On finite edge-primitive and edge-quasiprimitive graphs

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Abstract

Many famous graphs are edge-primitive, for example, the Heawood graph, the Tutte–Coxeter graph and the Higman–Sims graph. In this paper we systematically analyse edge-primitive and edge-quasiprimitive graphs via the O’Nan–Scott Theorem to determine the possible edge and vertex actions of such graphs. Many interesting examples are given and we also determine all $G$-edge-primitive graphs for $G$ an almost simple group with socle $\text{PSL}(2, q)$.

1 Introduction

Let $\Gamma$ be a finite connected graph and $G \leq \text{Aut}(\Gamma)$. We say that $\Gamma$ is $G$-edge-primitive if $G$ acts primitively on the set of edges of $\Gamma$, that is, if $G$ preserves no nontrivial partition of the edge set. If $\Gamma$ is $\text{Aut}(\Gamma)$-edge-primitive we call $\Gamma$ edge-primitive. The aim of this paper is to initiate a systematic study of edge-primitive graphs and the wider class of edge-quasiprimitive graphs, that is graphs with a group of automorphisms which acts quasiprimively on edges. (A transitive permutation group is said to be quasiprimitive if every nontrivial normal subgroup is transitive).

The Atlas [3] notes many edge-primitive graphs with a sporadic simple group as a group of automorphisms. These include the Hoffman–Singleton and Higman–Sims graphs, and the rank three graphs of the sporadic simple groups $J_2$, $\text{McL}$, $\text{Ru}$, $\text{Suz}$ and $\text{Fi}_{23}$. Weiss [18] has determined all edge-primitive graphs of valency three. These are the complete bipartite graph $K_{3,3}$, the Heawood graph, the Biggs–Smith cubic distance-transitive graph on 102 vertices and the Tutte–Coxeter graph (also known as Tutte’s 8-cage or the Levi graph). All but the Biggs–Smith graph are bipartite. We say that $\Gamma$ is $s$-arc-transitive if the automorphism group acts transitively on the set of $s$-arcs of

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Γ, that is, on the set of \((s + 1)\)-tuples \((v_0, v_1, \ldots, v_s)\) where \(v_i\) is adjacent to \(v_{i+1}\) and \(v_i \neq v_{i+2}\). Of the four edge-primitive cubic graphs, \(K_{3,3}\) is 3-arc-transitive, the next two are 4-arc-transitive while the Tutte–Coxeter graph is 5-arc-transitive.

Whereas any primitive permutation group with a nontrivial self-paired orbital gives rise to a vertex-primitive graph, the existence of edge-primitive graphs is far more restrictive. Given a group \(G\) there is a \(G\)-edge-primitive graph if and only if there exists a maximal subgroup \(E\) of \(G\) with an index two subgroup properly contained in some corefree subgroup \(H\) of \(G\) with \(H \neq E\) (see Proposition 2.5 and Lemma 3.4).

One of the main motivations for our study of edge-primitive and edge-quasiprimitive graphs is the study of graph decompositions [9]. Given a graph \(Γ\) and a group of automorphisms \(G\), we say that a partition \(P\) of the edge set is a \(G\)-transitive decomposition if \(P\) is \(G\)-invariant and \(G\) acts transitively on \(P\). A \(G\)-transitive decomposition \(P\) of a graph \(Γ\) is called a homogeneous factorisation if the kernel of the action of \(G\) on \(P\) is vertex-transitive. Homogeneous factorisations have been studied in \([7, 8, 13]\). Let \(Γ\) be a \(G\)-edge-transitive graph. Then \(Γ\) is \(G\)-edge-primitive if and only if \(G\) has no \(G\)-transitive decompositions. If \(G\) is edge-quasiprimitive then the \(G\)-transitive decompositions of \(Γ\) are not homogeneous factorisations. Conversely, if none of the \(G\)-transitive decompositions of \(Γ\) are homogeneous factorisations then the kernel of each \(G\)-transitive decomposition is vertex-intransitive.

If \(Γ\) is a bipartite graph with a vertex-transitive group of automorphisms \(G\), then \(G\) has a normal subgroup \(G^+\) of index two which fixes each of the bipartite halves setwise. We say that a transitive group \(G\) is biprimitive if it is imprimitive and all nontrivial systems of imprimitivity have precisely two parts, while we say that \(G\) is biquasiprimitive if \(G\) is not quasiprimitive and every normal subgroup has at most two orbits. We note here that some authors’ definition of biprimitive as a transitive permutation group \(G\) with index two subgroup \(G^+\) acting primitively on both of its orbits is not equivalent to ours. All our biprimitive groups are biprimitive in this sense but not all biprimitive groups in this alternative sense are biquasiprimitive. For example \(S_n \times S_2\) acting imprimitively on \(2n\) points for \(n \geq 3\) has a system of imprimitivity with \(n\) parts of size 2 while the index two subgroup \(S_n\) acts primitively on each of its orbits. Given property \(P\), we say that a graph \(Γ\) with a group of automorphisms \(G\) is \(G\)-locally \(P\) if for each vertex \(v\), the vertex stabiliser \(G_v\) has property \(P\) on the set \(Γ(v)\) of all vertices adjacent to \(v\). In particular, \(Γ\) is called \(G\)-locally primitive if \(G_v\) acts primitively on \(Γ(v)\) for all vertices \(v\).

For any positive integer \(n\) and prime \(p\), the star \(K_{1,n}\) and the cycle \(C_p\) are both edge-primitive. We call these two examples trivial. Disconnected edge-primitive graphs are easily reduced to connected ones (see Lemma 3.1). We see in Lemma 3.4 that except for the trivial examples, edge-primitivity implies arc-transitivity.

Let \(Γ\) be a connected \(G\)-arc-transitive graph and let \(B\) be a \(G\)-invariant partition of \(VT\). We define the quotient graph \(Γ_B\) to be the graph with vertex set \(B\) such that \(B, C \in \mathcal{B}\) are adjacent if and only if \(Γ\) has an edge \(\{v, w\}\) with \(v \in B\) and \(w \in C\). It easily follows that \(Γ_B\) is arc-transitive. We are interested in the special case where for an arc \((B, C)\) of \(Γ_B\), there is only one arc \((v, w)\) of \(Γ\) with \(v \in B\) and \(w \in C\). In this case we call \(Γ\) a spread of \(Γ_B\).

We will see in Lemma 3.5 that if \(G\) is edge-primitive and vertex-transitive then it is either vertex-quasiprimitive or vertex-biquasiprimitive on vertices. In fact we can reduce to the vertex-primitive or vertex-biprimitive cases.
Theorem 1.1. Let $\Gamma$ be a connected nontrivial $G$-edge-primitive graph. Then $\Gamma$ is $G$-arc-transitive, and one of the following holds.

1. $\Gamma$ is $G$-vertex-primitive.
2. $\Gamma$ is $G$-vertex-biprimitive.
3. $\Gamma$ is a spread of a $G$-edge-primitive graph which is $G$-locally imprimitive.

Conversely, a $G$-edge-primitive, $G$-locally imprimitive graph $\Sigma$ is a quotient graph of a larger $G$-edge-primitive graph $\Gamma$ with $G^E \Sigma \cong G^E \Gamma$.

This reduces the study of edge-primitive graphs to those which are also vertex-primitive or vertex-biprimitive.

The actions of primitive permutation groups are described by the O’Nan–Scott Theorem. We follow the subdivision in [15] of primitive groups into 8 types and these are described in Section 4. By playing the edge-primitive action of $G$ against the vertex-primitive action of $G$ or $G^+$ we see that the possible actions for edge-primitive graphs are quite restrictive.

Theorem 1.2. Let $\Gamma$ be a connected nontrivial $G$-edge-primitive graph with $G^E \Gamma$ primitive of type $X$ such that $G^V \Gamma$ is either primitive or biprimitive. Then one of the following holds.

1. $\Gamma = K_{n,n}$.
2. $G^V \Gamma$ is primitive of type $X$ and $X \in \{\text{AS, PA}\}$.
3. $G^V \Gamma$ is biprimitive and $G^+$ is primitive of type $X$ on each orbit with $X \in \{\text{AS, PA}\}$.
4. $G^E \Gamma$ is of type SD or CD, $\Gamma$ is bipartite and arises from Construction 5.6, and $G^+$ is primitive of type CD on each orbit.

We see in Sections 2 and 5 that examples exist in all cases. Moreover, we can find $G$-locally imprimitive examples in each case. A characterisation of all groups which act edge-primitively on $K_{n,n}$ is given in Theorem 3.7. We also see in Proposition 6.15 that the existence of $G$-edge-primitive graphs with $G$ of type PA relies on the existence of edge-primitive graphs where the action on edges is of type AS.

We undertake much of our analysis in the context of vertex-quasiprimitive graphs and only specialise to the edge-primitive case when we are able to obtain stronger conclusions. There are, however, a couple of notable differences between the two classes. There are many $G$-edge-quasiprimitive graphs with $G$ not vertex-transitive, for example any bipartite graph with an edge-transitive simple group $G$ of automorphisms is $G$-edge-quasiprimitive while $G$ has two orbits on vertices. Vertex-transitive, edge-quasiprimitive graphs are still either vertex-quasiprimitive or vertex-biquasiprimitive but we are no longer able to reduce to the vertex-primitive or vertex-biprimitive cases. Theorem 6.12 is an analogue of Theorem 1.2 in the $G$-vertex-transitive, $G$-edge-quasiprimitive case.

It appears feasible to determine all edge-primitive graphs for certain families of almost simple groups, for example, for low rank groups of Lie type. We begin this process in Section 8 by determining all $G$-edge-primitive graphs where $\text{soc}(G) = \text{PSL}(2,q)$. The socle (denoted $\text{soc}(G)$) of a group $G$ is the product of all of its minimal normal subgroups.
Theorem 1.3. Let $\Gamma$ be a $G$-edge-primitive graph with $soc(G) = PSL(2,q)$, such that $q = p^f$ for some prime $p$ and $q \neq 2, 3$. Then either $\Gamma$ is complete and $G$ is listed in Table 2, or $\Gamma$ and $G$ are given in Table 1.

In some rows of Table 1 we just state the edge stabiliser $E$ and vertex stabiliser $H$ along with $H \cap E$ as by Proposition 2.5, a $G$-edge-transitive graph is uniquely determined by the vertex stabiliser and edge stabiliser. Note for the first two examples $PGL(2,7) \cong Aut(PSL(3,2))$, for the fourth example note $P\Gamma L(2,9) \cong Aut(PSp(4,2))$, while for the eighth example $PSL(2,25) \cong P\Omega^-(4,5)$. Apart from complete graphs and $K_{6,6}$, we get two infinite families and seven sporadic examples. All of the graphs listed in Table 1 are 2-arc-transitive except for the eighth one.

