$ it follows that $W_{C} = N_{w}N_{S}/N_{S} \cong N_{w}/(N_{w} \cap N_{S}) \cong N_{w}$. Since $G$ acts quasiprimitively of type PA on $\Delta_{1}$ it follows that $N_{w} \neq 1$ and is not isomorphic to $T^{l}$ for any $l \leq k$. Hence $G_{K,S}$ acts quasiprimitively of type PA on $\Delta_{2}'$ and so $\Gamma(A,d,S)$ is of $\{\text{SD,PA}\}$-type.

The examples in [6, Example 4.2] can be obtained from Construction 3.10 by taking $K$ to be a subgroup of $\text{AGL}(1,q)$ of order 2 and $S$ a nontrivial orbit.

We also need the following lemma which combined with Propositions 3.12 and 3.9, and Lemma 3.8 determines the largest $s$ for which $\Gamma(A,d,S)$ is locally $(G_{K,S},s)$-arc transitive.

**Lemma 3.13** Let $\Gamma$ be a graph, $G \leq \text{Aut}(\Gamma)$, $N \triangleleft G$ intransitive on $V\Gamma$ and suppose that $\Gamma$ is a cover of $\Gamma_{N}$. If $\Gamma_{N}$ is locally $(G/N,s)$-arc transitive then $\Gamma$ is locally $(G,s)$-arc transitive.

**PROOF.** Let $(v_{0},v_{1},\ldots,v_{s})$ and $(v_{0},w_{1},\ldots,w_{s})$ be $s$-arcs in $\Gamma$ starting at $v_{0}$. Let $B_{i}$ be the $N$-orbit containing $v_{i}$ and $C_{i}$ be the $N$-orbit containing $w_{i}$. Then $(B_{0},B_{1},\ldots,B_{s})$ and $(B_{0},C_{1},\ldots,C_{s})$ are $s$-arcs in $\Gamma_{N}$ starting at $B_{0}$. Thus there exists $g \in G$ such that $(B_{0},B_{1},\ldots,B_{s})g^{N} = (B_{0},C_{1},\ldots,C_{s})$. Then $v_{0}g \in B_{0}$ and since $B_{0}$ is an $N$-orbit, there exists $n \in N$ such that $v_{0}g^{n} = v_{0}$. Moreover, $v_{1}g \in \Gamma(v_{0}) \cap C_{1} = \{w_{1}\}$ as $\Gamma$ is a cover of $\Gamma_{N}$. Hence $v_{1}g^{n} = w_{1}$. Similarly, we see that $v_{i}g^{n} = w_{i}$ for all $i = 2,\ldots,s$ and so $(v_{0},v_{1},\ldots,v_{s})g^{n} = (v_{0},w_{1},\ldots,w_{s})$. Thus $\Gamma$ is locally $(G,s)$-arc transitive.
Corollary 3.14  (1) Each graph $\Gamma(\text{PGL}(n, q), 1, S)$, for $n \geq 3$, is locally $(G_{K,S}, 2)$-arc transitive but not locally $(G_{K,S}, 3)$-arc transitive.

(2) If $A$ is one of $\text{PGU}(3, q)$, $\text{Ree}(q)$ or $\text{Sz}(q)$, then each graph $\Gamma(A, d, S)$ is locally $(G_{K,S}, 3)$-arc transitive but not locally $(G_{K,S}, 4)$-arc transitive.

(3) Each graph $\Gamma(\text{PGL}(2, q), d, S)$ is locally $(G_{K,S}, 3)$-arc transitive. Moreover, if either $d \geq 2$, $q$ is odd, or $S \neq \text{GF}(q)$ then $\Gamma(\text{PGL}(2, q), d, S)$ is not locally $(G_{K,S}, 4)$-arc transitive.

4 Proof of Theorem 1.2

We make the following general hypothesis.

