\(\frac{3}{2}\)-transitive permutation groups

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Background

\[ G \leq Sym(\Omega), \ |\Omega| \text{ finite.} \]

\(G\) is called \textbf{transitive} if for all \(\alpha, \beta \in \Omega\) there exists \(g \in G\) such that \(\alpha^g = \beta\).

\[ \begin{array}{cccc}
\alpha & \xrightarrow{g} & \beta \\
\end{array} \]

\(G\) is called \textbf{2-transitive} if \(G\) is transitive on the set of all distinct ordered pairs.

\[ \begin{array}{cccc}
\alpha \to \beta_1 & \text{and} & \alpha_1 \to \beta_1 \\
\alpha_2 \to \beta_2 & \text{and} & \alpha_1 \to \beta_2 \\
\end{array} \]
$\frac{1}{2}$-transitive

$G$ is called $\frac{1}{2}$-transitive if either $|\Omega| = 1$ or all orbits of $G$ have the same length $m > 1$.

$G$ is called $\frac{3}{2}$-transitive if it is transitive and for $\alpha \in \Omega$, $G_\alpha$ is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$.

For a transitive group $G$, we call the lengths of the orbits of $G_\alpha$ the subdegrees of $G$. 
Examples

- Any Frobenius group (the stabiliser of any pair of points is trivial.)
- Any 2-transitive group.
- Any nonregular normal subgroup of a 2-transitive group.
- $\mathrm{PSL}(2, q)$, for $q$ even, acting on $q(q - 1)/2$ points.
- $A_7$ or $S_7$ acting on 2-subsets.
A permutation group is 2-transitive if and only if its permutation character is $1 + \chi$ where $\chi$ is irreducible over $\mathbb{C}$.

A permutation group is transitive on 2-subsets if and only if its permutation character is $1 + \chi$ where $\chi$ is irreducible over $\mathbb{R}$.

A transitive permutation group is called a $\text{Ql-group}$ if its permutation character is $1 + \chi$ where $\chi$ is irreducible over $\mathbb{Q}$.

eg $\text{PSL}(2, q)$ for $q = 2^k$ and $2^k - 1$ prime, acting on $\frac{q(q-1)}{2}$ points.

Dixon: A $\text{Ql-group}$ is $\frac{3}{2}$-transitive.
The orbits of a $\frac{3}{2}$-transitive permutation group on $\Omega \times \Omega$ form a pseudocyclic association scheme.
Primitive groups

$G$ is called primitive if there is no nontrivial partition of $\Omega$ preserved by $G$.

The structure of finite primitive groups is given by the O’Nan-Scott Theorem which classifies them into eight types.

Two important types:
- almost simple: $T \leq G \leq \text{Aut}(T)$, for some finite nonabelian simple group $T$.
- affine: $G \leq \text{AGL}(d, p)$ and contains all translations.
Burnside’s Theorem

Every 2-transitive group is primitive.

Burnside’s Theorem

A finite 2-transitive group is either almost simple or affine.
Theorem (Dixon 2005)

- A $Ql$-group is primitive and $\frac{3}{2}$-transitive. Moreover, it is almost simple or of affine type.
- A $Ql$-group of affine type is a subgroup of a 2-transitive group $G$ and contains the derived group of $G$. 
$\frac{3}{2}$-transitive groups

Wielandt (1965): A $\frac{3}{2}$-transitive permutation group is either primitive or Frobenius.

Problem

Determine the primitive $\frac{3}{2}$-transitive groups.

Theorem

A primitive $\frac{3}{2}$-transitive group is either almost simple or affine.

Passman (1967-1969): Classified all soluble $\frac{3}{2}$-transitive permutation groups.
Characterising the $\text{PSL}(2, q)$ examples:

Suppose that $G$ is $3/2$-transitive but not 2-transitive and is not of affine type.

**McDermott (1977):** If $G_{\alpha,\beta} = C_2$ for all $\alpha, \beta \in \Omega$ then $G = \text{PSL}(2, 2^f)$.

**Camina (1979):** If $G_{\alpha,\beta}$ is a cyclic 2-group then $\text{soc}(G) = \text{PSL}(2, 2^f)$.

**Zieschang (1992):** If all two point stabilisers are conjugate then $\text{soc}(G) = \text{PSL}(2, 2^f)$. 
An elementary observation

- $G$ a transitive permutation group on $\Omega$.
- $r$ a prime dividing $|\Omega|$.

If $G$ has a subdegree divisible by $r$ then $G$ is not $\frac{3}{2}$-transitive.
Groups of Lie type

Lemma (Tits)

A maximal subgroup of a group of Lie type of characteristic $p$ with index coprime to $p$ is a parabolic subgroup.

Problem A

Determine the groups of Lie type of characteristic $p$ which have a primitive action with no subdegrees divisible by $p$. 
Theorem (Neumann-Praeger 1996)

Let $G \leq \text{Sym}(\Omega)$. If there exists a $k$-set $\Gamma$ of $\Omega$ such that there is no $g \in G$ with $\Gamma^g \cap \Gamma = \emptyset$ then $\Gamma$ intersects nontrivially a $G$-orbit of length at most $k^2 - k + 1$.

Lemma

Let $G$ be a permutation group with point stabiliser $H$ and let $x \in H$ have order a power of $p$. If

$$|x^G| > |x^G \cap H|^2 - |x^G \cap H| + 1$$

then $G$ has a subdegree divisible by $p$. 
Solution to Problem A

Theorem

If $G$ is a group of Lie type of characteristic $p$ which does not have a subdegree divisible by $p$, then either $|H|$ is coprime to $p$ or one of the following holds:

1. $\text{soc}(G) = \text{PSL}(2, 2^f)$ acting on $2^{f-1}(2^f - 1)$ points and either $|G : \text{PSL}(2, 2^f)|$ is odd or $f = 2$.
2. $G = \text{Sp}(2d, 2)$ acting on cosets of $O^\pm(2d, 2)$.
3. $\text{soc}(G) = \text{PSU}(3, 5)$ acting on cosets of $N_G(A_7)$.
4. $\text{soc}(G) = \text{PSp}(4, 3)$ acting on cosets of $N_G(2^4.\Omega^-(4, 2))$.

This leaves parabolics and the case where $|H|$ coprime to $p$. 
Classification of almost simple examples

Theorem

Let $G$ be an almost simple $\frac{3}{2}$-transitive permutation group. Then one of the following holds:

1. $G$ is 2-transitive.
2. $\text{soc}(G) = \text{PSL}(2, 2^f)$ acting on $2^{f-1}(2^f - 1)$ points and either $G = \text{soc}(G)$ or $f$ is prime.
3. $G$ is $S_7$ or $A_7$ acting on 2-subsets.