### 2 Some examples

If $G \leq S_n$ acts arc-transitively on $K_n$ then $G$ is 2-transitive on vertices. Moreover, $G$ is edge-primitive if and only if $G$ acts primitively on 2-subsets. The following theorem, which is essentially [17, Theorem 6], classifies all such $G$.

**Theorem 2.1.** Let $G$ be a 2-transitive subgroup of $S_n$ such that $G$ is primitive on 2-subsets. Then $G$ and $n$ are as in Table 2.

**Proof.** By Burnside’s Theorem (see for example [5, Theorem 4.1B]), $G$ is either almost simple or a subgroup of $AGL(d,p)$ with $n = p^d$ for some prime $p$. Sibley [17] classified all $G$-transitive decompositions of $K_n$ for $G$ a 2-transitive simple group and so this yields a classification of almost simple groups acting edge-primitively on $K_n$. Suppose now that $G \leq AGL(d,p)$ and let $u, v$ be a pair of points of $AG(d,p)$. Then $\{u, v\}$ lies on a unique line $l$ and so $G_{\{u,v\}} \leq G_l \leq G_B \leq G$, where $B$ is the parallel class containing $l$. Thus for $d \geq 2$, $G$ is not primitive on 2-subsets. Note that this includes $A_4$ and $S_4$. When $d = 1$, there is a unique parallel class and $G_{\{u,v\}} \cong C_2$. In this case, $G$ is primitive on 2-subsets if and only if $p = 2$ or 3. Here $G \cong S_2, S_3$ respectively. \(\square\)
Table 2: 2-transitive groups which are primitive on 2-subsets

<table>
<thead>
<tr>
<th>$n$</th>
<th>$G$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$S_n$</td>
<td>$n \neq 4$</td>
</tr>
<tr>
<td>$n$</td>
<td>$A_n$</td>
<td>$n \geq 5$</td>
</tr>
<tr>
<td>$q + 1$</td>
<td>$\text{soc}(G) = \text{PSL}(2, q)$</td>
<td>$q \geq 7$ $G \neq \text{PSL}(2, 7), \text{PSL}(2, 9), \text{PGL}(2, 9)$ or $\text{PSL}(2, 11)$.</td>
</tr>
<tr>
<td>$q^2 + 1$</td>
<td>$\text{soc}(G) = \text{Sz}(q)$</td>
<td>$q = 2^{2d+1}$</td>
</tr>
<tr>
<td>11</td>
<td>$\text{PSL}(2, 11)$</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$M_{11}$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$M_{11}$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$M_{12}$</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>$M_{22}, \text{Aut}(M_{22})$</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>$M_{23}$</td>
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<tr>
<td>24</td>
<td>$M_{24}$</td>
<td></td>
</tr>
<tr>
<td>176</td>
<td>$HS$</td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>$Co_3$</td>
<td></td>
</tr>
</tbody>
</table>

There are many geometrical constructions of edge-primitive graphs with the following being just a couple.

**Example 2.2.** Let $T = \text{PSL}(d, q)$ for $d \geq 3$ and $G = \text{Aut}(T)$. Let $\Delta_1$ be the set of $r$-dimensional subspaces of a $d$-dimensional vector space over $\text{GF}(q)$ with $1 \leq r < d/2$ and let $\Delta_2$ be the set of $(d - r)$-dimensional subspaces. We define $\Gamma$ to be the bipartite graph with vertex set $\Delta_1 \cup \Delta_2$ with adjacency given by inclusion. Then $G \leq \text{Aut}(\Gamma)$ and acts biprimitively on vertices such that the stabiliser $G^+$ of the bipartition is equal to $\text{PGL}(d, q)$. Moreover, the stabiliser $E$ of an edge is a maximal subgroup of $G$ and so $\Gamma$ is $G$-edge-primitive. When $(d, r) = (3, 1)$, the graph obtained is 4-arc transitive and when $(d, r, q) = (3, 1, 2)$, the graph obtained is the Heawood graph.

Alternatively, we can define an $r$-space to be adjacent to a $(d - r)$-space if they are complementary. This also gives us a $G$-edge-primitive graph with $G$ acting biprimitively on vertices and when $(d, r, q) = (3, 1, 2)$ we get the co-Heawood graph.

**Example 2.3.** Let $V$ be a 4-dimensional vector space over $\text{GF}(q)$ with $q$ even and let $B$ be a nondegenerate alternating form. Let $\Delta_1$ be the set of totally isotropic 1-spaces and $\Delta_2$ be the set of totally isotropic 2-spaces. Define $\Gamma$ to be the graph with vertex set $\Delta_1 \cup \Delta_2$ and adjacency defined by inclusion. Then $\text{PGL}(4, q)$ is an edge-transitive group of automorphisms of $\Gamma$ but has two orbits on vertices. Let $\tau$ be a duality of the polar space interchanging $\Delta_1$ and $\Delta_2$. Then $G = \langle \text{PGL}(4, q), \tau \rangle$ is an arc-transitive group of automorphisms of $\Gamma$ which is vertex-biprimitive. Moreover, an edge stabiliser $G_e$ is a maximal subgroup of $G$ and so $\Gamma$ is $G$-edge-primitive. When $q = 2$, $\Gamma$ is the Tutte–Coxeter graph.

There are also many other geometrical constructions of infinite families of edge-primitive graphs involving sesquilinear or quadratic forms. We give one such example here.
Example 2.4. Let $V$ be a vector space of dimension $d$ over the field $GF(q)$, with $q = 3$ or 5, and let $Q$ be a nondegenerate quadratic form on $V$ with associated bilinear form $B$. Let $\Gamma$ be the graph whose vertex set is the set of all nonsingular 1-spaces upon which the quadratic form is a square with adjacency given by orthogonality with respect to $B$. By Witt’s Lemma, the group $\Gamma = \text{PO}(d,q)$ of all isometries of $Q$ is an arc-transitive automorphism group of $\Gamma$.

Let $e = \{\langle v \rangle, \langle w \rangle\}$ be an edge of $\Gamma$. If $q = 5$ then $\langle v, w \rangle$ is a hyperbolic line while if $q = 3$ then $\langle v, w \rangle$ is anisotropic. Moreover, in both cases $\langle v \rangle$, $\langle w \rangle$ are the only 1-spaces of $\langle v, w \rangle$ upon which $Q$ is a square. Thus $G_e = G_{\langle v, w \rangle}$. By [10], it follows that if $q = 5$ then $G_e$ is maximal in $G$ except when $d = 4$ and $Q$ is hyperbolic. Also, if $q = 3$ then $G_e$ is maximal in $G$ except when $d = 4$ or 5.

Edge-primitive graphs can be defined via group theoretic means using the coset graph construction. Let $G$ be a group with a core-free subgroup $H$. Let $g \in G$ such that $g$ does not normalise $H$ and $g^2 \in H$. We define the coset graph $\Gamma = \text{Cos}(G,H,HgH)$ to have vertex set, the set $[G : H]$ of right cosets of $H$ in $G$ with two vertices $Hx,Hy$ being adjacent if and only if $xy^{-1} \in HgH$. The graph $\Gamma$ is connected if and only if $\langle H, g \rangle = G$. Moreover, $G$ acts as an arc-transitive group of automorphisms of $\Gamma$ via right multiplication. The valency of $\Gamma$ is $|H : H \cap H^g|$ while the stabiliser of the edge $\{H, Hg\}$ is $\langle H \cap H^g, g \rangle$. Conversely, suppose that $\Gamma$ is a graph with adjacent vertices $v$ and $w$. Let $G \leq \text{Aut}(\Gamma)$ be arc-transitive and let $g \in G$ interchange $v$ and $w$. Then $\Gamma \cong \text{Cos}(G, G_v, G_vgG_v)$. We have the following characterisation of arc-transitive edge-primitive graphs.

Proposition 2.5. Let $G$ be a group with a maximal subgroup $E$. Then there exists a $G$-edge-primitive, arc-transitive graph $\Gamma$ with edge stabiliser $E$ if and only if $E$ has a subgroup $A$ of index two, and $G$ has a corefree subgroup $H$ such that $A < H \neq E$; in this case $\Gamma = \text{Cos}(G,H,HgH)$ for some $g \in E \setminus A$.

Proof. Suppose first that $G, E, A, H$ and $g$ are as in the statement. Since $E$ is maximal in $G$ and $H$ is not contained in $E$ it follows that $E < \langle H, g \rangle = G$. As $H$ is corefree in $G$ we have that $g$ does not normalise $H$. Let $\Gamma = \text{Cos}(G,H,HgH)$, let $v = H$, $w = Hg$ and $e = \{v, w\}$. Then $\Gamma$ is connected, $G_v = H$, $G_w = H^g$, $G_{vw} = H \cap H^g$ and $G_e = \langle H \cap H^g, g \rangle$. Since $g$ does not normalise $H$, but does normalise $A$ we have $A \leq H \cap H^g < H$ and so $E \leq \langle H \cap H^g, g \rangle = G_e$. The maximality of $E$ implies that $G_e = E$ and $\Gamma$ is edge-primitive.

Conversely, suppose that $\Gamma$ is a $G$-arc-transitive, $G$-edge-primitive graph. Let $e = \{v, w\}$ be an edge of $\Gamma$. Then $H = G_e$ is corefree in $G$. Since $G$ is arc-transitive, there exists $g \in G$ such that $v^g = w$ and $w^g = v$. Moreover, $\Gamma \cong \text{Cos}(G,H,HgH)$. Now $G_{vw} = H \cap H^g$ which is an index two subgroup of $G_e = \langle H \cap H^g, g \rangle$. Since $G$ is edge-primitive, $E = G_e$ is maximal in $G$ and $A = H \cap E = G_{vw}$ has index 2 in $E$.

We also have the following lemma.

Lemma 2.6. Let $\Gamma = \text{Cos}(G,H,HgH)$. Then for any subgroup $L \leq G$ such that $H \cap H^g < L < H$, the graph $\text{Cos}(G,L, LgL)$ is a spread of $\Gamma$. 