(\text{SDPA}) 

$\Gamma$ is a locally $(G, 2)$-arc transitive connected graph such that $G$ has two orbits $\Delta_1$ and $\Delta_2$ on vertices, $G$ acts faithfully on both orbits, quasiprimively on $\Delta_1$ with type SD and quasiprimively on $\Delta_2$ with type PA. Furthermore, $N = \text{soc}(G) \cong T^k$ for some finite nonabelian simple group $T$ and positive integer $k \geq 2$. Each vertex of $\Gamma$ has valency at least 3 and $v, w$ are a pair of adjacent vertices with $v \in \Delta_1$ and $w \in \Delta_2$.

We denote by $\pi_i$ the projection homomorphism of $N$ onto its $i$th coordinate. We also let $T_i$ be the normal subgroup of $N$ for which $\pi_i(T_i) = T$ and $\pi_j(T_i) = 1$ for all $j \neq i$. Then for any subset $I$ of $\{1, \ldots, k\}$ we let $T_I = \prod_{i \in I} T_i$.

First we note [4, Lemma 6.2].

Lemma 4.1 Let $\Gamma$ be a locally $(G, s)$-arc transitive graph with $s \geq 2$ such that $G$ acts quasiprimively and faithfully on both orbits on vertices. Let $N \triangleleft G$. If $N$ is not regular on $\Delta_1$ then $N^\Gamma(v)$ is transitive for all $v \in V\Gamma$.

Corollary 4.2 Under the hypothesis of Lemma 4.1, $\Gamma \cong \text{Cos}(N, N_v, N_w)$.

PROOF. Since $N_v$ acts transitively on $\Gamma(v)$ and $N$ acts transitively on $\Delta_1$, it follows that $N$ is edge transitive. Thus the result follows from Lemma 2.1.

Lemma 4.3 Let $\Gamma$ be as in (SDPA). Then replacing $G$, if necessary, by a conjugate in $\text{Sym}(\Delta_1 \cup \Delta_2)$, we may assume that the following all hold.

1. $N_v = \{(t, \ldots, t) \mid t \in T\}$.
2. $N_{vw} = \{(t, \ldots, t) \mid t \in H\}$ for some maximal subgroup $H$ of $T$ and the action of $T$ on the set of right cosets of $H$ is the action of the socle of a 2-transitive almost simple group.
(3) \( N_w \) is a subdirect subgroup of \( H^k \), acts transitively on \( \Gamma(w) \) and \( N_w \neq N_{vw} \).

**PROOF.** 1). As the action of \( G \) on \( \Delta_1 \) is quasiprimitive of type SD, replacing \( G \) if necessary by a conjugate in \( \text{Sym}(\Delta_1 \cup \Delta_2) \), we may choose \( v \) such that \( N_v = \{(t, \ldots, t) \mid t \in T\} \).

2). By Lemma 4.1,

\[
1 \neq N_v^T < G_v^T
\]

and \( G_v^T \) is a 2-transitive group. As \( N_v \) is simple it follows that \( N_v^T \cong T \). Then a theorem of Burnside (see [2, Theorem 4.3]) implies that \( N_v^T \) is primitive and \( G_v^T \) is an almost simple 2-transitive group with socle \( T \). Thus (2) holds.

3). Now \( N_{vw} \leq N_w \) and so \( H = \pi_i(N_{vw}) \leq \pi_i(N_w) \) for each \( i = 1, \ldots, k \). As \( G \) is quasiprimitive of type PA on \( \Delta_2 \), \( \pi_i(N_w) \neq T \) for all \( i \) and so the maximality of \( H \) in \( T \) yields that \( \pi_i(N_w) = H \) for all \( i \). Hence \( N_w \) is a subdirect subgroup of \( H^k \). By Lemma 4.1, \( N_w^T \) is transitive on \( \Gamma(w) \). Since \( |\Gamma(w)| \geq 3 \) it follows that \( N_w \neq N_{vw} \).

**Lemma 4.4** Let \( \Gamma \) be as in (SDPA). Then \( G \) acts by conjugation on the set \( T = \{T_1, T_2, \ldots, T_k\} \), and the permutation groups \( G_T, G_v^T, G_w^T \) and \( G_{vw}^T \) are all transitive and pairwise permutationally isomorphic.

