Point regular automorphism groups of generalised quadrangles

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AustMS 2010, University of Queensland
Generalised quadrangles

A generalised quadrangle is a point-line incidence geometry $Q$ such that:

1. any two points lie on at most one line, and
2. given a line $\ell$ and a point $P$ not incident with $\ell$, $P$ is collinear with a unique point of $\ell$.

If each line is incident with $s + 1$ points and each point is incident with $t + 1$ lines we say that $Q$ has order $(s, t)$. 
The Classical GQ’s

<table>
<thead>
<tr>
<th>Name</th>
<th>Order</th>
<th>Automorphism group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(3, q^2)$</td>
<td>$(q^2, q)$</td>
<td>$\text{PΓU}(4, q)$</td>
</tr>
<tr>
<td>$H(4, q^2)$</td>
<td>$(q^2, q^3)$</td>
<td>$\text{PΓU}(5, q)$</td>
</tr>
<tr>
<td>$W(3, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PΓSp}(4, q)$</td>
</tr>
<tr>
<td>$Q(4, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PΓO}(5, q)$</td>
</tr>
<tr>
<td>$Q^-(5, q)$</td>
<td>$(q, q^2)$</td>
<td>$\text{PΓO}^-(6, q)$</td>
</tr>
</tbody>
</table>

Take a sesquilinear or quadratic form on a vector space

- Points: totally isotropic 1-spaces
- Lines: totally isotropic 2-spaces
A group $G$ acts regularly on a set $\Omega$ if $G$ is transitive on $\Omega$ and $G_\omega = 1$ for all $\omega \in \Omega$.

The study of regular groups of automorphisms of combinatorial structures has a long history:

- A graph $\Gamma$ is a Cayley graph if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup.
- A group $G$ acts regularly on a symmetric block design if and only if $G$ has a difference set.
Groups acting regularly on GQ’s

Ghinelli (1992): A Frobenius group or a group with nontrivial centre cannot act regularly on a GQ of order \((s, s)\), \(s\) even.

de Winter, K. Thas (2006): A finite thick GQ with a regular abelian group of automorphisms is a Payne derivation of a TGQ of even order.

Yoshiara (2007): No GQ of order \((s^2, s)\) admits a regular group.
A $p$-group $P$ is called special if $Z(P) = P' = \Phi(P)$.
A special $p$-group is called extraspecial if $|Z(P)| = p$.
Extraspecial $p$-groups have order $p^{1+2n}$, and there are two isomorphism classes for each order.
For $p$ odd, one has exponent $p$ and one has exponent $p^2$.
$Q_8$ and $D_8$ are extraspecial of order 8.
Theorem

If $Q$ is a finite classical GQ with a regular group $G$ of automorphisms then

- $Q = Q^{-}(5, 2)$, $G$ extraspecial of order 27 and exponent 3.
- $Q = Q^{-}(5, 2)$, $G$ extraspecial of order 27 and exponent 9.
- $Q = Q^{-}(5, 8)$, $G \cong C_{513} \rtimes C_9$.

Use classification of all regular subgroups of primitive almost simple groups by Liebeck, Praeger and Saxl (2010)

Alternative approach independently done by de Winter, K. Thas and Shult.
Payne derived quadrangles

Begin with $W(3, q)$ defined by the alternating form

$$\beta(x, y) := x_1 y_4 - y_1 x_4 + x_2 y_3 - y_2 x_3$$

Let $x = \langle (1, 0, 0, 0) \rangle$.

Define a new GQ, $Q^x$, with

- **points:** the points of $W(3, q)$ not collinear with $x$, that is, all $\langle (a, b, c, 1) \rangle$
- **lines:**
  - (a) lines of $W(3, q)$ not containing $x$, and
  - (b) the hyperbolic 2-spaces containing $x$.

$Q^x$ is a GQ of order $(q - 1, q + 1)$ known as the Payne derivation of $W(3, q)$. 
Automorphisms

\[ \operatorname{PGSp}(4, q)_x \leq \operatorname{Aut}(Q^x) \]

Grundhöfer, Joswig, Stroppel (1994): Equality holds for \( q \geq 5 \).

For \( q = 3 \), \( Q^x \cong Q^{-}(5, 2) \).

Note \( \operatorname{Sp}(4, q)_x \) consists of all matrices

\[
\begin{pmatrix}
  \lambda & 0 & 0 \\
  u^T & A & 0 \\
  z & v & \lambda^{-1}
\end{pmatrix}
\]

with

\[
A \in \operatorname{GL}(n, q), z \in \operatorname{GF}(q), u, v \in \operatorname{GF}(q)^2,
\]

\[ AJA^T = J, u = \lambda v JA^T \]

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).
An obvious regular subgroup

Let

\[ E = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \mid a, b, c \in \text{GF}(q) \right\} \triangleleft \Gamma \text{Sp}(4, q)_x \]

- \(|E| = q^3\),
- acts regularly on the points of \(Q^x\),
- elementary abelian for \(q\) even,
- special of exponent \(p\) for \(q\) odd (Heisenberg group).
But there are more . . .

<table>
<thead>
<tr>
<th>$q$</th>
<th># conjugacy classes</th>
<th># isomorphism classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2 (this is $\mathbb{Q}^-(5, 2)$)</td>
</tr>
<tr>
<td>4</td>
<td>58</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>231</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>23</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>7</td>
<td>-</td>
</tr>
</tbody>
</table>
For $\alpha \in \mathbb{GF}(q)$, let

$$\theta_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix} \in \text{Sp}(4, q)_x.$$
Construction 1

Let \( \{\alpha_1, \ldots, \alpha_f\} \) be a basis for \( \text{GF}(q) \) over \( \text{GF}(p) \).

\[
P = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & 0 & 1 \end{pmatrix}, \theta_{\alpha_1}, \ldots, \theta_{\alpha_f} \right\rangle
\]

- \( P \) acts regularly on points
- \( P \) is not normal in \( \text{Aut}(Q^x) \)
- nonabelian for \( q > 2 \)
- special of exponent 9 for \( p = 3 \)
- \( P \cong E \) for \( p \geq 5 \)
- for \( p \geq 5 \), any regular subgroup on \( W(3, p) \) must be isomorphic to \( E \).
Construction 2

Let $U \oplus W$ be a decomposition of $\text{GF}(q)$, $q = p^f$, $f \geq 2$.
Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for $U$ over $\text{GF}(p)$.

$$S_{U,W} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ -w & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & w & 1 \end{pmatrix}, \theta_{\alpha_1}, \ldots, \theta_{\alpha_k} \right\rangle$$

- $S_{U,W}$ acts regularly on points,
- for $U$ a 1-space:
  - $E \not\cong S_{U,W} \not\cong P$
  - $S_{U,W}$ is not special
Summing up

The class of groups that can act regularly on the set of points of a GQ is richer/wilder than previously thought.

Such a group can be

- a nonabelian 2-group,
- a 2-group of nilpotency class 7,
- $p$-groups that are not Heisenberg groups,
- $p$-groups that are not special.