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Proof. Let $v$ be the vertex of $\Sigma = \text{Cos}(G, L, LgL)$ corresponding to $L$ and $w$ the vertex adjacent to $v$ corresponding to $Lg$. Then $B = v^H$ is a block of imprimitivity for $G$ on $V\Sigma$ containing $v$ and the corresponding block containing $w$ is $B^g$. Let $B = \{B^k \mid k \in G\}$. Since $\Gamma$ is $G$-arc-transitive, so is $\Sigma_B$ and $H$ is the stabiliser of the vertex of $\Sigma_B$ given by the block $B$. Hence $\Sigma_B = \text{Cos}(G, H, HgH) = \Gamma$. Now the stabiliser of the block $B^g$ is $H^g$ and $H \cap H^g < L$. Let $(x, y)$ be an arc of $\Sigma$ with $x \in B$ and $y \in B^g$. Then there exists $h \in G$ mapping $v$ to $x$ and $w$ to $y$. Since $B$ is a block of imprimitivity, $h \in H \cap H^g < L$ and so $h \in L \cap L^g$. Thus $h$ fixes $v$ and $w$ and so $\{v, w\}$ is the only edge between the blocks $B$ and $B^g$. Hence $\Sigma$ is a spread of $\Gamma$. \qed

One easy way of constructing edge-primitive graphs is to look for novelty maximal subgroups. Given a group $G$ with a normal subgroup $N$, we say that a maximal subgroup $E$ of $G$ not containing $N$ is a novelty if $E \cap N$ is not maximal in $N$. Thus if $N$ is an index two subgroup of $G$, every novelty maximal subgroup $E$ of $G$ gives rise to a $G$-edge-primitive graph with edge stabiliser $E$, arc stabiliser $A = E \cap N$ and vertex stabiliser $H$, where $H$ is a proper subgroup of $N$ properly containing $A$. This phenomenon lies behind Examples 2.2 and 2.3. We also have the following example.

**Example 2.7.** Let $T$ be the Mathieu group $M_{12}$ and $G = \text{Aut}(T)$. From the Atlas [3, p 33], $G$ has maximal subgroups $E \cong S_5$ and $H \cong \text{PGL}(2, 11)$ such that $A = E \cap H \cong A_5$ and $H \cap T = \text{PSL}(2, 11)$ is a maximal subgroup of $T$. The subgroup $E$ is a novelty maximal. Let $g \in E \setminus A$. Then by Proposition 2.5, $\Gamma = \text{Cos}(G, H, HgH)$ is $G$-edge-primitive. As $H$ is maximal in $G$ it follows that $G$ acts primitively on $VT$. Note that $A \leq T$ and so $TA \neq G$. Hence $T$ acts transitively on vertices and edges but not on arcs. Moreover, as $A$ is selfnormalising in $T$, we have $A < H \cap T < T$ and $A$ is the stabiliser in $T$ of an edge. Thus $\Gamma$ is $T$-edge-quasiprimitive, but not $T$-edge-primitive. Moreover, $\Gamma$ is $G$-locally imprimitive and letting $B = H \cap T$, we see that $\Gamma$ is the quotient graph of the bipartite graph $\Sigma = \text{Cos}(G, B, BgB)$. The graph $\Sigma$ is $G$-edge-primitive and $(G, 2)$-arc-transitive such that $G^{E\Sigma} = G^{E\Gamma}$ and is $G$-vertex-biquasiprimitive, but not $G$-vertex-biprimitive. There is a partition $\mathcal{P}$ of $V\Sigma$ into blocks of size two such that $\Sigma_{\mathcal{P}} = \Gamma$. Each block of $\mathcal{P}$ has one vertex in each bipartite half of $\Sigma$, and there is at most one edge between any two blocks.

We have the following general construction of locally imprimitive, edge-primitive graphs.

**Construction 2.8.** Let $E$ be an almost simple primitive permutation group of degree $n$ such that $E$ has an index 2 subgroup $A$ which preserves a nontrivial partition of the $n$ points into $l$ parts of size $k$. Let $H = S_k \text{wr} S_l$ and $G = S_n$. Suppose that $E$ is a maximal subgroup of $G$ and let $g \in E \setminus A$. Then by Proposition 2.5, the graph $\Gamma = \text{Cos}(G, H, HgH)$ is $G$-edge-primitive. If $A$ is not maximal in $H$ then $\Gamma$ is $G$-locally imprimitive.

The requirements for $A$ and $E$ are often satisfied. An infinite family of examples is where $E = \text{Aut}(\text{PSL}(d, q))$ for $d \geq 3$ and $A = \text{PGL}(d, q)$. Let $n = (q^d - 1)(q^{d-1} - 1)/(q - 1)^2$, the number of point-hyperplane incident pairs. Then by [14], $E$ is maximal in $G = S_n$. However, $A$ is imprimitive and preserves a partition of $l = (q^d - 1)/(q - 1)$ parts of size $k = (q^{d-1} - 1)/(q - 1)$. Moreover, $A$ is not maximal in $H = S_l \text{wr} S_k$ since it is contained in $S_k \text{wr} \text{PGL}(d, q)$. Thus $\Gamma$ is $G$-locally imprimitive.
3 Initial Analysis

We begin by noting the following lemma.

**Lemma 3.1.** If $\Gamma$ is a disconnected $G$-edge-primitive graph then either $\Gamma$ is a union of isolated vertices and single edges, or $\Gamma$ is a union of isolated vertices and a connected $G$-edge-primitive graph.

*Proof.* Each connected component which contains an edge forms a block of imprimitivity for $G$ on edges. Thus either each connected component consists of zero or one edge, or there is a unique connected component with at least one edge. \qed

Next we look at vertex-transitivity.

**Lemma 3.2.** Let $\Gamma$ be a connected $G$-edge-quasiprimitive graph. Then either $G$ is vertex-transitive, or $\Gamma$ is bipartite and $G$ has two orbits on vertices. Moreover, in the latter case, either $\Gamma$ is a star or $G$ acts faithfully and quasiprimively on each of its two orbits.

*Proof.* Since $G$ is edge-transitive, either $G$ is vertex-transitive or $\Gamma$ is bipartite and the two orbits $\Delta_1$, $\Delta_2$ of $G$ on $V\Gamma$ are the two parts of the bipartition. Suppose that we are in the latter case and let $N$ be a nontrivial normal subgroup of $G$. Then $N$ acts transitively on $E\Gamma$ and so, since $\Gamma$ is connected, $N$ acts transitively on both $\Delta_1$ and $\Delta_2$. Thus either $|\Delta_1| = 1$ and $\Gamma$ is a star, or $G$ acts faithfully and quasiprimively on each of its two orbits. \qed

In the edge-primitive case things are more restricted.

**Lemma 3.3.** Let $\Gamma$ be a connected $G$-edge-primitive graph. Then either $\Gamma$ is a star or $G$ is vertex-transitive.

*Proof.* Suppose that $G$ is vertex-intransitive. Then as $G$ is edge-transitive, $\Gamma$ is a bipartite graph with the orbits of $G$ being the two bipartite halves $\Delta_1$ and $\Delta_2$. Let $v \in \Delta_1$ and $B = \{\{v, w\} | w \in \Gamma(v)\}$. Then $B$ forms a block of imprimitivity for $G$ on edges. Thus either $|\Gamma(v)| = 1$ or $\Delta_1 = \{v\}$. Since $\Gamma$ is connected, it follows that $\Gamma$ is a star. \qed

We can now show that all nontrivial edge-primitive graphs are arc-transitive.

**Lemma 3.4.** Let $\Gamma$ be a connected $G$-edge-primitive graph. Then one of the following holds:

1. $\Gamma$ is a star;
2. $\Gamma$ is a cycle of prime length $p$, and $G$ is a cyclic group of order $p$;
3. $\Gamma$ is $G$-arc-transitive.

*Proof.* By Lemma 3.3, either case (1) holds or $G$ is vertex-transitive. Suppose now that $G$ is vertex-transitive but not arc-transitive. Then for an edge $e = \{v, w\}$ we have $G_e = G_{vw} = G_v \cap G_w$. However, as $G$ acts primitively on edges, $G_e$ is a maximal subgroup of $G$. Thus $G_v = G_w$ for every pair of adjacent vertices. But $\Gamma$ is connected, and so $G_v$ fixes every vertex of $\Gamma$. This implies that $G_v = 1 = G_e$ and so $G$ acts
regularly on vertices and on edges. Thus $\Gamma$ has the same number of edges as vertices and so the connectivity of $\Gamma$ implies that it is a cycle. Furthermore, as $G$ is primitive on edges this cycle has a prime number of edges and hence vertices. Moreover, as $G$ is not arc-transitive, $G$ is cyclic. Thus either case (2) or (3) holds.

Lemma 3.4 does not hold for $G$-edge-quasiprimitive graphs. In particular, the graph $\Gamma$ in Example 2.7 is $T$-edge-quasiprimitive, $T$-vertex-transitive but not $T$-arc-transitive.

Next we look at the action of $G$ on vertices.

**Lemma 3.5.** Let $\Gamma$ be a connected $G$-vertex transitive, $G$-edge-quasiprimitive graph. Then $G$ is either quasiprimitive or biquasiprimitive on the set of vertices of $\Gamma$.

**Proof.** Let $N$ be a nontrivial normal subgroup of $G$. Then $N$ is transitive on edges and so is either transitive on vertices or $\Gamma$ is bipartite and $N$ has two orbits on the vertex set. Thus $G$ is either quasiprimitive or biquasiprimitive on $VT$.

In the edge-primitive case we can actually reduce to the situation where $G$ is either primitive or biprimitive on vertices.

**Proof.** (of Theorem 1.1) By Lemma 3.4 $G$ is arc-transitive. Suppose that $G$ is neither primitive nor biprimitive on $VT$. Then there exists a $G$-invariant partition $B$ of $VT$ with at least three parts. Since $\Gamma$ is connected and edge-transitive, the edges of $\Gamma$ occur between the parts of $B$, that is, there are no edges within parts. Let $\Gamma_B$ be the quotient graph of $\Gamma$ with respect to the partition $B$. Given $B_1, B_2 \in B$ which are adjacent in $\Gamma_B$, the set of edges of $\Gamma$ between vertices of $B_1$ and vertices of $B_2$ forms a block of imprimitivity for $G$. Hence there is a unique edge in $\Gamma$ between vertices of $B_1$ and vertices of $B_2$. Thus $\Gamma$ is a spread of $\Gamma_B$ and $G^{ET} \cong G^{ET_B}$. Moreover, if $g \in G$ fixes each part of $B$, then $g$ fixes each edge of $\Gamma$. Thus $G$ acts faithfully on $B$. Moreover, by choosing $B$ to be a maximal $G$-invariant partition with at least three parts, $G$ is either primitive or biprimitive on the set of vertices of $VT_B$. Let $v \in B_1$ and $w \in B_2$ be the unique pair of adjacent vertices in $B_1 \cup B_2$. Then $G_{B_1B_2} = G_v < G_w < G_{B_1}$, since $G$ is arc-transitive and $|B_1| > 1$. Hence $\Gamma_B$ is $G$-locally imprimitive.

