Generalised quadrangles with a group acting primitively on points and lines

Michael Giudici

joint work with John Bamberg, Joy Morris, Gordon F. Royle and Pablo Spiga

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Generalised polygons

A generalised $n$-gon is a point-line incidence structure whose incidence graph has diameter $n$ and girth $2n$.

- **order** $(s, t)$ if every point lies on $t + 1$ lines and every line has $s + 1$ points.
- **thick** if $s, t \geq 2$. 
Generalised polygons were introduced by Tits as a model for rank 2 simple groups of Lie type.

Feit-Higman (1964): Finite and thick implies \( n \in \{2, 3, 4, 6, 8\} \).
Classical examples

- $n = 2$: complete bipartite graphs
- $n = 3$: projective planes: Desarguesian planes $\text{PG}(2, q)$.

Many other examples of projective planes known.
## $n = 4$: Generalised quadrangles

<table>
<thead>
<tr>
<th>Quadrangle</th>
<th>Order</th>
<th>Aut group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(3, q^2)$</td>
<td>$(q^2, q)$</td>
<td>$\text{PΓU}(4, q)$</td>
</tr>
<tr>
<td>$H(4, q^2)$</td>
<td>$(q^2, q^3)$</td>
<td>$\text{PΓU}(5, q)$</td>
</tr>
<tr>
<td>$W(3, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PΓSp}(4, q)$</td>
</tr>
<tr>
<td>$Q(4, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PΓO}(5, q)$</td>
</tr>
<tr>
<td>$Q^-(5, q)$</td>
<td>$(q, q^2)$</td>
<td>$\text{PΓO}^-(6, q)$</td>
</tr>
</tbody>
</table>

Take a vector space equipped with a hermitian, alternating or quadratic form:

- points are the totally singular 1-spaces
- lines are the totally singular 2-spaces

Many other examples of generalised quadrangles known.
Generalised hexagons and octagons

- $n = 6$: generalised hexagons: associated with $G_2(q)$ and $^3D_4(q)$.
- $n = 8$: generalised octagons: associated with $^2F_4(q)$.

These are the only known examples.
A generalised polygon is flag-transitive if its automorphism group is transitive on incident point-line pairs.

All classical examples are flag-transitive and primitive on both points and lines.

Buekenhout-van Maldeghem (94)

- A thick generalised polygon with a group acting distance-transitively on the points is either classical or the unique generalised quadrangle of order $(3, 5)$.
- Distance-transitive implies primitive on points.
Projective planes

Let $\pi$ be a projective plane of order $n$, $G \leq \text{Aut}(\pi)$.

- **Ostrom-Wagner (1959):** 2-transitive on points implies $\pi$ is Desarguesian.
- **Higman-McLaughlin (1961):** Flag-transitive implies point-primitive.
- **Kantor (1987):** Point-primitive implies $\pi$ Desarguesian, or $G$ is regular or Frobenius with $n^2 + n + 1$ a prime.
- **Gill (2007):** Transitive on points implies $\pi$ Desarguesian or every minimal normal subgroup of $G$ is elementary abelian.
- **K. Thas and Zagier (2008):** A non-Desarguesian, flag-transitive plane has at least $4 \times 10^{22}$ points.
- **Gill (?):** An insoluble group acting transitively on points of a non-Desarguesian plane has $G/O(G) \cong \text{SL}(2,5)$ or $\text{SL}(2,5)$.
Generalised hexagons and octagons

Schneider-van Maldeghem (2008): A group acting flag-transitively, point-primitively and line-primitively on a generalised hexagon or octagon is almost simple of Lie type.
A generalised quadrangle is a point-line incidence geometry $\mathcal{Q}$ such that:

1. any two points lie on at most one line, and
2. given a line $\ell$ and a point $P$ not incident with $\ell$, $P$ is collinear with a unique point of $\ell$.

A GQ of order $(s, t)$ has $(s + 1)(st + 1)$ points and $(t + 1)(st + 1)$ lines.
O’Nan-Scott Theorem

Primitive groups are divided into 8 types.