**PROOF.** Since \( N \) is a minimal normal subgroup of \( G \), the group \( G_T \) is transitive. Furthermore, \( N \) acts transitively on \( \Delta_1 \), and so \( G = NG_v \). Then as \( N \) acts trivially on \( T \) it follows that \( G_T \) and \( G_v^T \) are permutationally isomorphic. Similarly, \( G_T \) and \( G_w^T \) are permutationally isomorphic. Furthermore, by Lemma 4.3, \( N_w \) acts transitively on \( \Gamma(w) \) and so \( G_w = N_wG_{vw} \). Hence \( G_v^T \) and \( G_{vw}^T \) are permutationally isomorphic.

We now have the following theorem which determines the possibilities for \( T \) and \( H \).

**Theorem 4.5** Let \( \Gamma \) be as in (SDPA), \( N \) be as in Lemma 4.3 and \( H \) be the subgroup of \( T \) isomorphic to \( N_{vw} \). Then \( T \) and \( H \) are as in one of the rows of Table 1.

**PROOF.** Suppose that \( H \) is an almost simple group with socle \( M \) and let \( S \) be the socle of \( N_w \). Then \( S \trianglelefteq G_w \), as it is a characteristic subgroup of \( N_w \). By Lemma 4.3, \( N_w \) is a subdirect subgroup of \( H^k \) and so \( \pi_i(S) \trianglelefteq \pi_i(H) \) for each \( i = 1, \ldots, k \). By Lemma 4.4, \( G_w \) acts transitively by conjugation on
the $k$ simple direct factors of $N$, so the $\pi_i(S)$, for $i = 1, 2, \ldots, k$, are pairwise isomorphic. Then as $M$ is the unique minimal normal subgroup of $H$, it follows that $M \leq \pi_i(S)$ for all $i$. Furthermore, if $1 \neq R \vartriangleleft N_w$ then $R \cap M^k \vartriangleleft N_w$, and also (since $S$ is the socle of $N_w$) $R \cap S \neq 1$ from which it follows that $R \cap M^k \neq 1$. Thus every minimal normal subgroup of $N_w$ is contained in $M^k$ and so $S$ is also contained in $M^k$. Hence $S$ is a subdirect subgroup of $M^k$. Moreover, a well known lemma, (see for example [13, Page 328]) together with the facts that $G_w$ normalises $S$ and is transitive on the simple direct factors of $N$, imply that there exists a divisor $r$ of $k$ such that $S = D_1 \times \cdots \times D_r$, where $D_1 = \{(l, l^{p_2}, \ldots, l^{p_k/r}) \mid l \in M\} \leq M^{k/r}$, for some automorphisms $\sigma_i$ of $M$, and the $D_j$ are permuted transitively by $G_w$. Hence $S \cong M^r$.

Let $K$ be the kernel of the action of $S$ on $\Gamma(w)$. As $K$ is the intersection of $S$ and the kernel of the action of $G_w$ on $\Gamma(w)$, both of which are normal in $G_w$, we have $K \triangleleft G_w$. Now $G_w$ acts transitively on the $k$ simple direct factors of $N$ and as $S = D_1 \times \cdots \times D_r$, where each $D_j$ is a nonabelian simple group isomorphic to $M$, either $K = 1$ or $K = S$. If $K = 1$ then $S^{\Gamma(w)} \cong S \cong M^r$. Now $S^{\Gamma(w)} \cong G_w^{\Gamma(w)}$, a 2-transitive group, and so by a theorem of Burnside, (see [2, Theorem 4.3], $S^{\Gamma(w)} \cong M^r$ is a nonabelian simple group. Hence in the case where $K = 1$ we have $r = 1$. If on the other hand $K = S$, then $K \leq N_{vw} \cong H$. As $M$ is the unique minimal normal subgroup of $H$, and $M$ is simple, it follows that $r = 1$ in this case also. Thus in either case $M \cong S \triangleleft N_w$. As $S = \soc(N_w)$ and $M$ is nonabelian, it follows that $C_{N_w}(S) = 1$ and so $N_w \leq \Aut(M)$. Hence $N_w \cong H$ and $N_w = N_{vw}$. This contradicts Lemma 4.3 and so $H$ is not almost simple. Then by Theorem 2.2, $T$ and $H$ are as in one of the lines of Table 1.