Conversely, suppose that $\Sigma$ is a $G$-edge-primitive, $G$-locally imprimitive graph. Let $\{\alpha, \beta\} \in E \Sigma$. Then there exists a subgroup $H$ such that $G_{\alpha \beta} < H < G_{\alpha}$. Since $G$ is arc-transitive, there exists $g \in G$ such that $g$ interchanges $\alpha$ and $\beta$. Thus $H \cap H^g \leq G_{\alpha} \cap G_{\beta}$, but since $g$ normalises $G_{\alpha \beta}$ we have $H \cap H^g = G_{\alpha \beta}$. Moreover, $g^2 \in G_{\alpha \beta} \leq H$. Thus we can define the graph $\Gamma = \text{Cos}(G, H, HgH)$. Let $v$ be the vertex of $\Gamma$ given by $H$ and $w$ the vertex given by the coset $Hg$. Then $e = \{v, w\}$ is an edge and $G_e = \langle H^g \cap H, g \rangle = \langle G_{\alpha \beta}, g \rangle = G_{\{\alpha, \beta\}}$. Hence $G^{ET} \cong G^{E \Sigma}$ and so $\Gamma$ is $G$-edge-primitive. Since $H = G_v < G_{\alpha} < G$, it follows that $B_1 = v^{G_{\alpha}}$ is a block of imprimitivity for $G$ on $VT$. Let $B$ be the corresponding system of imprimitivity. Now $v^{G_{\alpha}g} = v^{g^{-1}G_{\alpha}g} = w^{G_{\beta}}$ and so $B_2 = w^{G_{\beta}}$ is the block of $B$ containing $w$. Moreover, $(v, w)$ is the unique edge between the two blocks $B_1$ and $B_2$. Then as $G_v = G_{B_1}$ and $g$ interchanges the edge $\{B_1, B_2\}$ of the quotient graph $\Gamma_B$ we have that $\Gamma_B \cong \text{Cos}(G, G_v, G_v^gG_v) \cong \Sigma$.

We also have the following lemma in the vertex-biquasiprimitive case.

**Lemma 3.6.** Let $\Gamma$ be a $G$-vertex-biquasiprimitive graph which is not complete bipartite. Then $G^+$ is faithful on each orbit.
Proof. Let \( \Delta_1 \) and \( \Delta_2 \) be the two orbits of \( G^+ \) on vertices and suppose that \( G^+ \) is unfaithful on \( \Delta_1 \). Let \( K_1 \) be the kernel of the action of \( G^+ \) on \( \Delta_1 \) and \( K_2 \) be the kernel of the action of \( G^+ \) on \( \Delta_2 \). Then as \( G \) is vertex-transitive, there exists \( g \in G \) such that \( K_1^g = K_2 \). Moreover, \( 1 \neq K_1 \times K_2 < G \). Since \( G \) is vertex-biquasiprimitive, it follows that \( K_1 \) is transitive on \( \Delta_2 \) and \( K_2 \) is transitive on \( \Delta_1 \). Since \( K_1 \) fixes each vertex in \( \Delta_1 \), we have that each vertex of \( \Delta_1 \) is adjacent to each vertex of \( \Delta_2 \). Thus \( \Gamma \) is complete bipartite.

We can determine all \( n \) and \( G \) such that \( K_{n,n} \) is \( G \)-edge-primitive and \( G^+ \) acts faithfully on each bipartite half.

**Theorem 3.7.** Let \( \Gamma = K_{n,n} \) be a \( G \)-edge-primitive graph. Then one of the following holds:

1. \( n = 6^k \) and \( \text{soc}(G^+) = A_6^k \).
2. \( n = 12^k \) and \( \text{soc}(G^+) = M_{12}^k \).
3. \( n = (q^2(q^2 - 1)/2)^k \) and \( \text{soc}(G^+) = \text{PSp}(4,q) \) with \( q \) even.
4. There exists a primitive group \( H \) of degree \( n \) with a transitive but not regular normal subgroup \( K \) and automorphism \( \phi \) such that \( G^+ = \langle \{hk_1, h^\phi k_2 \} \mid k_1, k_2 \in K, h \in H \rangle \), and \((g,1_H)(1,2) \in G\) for some \( g \in H \) interchanges the two \( G^+ \) orbits where \( \phi^2 \) is conjugation by \( g \).

**Proof.** Let \( \Delta_1 \) and \( \Delta_2 \) be the two bipartite halves of \( \Gamma \). Suppose that \( G^+ \) is imprimitive on \( \Delta_1 \) and let \( P_1 \) be a system of imprimitivity for \( G^+ \) on \( \Delta_1 \). Then there exists a system of imprimitivity \( P_2 \) of \( G^+ \) on \( \Delta_2 \) such that \( P_2 = P_1^g \) for all \( g \in G \setminus G^+ \). Let \( B_1 \in P_1 \) and \( B_2 \in P_2 \). Then \( C = \{(v,w) \mid v \in B_1, w \in B_2\} \) is a block of imprimitivity for \( G \) on \( ET \). Hence \( G^+ \) is primitive on each bipartite half.

Let \( v \in \Delta_1 \) and \( w \in \Delta_2 \). By Lemma 3.4, \( G \) is arc-transitive. Thus \( G_v \) is transitive on \( \Delta_2 \) and so \( G^+ = G_v G_w \). Suppose first that \( G^+ \) is faithful on \( \Delta_1 \) and \( \Delta_2 \). Since \( G_w = G_v^g \) for some \( g \in G \) with \( g^2 \in G^+ \), it follows that \( G^+, G_v \) and \( G_w \) are determined by \([2, \text{Theorem 1.1}]\). Either \( G = AGL(3,2) \wr K \) for some transitive subgroup \( K \) of \( S_k \), or \( \text{soc}(G^+) = T^k \) where \( T \) is one of \( \text{PGL}(8,q) \), \( \text{PSp}(4,q) \) \( q \geq 2 \) even, \( A_6 \) or \( M_{12} \).

If \( G = AGL(3,2) \wr K \), then \( G_{\{v,w\}} = \langle (C_2 \times C_3) \wr K, (\alpha,\ldots,\alpha) \rangle \) where \( \alpha \) is an automorphism of \( AGL(3,2) \) interchanging the two conjugacy classes of complements of \( C_2^3 \). Hence \( G_{\{v,w\}} < C_2^g \times G_{\{v,w\}} < G \) and so \( G \) is not edge-primitive.

If \( N = \text{soc}(G^+) = \text{PGL}(8,q)^k \) then \( G^+ \leq H^k \) where \( H \) is an extension of \( \text{PGL}(8,q) \) by field automorphisms, \( N_v = \text{PGL}(7,q), N_{vw} = G_2(q) \) and \( n = q^4(q^4 - 1) \) \([2, \text{Theorem 1.1}]\). Since \( N_v^g = N_v \) for some \( g \in G \setminus G^+ \) such that \( g^2 \in G^+ \), it follows that \( g \) does not induce a triality automorphism of \( \text{PGL}(8,q) \). Hence by \([11]\), \( G_{\{vw\}} \) is not maximal in \( G \), and so \( G \) is not edge-primitive. Thus \( \text{soc}(G^+) \) and \( n \) are as listed in the statement of the theorem.

Suppose next that \( G^+ \) is unfaithful on \( \Delta_1 \) and \( \Delta_2 \). Let \( K_1 \) be the kernel of the action of \( G^+ \) on \( \Delta_1 \) and \( K_2 \) be the kernel of the action of \( G^+ \) on \( \Delta_2 \). Then \( K_1 \times K_2 < G \) and so is transitive on \( ET \). Hence \( K_1 \) acts faithfully and transitively on \( \Delta_2 \) and \( K_2 \) acts transitively and faithfully on \( \Delta_1 \). Let \( H = (G^+)^{\Delta_1} \) and \( K = (K_2)^{\Delta_1} \). Then \( H \) is
a primitive permutation group with transitive normal subgroup $K$. Now $G \leq H \wr S_2$ and $G = \langle G^+, (g, 1_H)(1, 2) \rangle$ for some $g \in H$. Then $K_2 = \{(1_H, k) \mid k \in K\}$ and $K_1 = K_2^{(g, 1_H)(1, 2)} = \{(k, 1_H) \mid k \in K\}$. Furthermore, there exists an automorphism $\phi$ of $H$ such that $G^+ = \{(hk_1, h^\phi k_2) \mid h \in H, k_1, k_2 \in K\}$. Since $(g, 1_H)(1, 2)$ normalises $G^+$ it follows that $\phi^2$ is conjugation by $g$. If $K$ is regular then $H = K \rtimes H_v$ and so $G^+ = \langle K \times K \rangle \rtimes \{(h, h^\phi) \mid h \in H_v\}$. Moreover, $G_v = \langle \{(h, h^\phi) \mid h \in H_v\}, (g, 1_H)(1, 2) \rangle < \langle \{(h, h^\phi) \mid h \in H\}, (g, 1_H)(1, 2) \rangle < G$, contradicting $G_v$ being maximal in $G$. Thus $K$ is not regular. 

4 Primitive and quasiprimitive types

In this section we describe the subdivision of primitive and quasiprimitive groups into 8 types given in [15]. This description is in terms of the action of the minimal normal subgroups. If $N$ is a minimal normal subgroup of a group $G$ then $N \cong T^k$ for some finite simple group $T$. Moreover, if $G$ is quasiprimitive then $G$ has at most two minimal normal subgroups.

**HA:** A quasiprimitive group is of type HA if it has a unique minimal normal subgroup $N$ and $N$ is elementary abelian. If $|N| = p^d$ for some prime $p$, then $G$ can be embedded in $AGL(d, p)$ in its usual action on a $d$-dimensional vector space over $GF(p)$ with $N$ identified as the group of all translations.

**HS and HC:** These two classes consist of all quasiprimitive groups with two minimal normal subgroups. In both cases, the two minimal normal subgroups are regular and nonabelian. For type HS, the two minimal normal subgroups are simple, while for type HC the two minimal normal subgroups are isomorphic to $T^k$ for some $k \geq 2$ and $T$ nonabelian simple.

All quasiprimitive groups of type HA, HS and HC are in fact primitive. For the remaining five types the groups may or may not be primitive.