The possible O’Nan-Scott types for two faithful primitive actions of a group are:

<table>
<thead>
<tr>
<th>Primitive type on $\Omega_1$</th>
<th>Primitive type on $\Omega_2$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Almost Simple</td>
<td>Almost Simple</td>
<td></td>
</tr>
<tr>
<td>HA (affine)</td>
<td>HA</td>
<td>$</td>
</tr>
<tr>
<td>HS</td>
<td>HS</td>
<td>$</td>
</tr>
<tr>
<td>HC</td>
<td>HC</td>
<td>$</td>
</tr>
<tr>
<td>TW</td>
<td>TW, SD, CD, PA</td>
<td></td>
</tr>
<tr>
<td>SD</td>
<td>TW, SD, PA</td>
<td></td>
</tr>
<tr>
<td>CD</td>
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<td></td>
</tr>
<tr>
<td>PA</td>
<td>TW, SD, CD, PA</td>
<td></td>
</tr>
</tbody>
</table>
Some useful lemmas

Given a thick GQ of order \((s, t)\):

- If \(s = t\) and the number of points is equal to \(\delta^k\) with \(k \geq 2\), then \(s = 7\), \(\delta = 20\) and \(k = 2\).
- If \(s\) and \(t\) not coprime then an automorphism of order 2 fixes either a point or a line.

First uses number theoretic result of Nagell and Ljunggren that if \((s + 1)(s^2 + 1) = \delta^k\) then \(s = 7\), \(\delta = 20\) and \(k = 2\).
These results help eliminate many cases.

<table>
<thead>
<tr>
<th>Primitive type on points</th>
<th>Primitive type on lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>AS</td>
<td>AS</td>
</tr>
<tr>
<td>HA</td>
<td>HA</td>
</tr>
<tr>
<td>HS</td>
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</tbody>
</table>
PA on points and lines

Here $\mathcal{P} = \Delta^k$ and $\mathcal{L} = \Gamma^k$ with $|\Gamma| > |\Delta|$.

$G$ has unique minimal normal subgroup $N = T^k$ and $N_\ell = S^k$.

- $Q'$: points and lines fixed by $S^{k-1} \times 1$.
- $Q''$: points and lines fixed by $N_\ell$.
- Obtain GQ's

  $$ Q'' \subset Q' \subset Q $$

  of orders $(s, t'')$, $(s, t')$ and $(s, t)$ with $1 \leq t'' < t' < t$.

Payne-Thas implies $s^2 = t$ and allows us to obtain a contradiction.
Theorem

If $G$ acts primitively on the points and lines of a thick GQ then $G$ is almost simple.

Two known flag-transitive GQ's that are primitive on points but imprimitive on lines:

- unique GQ of order $(3,5)$,
- GQ of order $(15,17)$ arising from Lunelli-Sce hyperoval.

In both cases, action on points is primitive of type HA.
Theorem

An almost simple group with socle a sporadic cannot be point-primitive and line-primitive on a thick GQ.

$G$ must have maximal subgroups of index $(s + 1)(st + 1)$ and $(t + 1)(st + 1)$ with

- $s, t \geq 2$;
- $s \leq t^2$ and $t \leq s^2$;
- $s + t$ dividing $st(st + 1)$.

Only possibility is $Ru$ with $s = t = 57$ which can be eliminated.
$A_n$ and $S_n$

$G_P$ acting on $\{1, \ldots, n\}$ is either

- intransitive (points can be identified with subsets);
- imprimitive (points can be identified with partitions); or
- primitive.

If flag-transitive, $|G| \leq |G_P|^6$. Bounds on orders of primitive groups then imply $n \leq 47$.

**Theorem**

An almost simple group with socle $A_n$ cannot be flag-transitive and point-primitive on a GQ unless $n = 6$.

$A_6 \cong S_p(4, 2)'$
Theorem

If $G$ is flag-transitive, point-primitive and line-primitive on a GQ then $G$ is an almost simple group of Lie type.