By Theorem 2.2, each possibility for $H$ given in Theorem 4.5 has a unique minimal normal subgroup $M$ and $M$ is the centre of $O_p(H)$, where $p$ is the characteristic of the field over which $T$ is defined. Recall the Remarks 2.3 and 2.4, especially that we count $\PSL(2, 8)$ twice, once as $\PSL(2, 8)$ of degree 9 and once as $\Ree(3')$ of degree 28. Let $A = \PGL(n, q)$, $\PGU(3, q)$, $Sz(q)$ or $\Ree(q)$ such that $\soc(A) = T$. If we let $\Omega = [T : H]$ and $B = N_{\Sym(\Omega)}(T)$, then $B$ acts 2-transitively on $\Omega$ with socle $T$. We can choose $\alpha \in \Omega$ such that $T_\alpha = H$ and $B_\alpha$ induces $\TL(m, q)$ on $M$, where $M$ has order $q^m$. It transpires that the subgroup $X = N_w \cap M^k$ is crucial to our analysis.

**Proposition 4.6** Let $\Gamma$ be as in (SDPA), $N$ be as in Lemma 4.3, $B$ be as above and let $X = N_w \cap M^k$ and $Y = N_{vw} \cap M^k$. Furthermore, suppose that $G$ is the largest subgroup of $\Aut(\Gamma)$ of type $SD$ with socle $N$. Then, using the notation introduced above, the following all hold.

1. $X$ is normal in $G_w$ and acts transitively on $\Gamma(w)$, and $Y = \{(l, l, \ldots, l) \mid l \in M\}$.
2. $G_{vw} = N_F(X)$, where $F = \{(h, \ldots, h) \mid h \in B_\alpha\} \times S_k$. 

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(3) The action of $G_{vw}$ induced on $X/Y$ by conjugation is equivalent to the action of $G_{vw}$ on $\Gamma(w)$.

(4) There exists a positive integer $d$ such that $X$ is a $(d+1)$-dimensional vector space over a field of size $|M|$ and $|\Gamma(w)| = |M|^d$.

(5) There does not exist $I \subseteq \{1, 2, \ldots, k\}$ with $|I| \geq 2$ such that every element of $X$ is constant on $I$.

**Proof.** 1). As $M$ is the unique minimal normal subgroup of $H$, it is characteristic in $H$ and so $X = N_w \cap M^k$ is characteristic in $N_w$. Thus $X \triangleleft G_w$.

Now $G_w$ acts 2-transitively on $\Gamma(w)$ and so $X$ acts either transitively or trivially on $\Gamma(w)$. By Lemma 4.3, $N_{vw} = \{(h, \ldots, h) \mid h \in H\}$ and so $Y = N_{vw} \cap M^k = \{(l, \ldots, l) \mid l \in M\}$.

Suppose $X$ acts trivially on $\Gamma(w)$. Then $X \leq N_{vw}$ and so $X = Y = \{(l, \ldots, l) \mid l \in M\}$. By Lemma 4.3(3), $N_{vw} \neq N_w$. Thus there exists a proper subset $I$ of $\{1, \ldots, k\}$ such that $N_w \cap T_I \neq 1$ and we may choose $I$ to be a minimal such set. Without loss of generality we may assume that $I = \{1, \ldots, t\}$ for some $t < k$. As $I$ is minimal there exists a nontrivial subgroup $K$ of $H$ such that

$$N_w \cap T_I = \{(l_1, l_2^2, \ldots, l_t^r, 1, \ldots, 1) \mid l \in K\}$$

for some $\sigma_i \in \text{Aut}(K)$. Now $N_w \cap T_I \triangleleft N_w$ so for each $i \in I$,

$$1 \neq K \cong \pi_i(N_w \cap T_I) \triangleleft \pi_i(N_w) = H.$$

Then as $M$ is the unique minimal normal subgroup of $H$ we have $M \leq K$ and

$$\{(l_1, l_2^2, \ldots, l_t^r, 1, \ldots, 1) \mid l \in M\} \leq N_w \cap T_I \leq N_w \cap M^k = X.$$

This contradicts the fact that $X = \{(l, \ldots, l) \mid l \in M\}$ and so $X$ acts transitively on $\Gamma(w)$.