**AS:** This class consists of all groups $G$ such that $T \leq G \leq \text{Aut}(T)$ for some finite nonabelian simple group, that is, $G$ is an almost simple group. Note that any action of an almost simple group with $T$ transitive is quasiprimitive.

**TW:** This type consists of all quasiprimitive groups $G$ with a unique minimal normal subgroup $N \cong T^k$, for some finite nonabelian simple group $T$ and positive integer $k \geq 2$, such that $N$ is regular. Thus $G = N \rtimes G_\omega$ and can be constructed as a twisted wreath product (see [1]). If $G$ is primitive then $G_\omega$ normalises no nontrivial proper subgroup of $N$. The following lemma gives us a necessary and sufficient condition for a quasiprimitive TW group to be primitive.

Lemma 4.1. [1, Lemmas 3.1 and 3.2] Let $N \cong T^k$ for some finite nonabelian simple group $T$ and $G = N \rtimes P$. Let $Q$ be the normaliser in $P$ of a simple direct factor of $N$ and $\varphi : Q \to \text{Aut}(T)$ be the homomorphism induced by the action of $Q$ on this factor. Then $P$ is maximal in $G$ if and only if $\text{Inn}(T) \leq \varphi(Q)$ and there is no subgroup $H$ of $P$ with a homomorphism $\tilde{\varphi}$ from $H$ to $\text{Aut}(T)$ which extends $\varphi$. 

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Before describing the remaining three types of quasiprimitive groups we need some definitions. Let \( N = T_1 \times \cdots \times T_k \) for some nontrivial groups \( T_1, \ldots, T_k \). For each \( i = 1, \ldots, k \), let \( \pi_i: N \to T_i \) be the natural projection map. Given a subgroup \( K \) of \( N \), we say that \( K \) is a \textit{subdirect product} of \( N \) if \( \pi_i(K) = T_i \) for each \( i = 1, \ldots, k \), while we say that \( K \) is a \textit{diagonal subgroup} of \( N \) if \( K \) is isomorphic to each of its projections, that is, \( K \cong \pi_i(K) \) for all \( i = 1, \ldots, k \). If \( T_1 = T_2 = \cdots = T_k \) and \( \pi_i(g) = \pi_j(g) \) for all \( g \in K \), we call \( K \) a \textit{straight diagonal subgroup}. A \textit{full diagonal subgroup} of \( N \) is a subgroup which is both a subdirect product and a diagonal subgroup.

We call \( K \) a \textit{strip} of \( N \) if there exists some subset \( J \) of \( \{1, \ldots, k\} \) such that \( \pi_i(K) \cong K \) for all \( i \in J \) while \( \pi_i(K) = 1 \) for all \( i \notin J \). We refer to \( J \) as the \textit{support} of \( K \). Note that a strip is a diagonal subgroup of \( \prod_{i \in J} T_i \). We call \( K \) a \textit{full strip} if it is a full diagonal subgroup of \( \prod_{i \in J} T_i \), while we say that it is \textit{nontrivial} if \( |J| > 1 \). We say that two strips are \textit{disjoint} if their supports are disjoint. Note that disjoint strips commute.

If \( N = T_1 \times \cdots \times T_k \), where the \( T_i \) are pairwise isomorphic nonabelian simple groups, a well known lemma (see for example [16]) says that if \( K \) is a subdirect product of \( N \) then \( K \) is the direct product of pairwise disjoint full strips. The set of supports of these strips is a partition \( \mathcal{P} \) of \( \{1, \ldots, k\} \). Note that if \( K \) is normalised by a group \( G \), then \( G \) preserves \( \mathcal{P} \) and if \( G \) acts transitively by conjugation on the set \( \{T_1, \ldots, T_k\} \), then \( G \) acts transitively on \( \mathcal{P} \) and so the parts of \( \mathcal{P} \) all have the same size.

**SD:** A quasiprimitive group \( G \) acting on a set \( \Omega \) is of type \( \text{SD} \) if \( G \) has a unique minimal normal subgroup \( N, N \cong T^k \) for some nonabelian simple group \( T, k \geq 2 \) and given \( \omega \in \Omega \), the point stabiliser \( N_\omega \) is a full diagonal subgroup of \( N \). Conjugating \( G \), if necessary, by an element of \( \text{Sym}(\Omega) \) we may assume that \( N_\omega = \{(t, \ldots, t) \mid t \in T\} \) and \( G_\omega \leq \{(t, \ldots, t) \mid t \in \text{Aut}(T)\} \rtimes S_k \). Since \( N \) is a minimal normal subgroup of \( G \) and \( G = NG_\omega \), it follows that \( G_\omega \) acts transitively by conjugation on the set of \( k \) simple direct factors of \( N \). A quasiprimitive group \( G \) of type \( \text{SD} \) is primitive, if and only if \( G \) acts primitively on the set of \( k \) simple direct factors of \( N \).

**CD:** A quasiprimitive group \( G \) acting on a set \( \Omega \) is of type \( \text{CD} \) if \( G \) has a unique minimal normal subgroup \( N, N \cong T^k \) for some nonabelian simple group \( T, k \geq 2 \) and given \( \omega \in \Omega \), \( N_\omega \) is a product of \( \ell \geq 2 \) full strips of \( N \), that is, \( N_\omega \cong T^\ell \) and is a subdirect product of \( N \). Note that \( G \) acts transitively by conjugation on the set of \( k \) simple direct factors of \( N \) and preserves a partition \( \mathcal{P} \) of \( \{1, \ldots, k\} \) given by the set of supports of the full strips. The group \( G \) is a subgroup of \( H \wr S_\ell \) acting on \( \Omega = \Delta^\ell \), for some quasiprimitive group \( H \) of type \( \text{SD} \) on \( \Delta \) with unique minimal normal subgroup \( T^k/\ell \). In fact, given \( P \in \mathcal{P} \), the group \( G_P \) induces \( H \) on \( \Delta \). Moreover, \( G \) is primitive if and only if \( H \) is primitive and so \( G \) is primitive if and only if for \( P \in \mathcal{P} \), \( G_P \) acts primitively on \( \mathcal{P} \).

Given two partitions \( \mathcal{P}_1, \mathcal{P}_2 \) of a set \( \Omega \), we say that \( \mathcal{P}_1 \) refines \( \mathcal{P}_2 \) if each \( P \in \mathcal{P}_2 \) is a union of elements of \( \mathcal{P}_1 \). This defines a partial order on the set of all partitions of \( \Omega \) and we can define \( \mathcal{P}_1 \vee \mathcal{P}_2 \) to be the smallest partition of \( \Omega \) refined by both \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \). The following lemma will be very handy in our analysis of SD and CD groups.

**Lemma 4.2.** Let \( N = T_1 \times \cdots \times T_k \) for some nontrivial groups \( T_i \) and let \( K_1, K_2 \) be subgroups of \( N \). For each \( i = 1, 2 \), suppose that \( K_i \) is a product of strips such that the set...
of supports of these strips is the partition $\mathcal{P}_i$ of \{1, \ldots, k\}. Then $K_1 \cap K_2$ is a product of strips such that the set of supports of these strips is $\mathcal{P}_1 \cup \mathcal{P}_2$.

Proof. For each $P \in \mathcal{P}_1 \cup \mathcal{P}_2$, let

$$K_P = \{ g \in K_1 \cap K_2 \mid \pi_i(g) = 1 \text{ for all } i \notin P \}.$$ 

Then $X = \prod_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} K_P$ is a subgroup of $K_1 \cap K_2$.

Let $g \in K_1 \cap K_2$ such that $g \neq 1$, and let $J$ be the set of all $i \in \{1, \ldots, k\}$ such that $\pi_i(g) \neq 1$. Since $g \in K_1$ it follows that $J$ is a union of parts of $\mathcal{P}_1$ and since $g \in K_2$ it follows that $J$ is a union of parts of $\mathcal{P}_2$. Hence $J$ is a union of $\ell$ parts of $\mathcal{P}_1 \cup \mathcal{P}_2$ for some $\ell \geq 1$. Thus each $K_P$ is a strip. If $\ell = 1$ then $g \in K_P$ for some $P$ and so $g \in X$. If $\ell > 1$, let $P$ be one of the parts contained in $J$. Since $g \in K_1$ and $K_1$ is a product of strips, there exists $k_1 \in K_1$ such that $\pi_i(k_1) = \pi_i(g)$ for all $i \in P$ while $\pi_i(k_1) = 1$ for all $i \notin P$. Similarly, there exists $k_2 \in K_2$ such that $\pi_i(k_2) = \pi_i(g)$ for all $i \in P$ while $\pi_i(k_2) = 1$ for all $i \notin P$. Hence $k_1 = k_2 \in K_P \leq X$. Moreover, $gk_1^{-1} \in K_1 \cap K_2$ and has support $J \setminus P$, a union of $\ell - 1$ parts of $\mathcal{P}_1 \cup \mathcal{P}_2$. It follows that $g \in X$ and so $K_1 \cap K_2$ is a product of the strips $K_P$, whose supports are the parts of $\mathcal{P}_1 \cup \mathcal{P}_2$. \hfill \Box

PA: A quasiprimitive group $G$ acting on a set $\Omega$ is of type PA if $G$ has a unique minimal normal subgroup $N$, $N \cong T^k$ for some nonabelian simple group $T$, $k \geq 2$ and given $\omega \in \Omega$, $N_\omega$ is a subdirect product of $R^k$ for some $R < T$. The following two lemmas will be useful for determining primitivity. See for example, [5, Lemma 2.7A] for a proof of the first.

Lemma 4.3. Let $B$ be a group with subgroup $H \neq 1$. Then for each positive integer $k$, $H \wr S_k$ is maximal in $B \wr S_k$ if and only if $H$ is maximal in $B$.

Lemma 4.4. Let $T$ be a nonabelian simple group and let $T \leq A \leq B \leq \text{Aut}(T)$. Suppose that $H$ is a maximal subgroup of $B$ such that $B = TH$ and $T \cap H \neq 1$. Let

$$G = \langle A^k, (b, \ldots, b) \mid b \in B \rangle \rtimes S_k$$

and

$$L = \langle (A \cap H)^k, (h, \ldots, h) \mid h \in H \rangle \rtimes S_k.$$ 

Then $L$ is a maximal subgroup of $G$.