2). Since $G$ acts transitively on $\Delta_1$ with quasiprimitive type SD and $N_v = \{(t, \ldots, t) \mid t \in T\}$ we have

$$G_v \leq N_{\text{Sym}(\Delta_1)}(N_v) = \{(t, \ldots, t) \mid t \in \text{Aut}(T)\} \times S_k.$$

Furthermore, $G_v$ acts on $\Gamma(v)$ and $G_v^{(v)}$ is a 2-transitive group with socle $T$. Thus

$$G_v \leq \{(t, \ldots, t) \mid t \in B\} \times S_k,$$

where $B = N_{\text{Sym}(T)}(T)$, and $G_{vw} \leq F = \{(h, \ldots, h) \mid h \in B_\alpha\} \times S_k$. By Part 1, $X \triangleleft G_w$ and so is normalised by $G_{vw}$. Hence $G_{vw} \leq N_F(X)$ and it remains to prove equality. By Lemma 2.1, $\Gamma \cong \text{Cos}(N, N_v, N_w)$. Let $g \in N_F(X)$. Then as $g \in \text{Aut}(T)^k \rtimes S_k$ we have that $g$ normalises $N$. Furthermore, as $B_\alpha$ normalises $T$ it follows that $g$ normalises $N_v$. Finally, as $g \in N_F(X)$ and $B_\alpha$ normalises $H = T \cap B_\alpha$ we have that $g$ normalises $N_w = XN_{vw}$. Thus $g$ acts
on $\Delta_1 = [N : N_v]$ via the action $g : N_v x \mapsto N_v x^g$ and on $\Delta_2 = [N : N_w]$ via $g : N_w y \mapsto N_w y^g$ for all $x, y \in N$. Now $(x^g)(y^g)^{-1} = (xy^{-1})^g$ which in turn belongs to $N_v N_w$ if and only if $xy^{-1} \in N_v N_w$. Hence $g$ preserves adjacency in $\Gamma$. Then as $G$ is the largest subgroup of $\text{Aut}(\Gamma)$ of type SD with socle $N$ we have $g \in G_{vw}$. Thus $G_{vw} = N_F(X)$.

3). Since $X$ is abelian and acts transitively on $\Gamma(w)$ with point stabiliser $Y$ we can identify the elements of $\Gamma(w)$ with the cosets of $Y$ in $X$ such that $v$ corresponds to the coset $Y$. Furthermore, $G_w = XG_{vw}$ with $X$ acting on $\Gamma(w)$ by right multiplication and $G_{vw}$ acting by conjugation.

4). Regarding $M$ as the additive group of a field $F_M$, we note that if $T \neq \text{PSL}(2, q)$, $q$ odd, then $H$ contains a subgroup $H_\lambda$ such that the group of automorphisms of $M$ induced by $H_\lambda$ is the multiplicative group of $F_M$. Thus in these cases $N_{vw}$ contains the subgroup $\{(h, \ldots , h) \mid h \in H_\lambda\}$, which induces the multiplicative group of $F_M$ on $M^k$ and leaves $X$ invariant. (Recall $X = N_{vw} \cap M^k$ and so $X$ is invariant under $N_{vw}$.) Hence $X$ is a vector space over $F_M$. If $T = \text{PSL}(2, q)$, $q$ odd, then the group of automorphisms of $M$ induced by $H$ is an index two subgroup of the multiplicative group of $F_M$, that is, is the multiplicative group of all squares in $F_M$. Since the additive subgroup of $F_M$ generated by the $(q - 1)/2$ squares must have order dividing $q$, it follows that every element of $M$ is a sum of squares. Then as $X$ is closed under multiplication by the squares of $F_M$ and is an additive subgroup of $M^k$, it follows that $X$ is closed under the multiplicative group of $F_M$. (We are grateful to Tim Penttila for this argument.) Thus for all $T$, $X$ is a vector space over $F_M$. Now consider $Y = N_{vw} \cap M^k$. By Part (1), $Y = \{(l, \ldots , l) \mid l \in M\} < X$ and $Y \cong M$. Thus there exists a positive integer $d$ such that $|X| = |M|^{d+1}$. Since $X$ acts transitively on $\Gamma(w)$ with point stabiliser $Y$ it follows that $|\Gamma(w)| = |M|^d$.