Proof. Let $M$ be a subgroup of $G$ containing $L$ and let $X = M \cap B^k$. Since $S_k \leq M$ it follows that $\pi_i(X) \cong \pi_j(X)$ for all $i$ and $j$. Since $L \leq M$ we have $H \leq \pi_i(X)$ and since $H$ is maximal in $B$ it follows that $\pi_i(X) = B$ for all $i$. Hence $X \cap T^k$ is a subdirect product of $T^k$. However, $H \cap T \neq 1$ and $(H \cap T)^k \leq X$. Thus $X \cap T^k = T^k$. Since $B = TH$ we also have $A = T(A \cap H)$. Then as $(A \cap H)^k \leq X$ it follows that $A^k \leq X$. Thus $X = G \cap B^k$ and so $M = G$, that is, $L$ is maximal. \hfill \Box
5 Constructions

All the examples in Section 2 had \( G \) an almost simple group. In this section we provide some general constructions for \( G \)-edge-quasiprimitive graphs where \( G \) is not of type AS.

Our first construction takes a \( B \)-edge-primitive graph where \( B \) is an almost simple group such that \( B \neq \text{soc}(B) \), and builds a \( G \)-edge-primitive graph where \( G \) is primitive of type PA on edges and primitive of type PA on vertices.

**Construction 5.1.** (Primitive PA on vertices and primitive PA on edges) Let \( \Sigma \) be a \( B \)-edge-primitive, \( B \)-vertex-primitive graph such that \( B \) is an almost simple group with \( \text{socle} \ T < B \). Note that Example 2.7 is such a graph. Then there exist a maximal subgroup \( H \) of \( B \) and \( g \in B \setminus H \), such that \( g^2 \in H \) and \( \Sigma \cong \text{Cos}(B, H, HgH) \). Let

\[
G = \langle T^k, (b, \ldots, b) \mid b \in B \rangle \rtimes S_k
\]

and

\[
L = \langle (T \cap H)^k, (h, \ldots, h) \mid h \in H \rangle \rtimes S_k.
\]

Then letting \( \sigma = (g, \ldots, g) \) we define the coset graph \( \Gamma = \text{Cos}(G, L, L\sigma L) \).

**Lemma 5.2.** The graph \( \Gamma = \text{Cos}(G, L, L\sigma L) \) given by Construction 5.1 is \( G \)-edge-primitive of type PA and \( G \)-vertex-primitive of type PA. Moreover, if \( \Sigma \) given in Construction 5.1 is \( B \)-locally imprimitive then \( \Gamma \) is \( G \)-locally imprimitive.

*Proof.* Since \( T < B \) and \( B \) is primitive, it follows that \( H \neq 1 \). Then as \( H \) is a maximal subgroup of \( B \), Lemma 4.4 implies that the action of \( G \) on \( VT = [G : L] \) is primitive of type PA. Let \( v \) be the vertex given by the coset \( L \) and \( w \) be the adjacent vertex given by \( L\sigma \). Then \( G_w = L^\sigma \) and

\[
G_v \cap G_w = \langle (T \cap H \cap H^g)^k, (h, \ldots, h) \mid h \in H \cap H^g \rangle \rtimes S_k.
\]

Furthermore,

\[
G_{\{v,w\}} = \langle G_v \cap G_w, \sigma \rangle = \langle (T \cap H \cap H^g)^k, (h, \ldots, h) \mid h \in \langle H \cap H^g, g \rangle \rangle \rtimes S_k
\]

which by Lemma 4.4, is a maximal subgroup of \( G \) since \( \langle H \cap H^g, g \rangle \) is a maximal subgroup of \( B \). Hence \( G \) acts primitively on \( ET \) of type PA.

If \( \Sigma \) is \( B \)-locally imprimitive there exists a subgroup \( R \) such that \( H \cap H^g < R < H \). It follows that \( G_{vw} \) is not maximal in \( L \) and so \( \Gamma \) is \( G \)-locally imprimitive. \( \square \)

We have the following construction which takes a \( B \)-edge-primitive bipartite graph such that \( B \) is almost simple and \( B^+ \) is primitive on each bipartite half, and builds a \( G \)-edge-primitive bipartite graph with \( G \) primitive of type PA on edges and \( G^+ \) primitive of type PA on each of the bipartite halves.

**Construction 5.3.** (Primitive PA on edges and biprimitive on vertices with \( G^+ \) primitive of type PA) Let \( \Sigma \) be a bipartite connected \( B \)-edge-primitive graph such that \( B \) is an almost simple group with socle \( T \) such that \( B^+ \) acts primitively on each bipartite
half. Then there exist a corefree maximal subgroup $H$ of $B^+$ and $g \in B \setminus B^+$ such that $g^2 \in H$ and $\Sigma = \Cos(B, H, HgH)$. Let $\sigma = (g, \ldots, g)$,

$$G = \langle (B^+)^k, \sigma \rangle \rtimes S_k,$$

and $L = H^k \rtimes S_k$. Define $\Gamma = \Cos(G, L, L\sigma L)$.

**Lemma 5.4.** The connected bipartite graph $\Gamma = \Cos(G, L, L\sigma L)$ yielded by Construction 5.3 is $G$-edge-primitive of type PA and $G$-biprimitive on vertices such that $G^+$ acts primitively of type PA on both of its vertex orbits. Moreover, $\Gamma$ is $G$-locally primitive if and only if $\Sigma$ is $B$-locally primitive.

**Proof.** Since $\Sigma$ is connected we have $\langle H, g \rangle = B$. It follows that $\langle L, \sigma \rangle = G$ and so $\Gamma$ is connected. The stabiliser in $G$ of the edge $e = \{L, L\sigma\}$ is $\langle (H \cap H^g)^k, \sigma \rangle \rtimes S_k$ which by Lemma 4.4 is a maximal subgroup of $G$ since $\langle H \cap H^g, g \rangle$, the stabiliser in $B$ of an edge in $\Sigma$, is a maximal subgroup of $B$. Hence $G$ acts primitively of type PA on $ET$. The index two subgroup $G^+ = B^+ \wr S_k$ of $G$ has two orbits on $VT$. Hence $\Gamma$ is bipartite. Moreover, since $H$ is a maximal subgroup of $B^+$ it follows from Lemma 4.3 that $G^+$ acts primitively of type PA on each of the bipartite halves.

Since $\langle H \cap H^g, g \rangle$ is maximal in the almost simple group $B^+$, we have $H \cap H^g \neq 1$. Thus by Lemma 4.3 $\langle (H \cap H^g)^k, \sigma \rangle \rtimes S_k$ is maximal in $H^k \rtimes S_k$ if and only if $H \cap H^g$ is maximal in $H$, and $\Gamma \setminus \langle H, g \rangle$ is $G$-locally primitive if and only if $\Sigma$ is $B$-locally primitive. \qed

**Remark 5.5.** Suppose that in Construction 5.3, we let $k = 2$ and let $\overline{G} = (B^+)^2 \rtimes \langle (g, g)(1, 2) \rangle \leq G$. Then $\overline{G}_e = (H \cap H^g)^2 \rtimes \langle (g, g)(1, 2) \rangle$, which is a maximal subgroup of $\overline{G}$. Thus $\overline{G}$ is edge-primitive of type PA and biquasiprimitive on vertices. Moreover, $(\overline{G})^+ = (B^+)^2$ and $\overline{G}_v = H^2$. Hence $(\overline{G})^+$ is not quasiprimitive on each bipartite half of $\Gamma$.

We now give a general construction of $G$-edge-quasiprimitive graphs for which the action of $G$ on edges is of type SD or CD and $G$ is vertex-transitive.

**Construction 5.6.** (Quasiprimitive SD or CD on edges and vertex-transitive) Let $G$ be a quasiprimitive group on a set $\Omega$ of type SD or CD with socle $N = T^k$. Let $\omega \in \Omega$ and let $\mathcal{P}$ be the $G$-invariant partition of $\{1, \ldots, k\}$ given by the set of supports of the full strips of $N_\omega$. If $G$ is of type SD then $\mathcal{P} = \{\{1, \ldots, k\}\}$ while if $G$ is of type CD then $\mathcal{P}$ is a nontrivial system of imprimitivity for $G$. Suppose that $G$ has an index two subgroup $G^+$ which leaves invariant two distinct partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\{1, \ldots, k\}$ which are interchanged by $G$, and such that $\mathcal{P}_1 \lor \mathcal{P}_2 = \mathcal{P}$.

Let $L = G_\omega$. Conjugating by a suitable element of $\Sym(\Omega)$ we may assume that each $h \in L$ is of the form $(t_1, \ldots, t_k)\sigma$ where $t_i \in \Aut(T)$, $\sigma \in S_k$, $\sigma$ preserves $\mathcal{P}$, and if $i, j$ belong to the same part of $\mathcal{P}$ then $t_i = t_j$. Since $L^{(1, \ldots, k)} = G^{(1, \ldots, k)}$, it follows that $L$ has an index two subgroup $L^+$ which leaves $\mathcal{P}_1$ and $\mathcal{P}_2$ invariant. Moreover, $L = (L^+, g)$ for some element $g = (t_1, \ldots, t_k)\sigma \in G$, where $\sigma$ interchanges $\mathcal{P}_1$ and $\mathcal{P}_2$. For a subset $I$ of $\{1, \ldots, k\}$, let $T_I$ be the straight full strip of $N$ whose support is $I$. Let $N_I = \prod_{I \in \mathcal{P}_I} T_I$ and let $H = N_G(N_I)$. Then as $L^+$ leaves $\mathcal{P}_1$ invariant and $G^+$ is the stabiliser of $\mathcal{P}_1$ in $G$, we have $L^+ \leq H \leq G^+$. Moreover, since $G^+ = NL^+$, if $nl \in H$ with $n \in N$ and $l \in L$ then $n \in N_N(N_I) = N_I$. Thus $H = N_I L^+$. Furthermore, $H^g = N_2 L^+$ where $N_2 = \prod_{I \in \mathcal{P}_2} T_I$. Since $g^2 \in L^+$ it follows that $g^2 \in H$ and we can define $\Gamma = \Cos(G, H, HgH)$. 

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Lemma 5.7. The graph $\Gamma = \text{Cos}(G,H,HgH)$ obtained from Construction 5.6 is $G$-edge-quasiprimitive of type SD or CD such that $G$ is vertex-biquasiprimitive. Moreover, $\Gamma$ is $G$-locally primitive if and only if $P_1$ is the coarsest $L^+$-invariant partition of $P_1 \lor P_2$ refined by $P_1$.

Proof. Let $v$ be the vertex corresponding to the coset $H$ and $w$ be the vertex corresponding to $Hg$. Then $e = \{v,w\}$ is an edge and $G_{vw} = H \cap H^g = (N_1 \cap N_2)L^+$. Elements of $N_1 \cap N_2$ are constant on the parts of $P_1$ and the parts of $P_2$, hence are constant on the parts of $P_1 \lor P_2 = P$. Thus $N_1 \cap N_2 = L \cap N$. Hence $G_{vw} = L^+$ and $G_e = L$. It follows that $G^G \cong G^E$ and so $\Gamma$ is $G$-edge-quasiprimitive with $G^E$ of type SD or CD. Moreover, $G^+$ has two orbits on $V$ and so $\Gamma$ is bipartite. Since $N$ is the unique minimal normal subgroup of $G$ and has two vertex orbits it follows that $G$ is vertex-biquasiprimitive. Further, $G_{vw} = L^+$ is maximal in $G_v = H$ if and only if $P_1$ is the coarsest $L^+$-invariant partition of $P_1 \lor P_2$ refined by $P_1$. Hence the statement regarding local primitivity follows.

We now demonstrate the various vertex actions which can be yielded by Construction 5.6.

Example 5.8. A suitable choice for $G$ primitive of type SD in Construction 5.6, is $N \times K$, where $K = S_d \wr S_2$ for some $d \geq 3$, $N = T^{d^2}$ and $G^+ = N \times S_2^d$. Here $P_1$ corresponds to the set of orbits of $1 \times S_d$ on the $d^2$ simple direct factors of $N$ (that is, the “horizontal” blocks) while $P_2$ corresponds to set of orbits of $S_d \times 1$ (that is, the “vertical blocks”). Note that $G^+$ is primitive of type CD on each of its vertex orbits and $G$ is vertex-biprimitive.

Example 5.9. Let $G = T^4 \rtimes \langle (1,3,2,4) \rangle$. Here $P_1 = \{\{1,4\},\{2,3\}\}$, $P_2 = \{\{1,3\},\{2,4\}\}$, and $P = \{\{1,2,3,4\}\}$. Then $G$ is quasiprimitive but not primitive of type SD on edges and $G^+ = T^4 \rtimes \langle (1,2)\rangle$ is primitive of type HC on each vertex orbit.

Example 5.10. A suitable choice for $G$ primitive of type CD is $G = N \rtimes K$ where $N = T^{d^2m}$ and $K = (S_2^2)^m \times S_m$ such that $K$ preserves the partition $P$ of $m$ blocks of size $d^2$ with $d \geq 3$. Here $K$ has an index two subgroup $K_1 = S_2^d \wr S_m$ with two systems of imprimitivity $P_1$ and $P_2$ with $dm$ parts of size $d$, interchanged by $K$. The partition $P_1$ is the set of orbits of $(1 \times S_d)^m$ on the set of $d^2m$ simple direct factors of $N$ (the set of horizontal blocks in each part of $P$) while $P_2$ is the set of orbits of $(S_d \times 1)^m$ (the set of vertical blocks of each part of $P$). Moreover, $P_1 \lor P_2 = P$. Note that $G^+$ is primitive of type CD on each of its orbits on VT and $G$ is vertex-biprimitive.

Example 5.11. Let $G = T^8 \rtimes K$ where

$$K = \langle (1,2)(3,4)(5,6)(7,8), (1,5)(2,6)(3,7)(4,8), (1,3,2,4)(5,8,6,7) \rangle \cong D_8.$$ 

Then $K$ has an index 2 subgroup $K_1 = \langle (1,2)(3,4)(5,6)(7,8), (1,5)(2,6)(3,7)(4,8) \rangle$ which preserves the two partitions $P_1 = \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}$ and $P_2 = \{\{1,3\},\{2,4\},\{5,7\},\{6,8\}\}$. Moreover, $P_1 \lor P_2 = \{\{1,2,3,4\},\{5,6,7,8\}\}$. Thus $G^+ = T^8 \rtimes K_1$ is primitive of type HC on each vertex orbit while $G$ is quasiprimitive but not primitive of type CD on edges.
Example 5.12. If $G$ is edge-quasiprimitive but not edge-primitive it is not even necessary for $G^+$ to be quasiprimitive on each orbit. For example, let $G = N \rtimes K$ where $N = T^{4d}$ and $K = (S_d \wr S_2) \wr S_2$ such that $K$ preserves the partition $\{\{1, \ldots, d\}, \{d+1, \ldots, 2d\}, \{2d+1, \ldots, 3d\}, \{3d+1, \ldots, 4d\}\}$. Now $K$ has an index two subgroup $K^+ = (S_d \wr S_2)^2$ which has two orbits of size $2d$ on $\{1, \ldots, 4d\}$ and acts imprimitively on each orbit. Then with $G^+ = N \rtimes K^+$, and the two partitions $P_1 = \{\{1, \ldots, d\}, \{d+1, \ldots, 2d\}, \{2d+1, \ldots, 4d\}\}$ and $P_2 = \{\{1, \ldots, d\}, \{d+1, \ldots, 2d\}, \{2d+1, \ldots, 4d\}\}$, Construction 5.6 yields a $G$-edge-quasiprimitive graph such that $G^+$ is not quasiprimitive on either of its orbits (since the strips of $N_v$ are not all of equal length).

6 Analysing the quasiprimitive and primitive types

In this section we determine all the possible types of edge and vertex actions of edge-quasiprimitive graphs (Theorem 6.12). From this, after a bit more work we deduce Theorem 1.2. By Lemmas 3.2, 3.5 and 3.6 there are three types of vertex actions for $G$-edge-quasiprimitive graphs to consider:

- $G$-vertex-intransitive where $G$ acts faithfully and quasiprimively on both orbits;
- $G$-vertex-quasiprimitive;
- $G$-vertex-biquasiprimitive and $G^+$ faithful on each orbit.

We go through each of the 8 types of quasiprimitive groups as possibilities for the edge action and determine if there is a suitable vertex action in each case.

Lemma 6.1. Let $\Gamma$ be a connected $G$-edge-quasiprimitive graph such that $G$ is of type HA on edges. Then $\Gamma$ is either a cycle of prime length or a complete bipartite graph.

Proof. The unique minimal normal subgroup $N$ of $G$ is elementary abelian and $G$ is in fact edge-primitive. Since $N$ is edge-transitive, it is either vertex-transitive or has two orbits. If $N$ is vertex-transitive, then since $N$ is abelian it acts regularly on $V\Gamma$ and so $|V\Gamma| = |E\Gamma|$. Hence $\Gamma$ is a cycle and by the primitivity of $G$ on $E\Gamma$ it follows that $\Gamma$ has prime length. If $N$ has two orbits, then $G$ is quasiprimitive on vertices and so by Lemma 3.6, either $\Gamma$ is complete bipartite or $N$ acts faithfully on each orbit. In the latter case, $N$ acts regularly on each orbit and so there are twice as many vertices as edges. This contradicts the fact that $\Gamma$ is connected and so $\Gamma$ is complete bipartite. □

Lemma 6.2. Let $\Gamma$ be a connected $G$-edge-quasiprimitive graph such that $G$ is quasiprimitive of type HS or HC on edges. Then $\Gamma$ is a complete bipartite graph.

Proof. Let $N_1$ and $N_2$ be the two minimal normal subgroups of $G$. Since $G$ is of type HS or HC on edges, it is edge-primitive and so by Lemma 3.3 either $\Gamma$ is a star (and hence complete bipartite), or $G$ is vertex-transitive. We may assume that $G$ is vertex-transitive. Then by Lemma 3.5, $G^{VT}$ is either quasiprimitive or biquasiprimitive. If $G^{VT}$ is quasiprimitive then since $G$ has two minimal normal subgroups, $G^{VT}$ is of type HS or HC, respectively. Hence $N_1$ and $N_2$ are vertex-regular and so $|E\Gamma| = |V\Gamma|$. Thus $\Gamma$ is a cycle, contradicting $N_1$ being insoluble. Thus $G^{VT}$ is biquasiprimitive. Suppose
that \( \Gamma \) is not complete bipartite. Since neither \( N_1 \) nor \( N_2 \) has an index two subgroup, it follows that \( N_1, N_2 \leq G^+ \) and by Lemma 3.6, both act transitively and faithfully on each \( G^+ \) orbit. Since \( N_1 \) centralises \( N_2 \), it follows that \( N_1 \) and \( N_2 \) act regularly on each \( G^+ \) orbit ([5, Theorem 4.2A]). This implies that there are twice as many vertices as edges, contradicting \( \Gamma \) being connected. Hence \( \Gamma \) is a complete bipartite graph. \( \square \)

**Lemma 6.3.** Let \( \Gamma \) be a \( G \)-edge-quasiprimitive graph which is of type AS on edges. Then either \( G \) is quasiprimitive of type AS on vertices or \( \Gamma \) is bipartite and \( G^+ \) acts faithfully and quasiprimitively of type AS on both parts of the bipartition.

**Proof.** Noticing that any nontrivial normal subgroup of \( G \) is almost simple, the result follows by comparing isomorphism types and Lemmas 3.2 and 3.5. \( \square \)

Before dealing with the SD and CD cases we need the following lemma.

**Lemma 6.4.** Let \( \Gamma \) be a connected \( G \)-edge-quasiprimitive graph and let \( N \) be a normal subgroup of \( G \) such that \( N \cong T^k \) for some finite nonabelian simple group \( T \). Let \( e = \{v, w\} \) be an edge of \( \Gamma \). Then \( N_e \neq N_v \).

**Proof.** Suppose that \( N_e = N_v \). Since \( N \) is edge-transitive, it has at most two orbits on vertices. If \( N \) is vertex-transitive then \( |V\Gamma| = |E\Gamma| \) and so \( \Gamma \) is a cycle. This contradicts \( N \leq \text{Aut}(\Gamma) \). Hence \( N \) has two orbits on vertices and \( \Gamma \) is bipartite. Let \( \Delta_1 \) be the bipartite half containing \( v \). Then \( |\Delta_1| = |E\Gamma| \). This contradicts \( \Gamma \) being connected and so \( N_e \neq N_v \). \( \square \)

Next we deal with the SD and CD cases.

**Proposition 6.5.** Let \( \Gamma \) be a \( G \)-edge-quasiprimitive, \( G \)-vertex-transitive connected graph which is not complete bipartite and such that \( G \) is quasiprimitive of type SD or CD on edges. Let \( N \cong T^k \) be the unique minimal normal subgroup of \( G \), let \( e = \{v, w\} \) be an edge and let \( \mathcal{P} \) be the partition of the set of \( k \) simple direct factors of \( N \) given by the set of supports of the full strips of \( N_e \). Then the following all hold.