5). Suppose that there does exist $I \subseteq \{1, \ldots , k\}$ such that $|I| \geq 2$ and every element of $X$ is constant on $I$. We may assume that $I$ is a maximal such set. By part (1), $N_w = XN_{vw}$ and as every element of $N_{vw}$ is constant on $I$ it follows that each element of $N_w$ is constant on $I$. Then as $N_w \triangleleft G_w$ and $G_w$ permutes $T = \{T_1, \ldots , T_k\}$ transitively, it follows that $I$ forms a block of imprimitivity for $G_w$ in its action on $T$. Let $P$ be the associated system of imprimitivity. By Lemma 4.4, $G_{vw} = G_w$. Hence for all blocks $I' \in P$, there exists $g \in G_{vw} \leq \{(h, \ldots , h) \mid h \in B_k\} \times S_k$ such that $I' = I^g$. Now $X^g = X$ and it follows that each element of $X$ is constant on $I'$. Arguing as before we see that each element of $N_w$ is constant on each $I' \in P$. Furthermore, by Lemma 4.4, $P$ is also a system of imprimitivity for $G_w$. Since $G_v \leq \{(t, \ldots , t) \mid t \in B\} \times S_k$ it follows that for all $g \in G_v$, each element of $N_w^g$ is constant on each $I' \in P$. As $N_w \triangleleft G_w$, it follows that for all $g \in \langle G_v, G_w \rangle$, each element of $N_w^g$ is constant on $I$. Since $\Gamma$ is connected, by Lemma 2.1, $\langle G_v, G_w \rangle = G$. Thus each element of $N_0 = \langle N_w^g \mid g \in G \rangle$ is constant on $I$. Now $N_0 \triangleleft G$ and, as $N$ is a minimal normal subgroup of $G$, we deduce that $N_0 = N$. This contradicts the fact that
\[ |I| \geq 2. \] Thus \(|I| = 1.\]

We can now prove the following theorem which is the crucial part of the proof of Theorem 1.2. Recall that \(X = N_w \cap M^k\) and \(|X| = |M|^{d+1}.\) Recall that \(\pi : \text{AGL}(d, q) \to \text{GL}(d, q)\) is the projection homomorphism, \(\Phi\) is the isomorphism defined in (3.3), \(\Phi_u\) is the homomorphism defined in (3.5), and that \(T\) is the set of simple direct factors of \(N.\)

**Theorem 4.7** Let \(\Gamma\) be as in (SDPA), \(N\) as in Lemma 4.3 and suppose that \(|X| = |M|^{d+1}\), as in Proposition 4.6. Suppose also that \(G\) is the largest subgroup of \(\text{Aut}(\Gamma)\) of type SD with socle \(N = T^k\) and let \(A = PGL(n, q),\) \(PGU(3, q),\) \(\text{Ree}(q)\) or \(\text{Sz}(q)\) such that \(\text{soc}(A) = T.\) Then the following all hold.

1. There exists \(S \subseteq \text{GF}(q)^d\) of size \(k\) which spans \(\text{GF}(q)^d\) such that \(Y\) is the group consisting of the restrictions to \(S\) of all constant functions from \(\text{GF}(q)^d\) to \(M,\) and \(X\) is generated by \(Y\) together with the set of restrictions to \(S\) of all linear functions from \(\text{GF}(q)^d\) to \(M.\) Moreover, if \(T = PSL(n, q)\) with \(n \geq 3\) then \(d = 1.\)

2. If \(A \neq \text{PGU}(3, q)\) then there exists \(K \subseteq \text{AGL}(d, q)\) such that \(G^T = K^S,\) and \(\pi(K)\) is transitive on the set of 1-spaces of \(\text{GF}(q)^d.\) Moreover, if \(A \neq \text{PGU}(3, q)\) then

\[
G = \langle N, (t, \ldots, t) \mid t \in A \rangle \rtimes \Phi^{-1}(K),
\]

while if \(A = \text{PGU}(3, q)\) then

\[
G = \langle N, (t, \ldots, t) \mid t \in A \rangle \rtimes \Phi^{-1}_u(K).
\]

**Proof.** Note that \(Y \leq X \leq M^k\) and by Proposition 4.6(3), \(G_{vw}\) acts transitively by conjugation on the set of nontrivial cosets of \(Y\) in \(X.\) Moreover, by Proposition 4.6(2), \(G_{vw} = N_F(X)\) where \(F = \{(h, \ldots, h) \mid h \in B_\alpha\} \rtimes S_k.\) By Remark 2.3(5), \(B_\alpha\) induces \(\Gamma L(m, q)\) on \(M\) and so there exists a homomorphism \(\theta : F \to \text{Aut}(M^k)\) such that \(\theta(F) \leq \Gamma L(m, q) \times S_k\) and \(\theta(G_{vw})\) fixes \(X\) and \(Y\) setwise.

Suppose first that \(T \neq PSL(n, q)\) with \(n \geq 3.\) Then \(M\) is the additive group of the field \(\text{GF}(q).\) By Proposition 4.6(4), \(X\) is a vector space over \(M\) and for each \(h \in A_\alpha,\) the element \((h, h, \ldots, h)\) induces \(\text{GF}(q)\)-scalar multiplication on \(X.\) Thus \(Z := \{(h, h, \ldots, h) \mid h \in A_\alpha\} \subseteq G_{vw}.\) If, on the other hand \(T = PSL(n, q)\) with \(n \geq 3,\) we have that \(M\) is the additive group of the vector space \(\text{GF}(q)^{n-1},\) \(A_\alpha\) induces \(\text{GL}(n-1, q)\) on \(M.\) Moreover, \(H = A_\alpha \cap T\) also induces \(\text{GL}(n-1, q)\) on \(M.\) Recall that \(N_{vw} = \{(t, \ldots, t) \mid t \in H\}\) by Lemma 4.3(2). Then for all \(h \in A_\alpha,\) there exists \(h' \in H\) such that \(h\) and \(h'\) induce the
same element of $GL(n - 1, q)$ on $M$, and so the elements $(h', \ldots, h') \in N_{vw}$ and $(h, \ldots, h) \in F$ act in the same way on $M^k$. Since $N_{vw}$ fixes $X = N_w \cap M^k$ setwise, it follows that $G_{vw} = N_F(X)$ contains $Z := \{(h, h, \ldots, h) \mid h \in A_\alpha\}$ in this case also.

Now $\theta(Z) = GL(m, q)$ (with $\theta$ as defined in the first paragraph of the proof) and so $GL(m, q) \leq \theta(G_{vw}) \leq \Gamma L(m, q) \rtimes S_k$. By Proposition 4.6 and the fact that $G_{vw}$ acts transitively on $\Gamma(w) \setminus \{v\}$ we have that $\theta(G_{vw})$ acts transitively on the set of nontrivial cosets of $Y$ in $X$. Hence $X$ and $\theta(G_{vw})$ are determined by [3, Theorem 1.1(1)]. Thus $X$ is as stated in part (1). Moreover, if $T = PSL(n, q)$ with $n \geq 3$ then $M$ has dimension $m = n - 1 \geq 2$ over $GF(q)$ and so by [3, Theorem 1.1], $d = 1$. This completes the proof of part (1).

To complete the proof of part (2) we need to determine $G_{vw}$ and $G$ from $\theta(G_{vw})$. As we have said, $\theta(G_{vw})$ is determined by [3, Theorem 1.1(1)]. To put the results there in our context we need to establish some notation.

Let $C \cong M^d$ be the group all functions from $GF(q)^d$ to $M$. Then $M^k$ is the restriction to $S$ of all elements of $C$, $Y$ is the restriction to $S$ of all constant functions in $C$ and $X$ is the restriction to $S$ of all affine functions in $C$. Also $C$ has a group of automorphisms $\langle \sigma_g \mid g \in AGL(d, q) \rangle$ where $f^{\sigma_g}(a) = f(a^{g^{-1}})$ for all $f \in C$ and $a \in GF(q)^d$. Moreover, the Frobenius automorphism $\phi$ of $GF(q)$ defines a semilinear automorphism $\tau_\phi$ of $C$ such that $f^{\phi}(a) = (f(a))^{\phi}$. Then there exists an embedding $\Psi$ of $AGL(d, q)$ such that

\[
\Psi : AGL(d, q) \rightarrow (\tau_\phi) \times (\sigma_g \mid g \in AGL(d, q))
\]

\[
g \mapsto \sigma_g \text{ if } g \in AGL(d, q) \quad \text{(4.1)}
\]

\[
\phi \mapsto \tau_\phi \sigma_\phi.
\]

Furthermore, if $K \leq AGL(d, q)$ fixes $S$ setwise then $\Psi(K)$ acts on $M^k$ inducing a subgroup of $\Gamma L(m, q) \rtimes S_k$. Note that if $A \neq PGU(3, q)$ then $\theta(\Phi^{-1}(K))$ and $\Psi(K)$ induce the same group of automorphisms of $M^k$. When $A = PGU(3, q)$ then $\theta(\Phi^{-1}(K))$ induces a group $O$ of automorphisms of $M^k$ containing multiplication by $-1$. (Recall from Remark 2.4(4) that $\varphi^e$ induces multiplication by $-1$ on $M$). Then $O/\langle -1 \rangle$ is the the group of automorphism of $M^k$ induced by $\Psi(K)$.

Since $\theta(G_{vw})$ acts transitively on the nontrivial cosets of $Y$ in $X$, [3, Theorem 1.1(1)], implies that there exists $K \leq AGL(d, q)$ with orbit $S$ such that $\pi(K)$ acts transitively on the set of 1-spaces of $GF(q)^d$ and $\theta(G_{vw}) = GL(m, q) \rtimes \Psi(K)$. Since $G_{vw} = N_F(X)$ it then follows that $G_{vw} = \theta^{-1}(GL(m, q) \rtimes \Psi(K))$. Hence if $A \neq PGU(3, q)$, we have

\[
G_{vw} = \{(h, \ldots, h) \mid h \in A_\alpha\} \rtimes \Phi^{-1}(K)
\]
while if $A = \text{PGU}(3, q)$ then
\[ G_{vw} = \{(h, \ldots, h) \mid h \in A_{\alpha}\} \rtimes \Phi_u^{-1}(K). \]
Since $N$ acts transitively on $\Delta_A$ we have that $G = NG_v$. Furthermore, as $N_v$ acts transitively on $\Gamma(v)$ we have $G = N_vG_{vw}$ and so $G = NG_{vw}$. Hence $G^T = K$ and as $A = TA_{\alpha}$, $G$ is as stated in the part (2).

**Corollary 4.8** Let $\Gamma$ be as in (SDPA). Then either $\Gamma \cong \text{Cos}(A, d)$ as obtained from Construction 3.3 or $\Gamma \cong \text{Cos}(A, d, S)$ as yielded by Construction 3.10.

**PROOF.** By Theorem 4.7 there exists a group $K \leq \Gamma GL(d, q)$ with an orbit $S$ of size $k$ such that $S$ spans $\text{GF}(q)^d$ and $\pi(K)$ acts transitively on the set of 1-spaces of $\text{GF}(q)^d$. Moreover, $Y$ is the set of restrictions to $S$ of all constant functions $f : \text{GF}(q)^d \to M$, and $X$ is generated by $Y$ and all restrictions to $S$ of linear functions $f : \text{GF}(q)^d \to M$. If $S = \text{GF}(q)^d$ then $N$, $N_v$ and $N_w$ are as in $\Gamma(A, d)$ and so by Corollary 4.2, $\Gamma \cong \Gamma(A, d)$ as obtained from Construction 3.3. If $S \subset \text{GF}(q)^d$ (a proper subset) then $N$, $N_v$ and $N_w$ are as in $\Gamma(A, d, S)$ and so by Corollary 4.2, $\Gamma \cong \Gamma(A, d, S)$ as obtained from Construction 3.10.

Hence we have completed the proof of Theorem 1.2. Corollary 1.3 then follows from Proposition 3.9 and Corollary 3.14.

**References**