1. \( \Gamma \) is bipartite and \( G^+ \) acts faithfully on each bipartite half.

2. There exists a nontrivial \( G^+ \)-invariant partition \( \mathcal{P}_1 \) of \( \{1, \ldots, k\} \) such that \( N_v \) is the product of full strips whose supports are the parts of \( \mathcal{P}_1 \).

3. There exists a nontrivial \( G^+ \)-invariant partition \( \mathcal{P}_2 \) of \( \{1, \ldots, k\} \) such that \( N_w \) is the product of full strips whose supports are the parts of \( \mathcal{P}_2 \).

4. \( \mathcal{P}_1 \lor \mathcal{P}_2 = \mathcal{P} \) and \( G \) interchanges \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

5. \( \Gamma \) is isomorphic to the graph yielded by Construction 5.6 using \( G \), \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

**Proof.** Since \( N_e \cong T^l \) for some divisor \( l \) of \( k \), it does not have an index two subgroup and so \( N_e = N_{vw} \). Thus \( \pi_i(N_v) = T \) for each \( i \), and so by a well known lemma, (see for example [16, p 328]) there exists a partition \( \mathcal{P}_1 \) of \( \{1, \ldots, k\} \) such that \( N_v \) is the product of full strips whose supports are the parts of \( \mathcal{P}_1 \). Similarly, \( \pi_i(N_w) = T \) for each \( i \) and so there exists a partition \( \mathcal{P}_2 \) of \( \{1, \ldots, k\} \) such that \( N_w \) is the product of
full strips whose supports are the parts of $\mathcal{P}_2$. Since $N_e = N_v \cap N_w$, Lemma 4.2 implies that $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}$. By Lemma 6.4 $N_e \neq N_v$, and so $\mathcal{P}_1, \mathcal{P}_2 \neq \mathcal{P}$, hence $\mathcal{P}_1 \neq \mathcal{P}_2$. Thus $N$ is vertex-intransitive and $\mathcal{P}_1, \mathcal{P}_2$ are nontrivial partitions of $\{1, \ldots, k\}$. Since $N$ is edge-transitive, it follows that $\Gamma$ is bipartite with the two bipartite halves being $N$-orbits. By Lemmas 3.2 and 3.6, $G^+$ is faithful on each bipartite half and so (1) holds. Moreover, $G^+ = NG_v = NG_w$ and so $G^+$, $G_v$ and $G_w$ all induce the same permutation group on the set of $k$ simple direct factors of $N$. Hence $\mathcal{P}_1$ and $\mathcal{P}_2$ are $G^+$-invariant and so parts (2) and (3) hold. Furthermore, since $G$ is vertex-transitive there exists $g \in G$ such that $v^g = w$. Thus $G$ interchanges $\mathcal{P}_1$ and $\mathcal{P}_2$ and so part (4) holds. It remains to prove part (5).

Conjugating by a suitable element of $\text{Sym}(VT)$ we may assume that $N_v$ is a product of straight full strips corresponding to the parts of $\mathcal{P}_1$. Thus $N_v$ is the subgroup $N_1$ constructed in Construction 5.6. Since $G$ interchanges $\mathcal{P}_1$ and $\mathcal{P}_2$, it follows that $G_v \leq NG(N_v) \leq G^+$. Since $G^+ = NG_v$ and $N_v$ is selfnormalising in $N$, it follows that $G_v = N_G(N_v)$. Thus $G_v$ is the subgroup $H$ given in Construction 5.6. Letting $g \in G_e$ which interchanges $v$ and $w$ and hence $\mathcal{P}_1, \mathcal{P}_2$, it follows that $\Gamma \cong \text{Cos}(G, H, HgH)$, the graph constructed in Construction 5.6. Thus part (5) holds.

We have the following corollaries if $G$ is edge-primitive.

**Corollary 6.6.** Let $\Gamma$ be a connected $G$-edge-primitive graph which is not complete bipartite such that $G$ is primitive of type SD on edges. Then $\Gamma$ is bipartite and $G^+$ is faithful and quasiprimitive of type CD on each bipartite half.

**Proof.** Since $G^{ET}$ is primitive of type SD it follows that $\mathcal{P} = \{\{1, \ldots, k\}\}$ and $G$ acts primitively on the set of $k$ simple direct factors of $N$. Since $G^+ < G$ it follows that $G^+$ acts transitively on the the set of simple direct factors of $N$. Hence $N$ is a minimal normal subgroup of $G^+$ and so $G^+$ acts faithfully and quasiprimitively on each orbit. By Lemma 6.4, $N_e < N_v$ and so this action is of type CD.

**Corollary 6.7.** Let $\Gamma$ be a connected $G$-edge-primitive graph which is not complete bipartite such that $G$ is primitive of type CD on edges. Then $\Gamma$ is bipartite and $G^+$ is faithful and quasiprimitive of type CD on each bipartite half.

**Proof.** Let $N \cong T^k$ be the unique minimal normal subgroup of $G$. Let $e$ be an edge and let $\mathcal{P}$ be the system of imprimitivity of $\{1, \ldots, k\}$ given by the set of supports of the strips of $N_e$. Since $G$ is primitive of type CD on edges it follows that for $P \in \mathcal{P}$, $G_P$ acts primitively on $P$. Also $|G_P : G_P^+| \leq 2$. If $|G_P : G_P^+| = 1$ then $G_P^+$ acts primitively on $P$. However, by Proposition 6.5, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are preserved by $G^+$. Hence $P$ is a union of blocks of $\mathcal{P}_1$, contradicting $G_P^+$ acting primitively on $P$. Thus $|G_P : G_P^+| = 2$ and so $G^+$ is transitive on $\mathcal{P}$. Moreover, as $G_P$ is primitive on $P$ it follows that $G_P^+$ is transitive on $P$ and so $G^+$ is transitive on the set of $k$ simple direct factors on $N$. Hence $N$ is a minimal normal subgroup of $G^+$ and so $G^+$ acts faithfully and quasiprimitively on each orbit. By Lemma 6.4, $N_e < N_v$ and so this action is of type CD.

Next we investigate the case where $G$ is of type PA on edges.
Lemma 6.8. Let $\Gamma$ be a $G$-edge-quasiprimitive connected graph such that $G$ is of type PA on edges. Let $N$ be the unique minimal normal subgroup of $G$. Then $N_v \neq 1$.

**Proof.** Since $G$ is quasiprimitive of type PA on edges we have that $N_v \neq 1$. Suppose that $N_v = 1$. Then $|VT| \geq |N| > |E\Gamma|$, contradicting $\Gamma$ being connected. Thus $N_v \neq 1$. \qed

Corollary 6.9. Let $\Gamma$ be a $G$-edge-quasiprimitive connected graph such that $G$ is of type PA on edges. Suppose that $G$ is vertex-quasiprimitive. Then the quasiprimitive type of $G^{VT}$ is SD, CD or PA.

**Proof.** Let $N$ be the unique minimal normal subgroup of $G$. By Lemma 6.8, $N_v \neq 1$ and so $G$ is not of type TW on vertices. Since $G$ has a unique minimal normal subgroup which is not elementary abelian or simple, it follows that $G^{VT}$ is of type SD, CD or PA. \qed

Corollary 6.10. Let $\Gamma$ be a $G$-edge-quasiprimitive connected graph such that $G$ is of type PA on edges. Suppose that $G$ is vertex-intransitive. Then the quasiprimitive type of $G$ on each of its orbits is SD, CD or PA.

Corollary 6.11. Let $\Gamma$ be a $G$-edge-quasiprimitive connected graph such that $G$ is of type PA on edges. Suppose that $G$ is vertex-biquasiprimitive and $G^+$ is quasiprimitive on each orbit. Then the quasiprimitive type of $G^+$ on each of its orbits is HS, HC, SD, CD or PA.

Collecting together our results we have the following two theorems. We split the statements into the vertex-transitive and vertex-intransitive cases.

**Theorem 6.12.** Let $\Gamma$ be a $G$-edge-quasiprimitive, $G$-vertex-transitive connected graph of valency at least three such that $G^{ET}$ is of type $X$. Then one of the following holds.

1. $\Gamma$ is a complete bipartite graph.
2. $X \in \{SD, CD\}$ and $\Gamma$ can be obtained from Construction 5.6.
3. $X = PA$ and $G$ is quasiprimitive on VT of type SD, CD or PA.
4. $X = PA$ and $\Gamma$ is bipartite, such that $G^+$ is faithful and quasiprimitive on each of its orbits of type $Y \in \{HS, HC, SD, CD, PA\}$.
5. $X = PA$, $\Gamma$ is bipartite, and $G^+$ is not quasiprimitive on either orbit.
6. $X = AS$ and either $G^{VT}$ is quasiprimitive of type AS or $\Gamma$ is bipartite and $G^+$ is faithful and quasiprimitive of type AS on each of its orbits.
7. $X = TW$.

Moreover, examples occur in all cases.
Examples 5.8 and 5.10 provide edge-primitive examples for case (2), Construction 5.1 gives examples for case (3) where \( G \) is primitive of type PA on vertices, Construction 5.3 gives examples where \( G^+ \) is primitive of type PA on each orbit and Section 2 gives many example for case (6). An edge-primitive example for case (5) is given by Remark 5.5. Examples of edge-quasiprimitive but not edge-primitive are given in Section 7.

If \( G \) is edge-primitive we can sometimes deduce more information. For example, we can eliminate \( X = \text{TW} \).

**Proposition 6.13.** Let \( \Gamma \) be a \( G \)-edge-primitive graph such that \( G \) is of type \( \text{TW} \) on edges. Then \( \Gamma \) is a complete bipartite graph.

**Proof.** Let \( \Gamma \) be a \( G \)-edge-primitive graph such that \( G \) is of type \( \text{TW} \) on edges. Let \( N \) be the unique minimal normal subgroup of \( G \). Then \( N = T_1 \times \cdots \times T_k \) with each \( T_i \cong T \) for some finite nonabelian simple group \( T \) and \( G = N \rtimes G_e \). Moreover, \( G_e \) acts transitively by conjugation on the set of \( k \) simple direct factors of \( N \). Let \( (G_e)_1 \) be the normaliser of \( G_e \) on the set \( T \) such that each \( T \) contains a partition \( (G_e)_1 \cap G_w \vartriangleleft G_w \) such that \( \varphi \) extends to \( R \). Thus \( \varphi \) would extend to \( \langle (G_e)_1, R \rangle \leq G_e \). Since \( \varphi \) does not extend to any overgroup of \( (G_e)_1 \) in \( G_e \) it follows that \( R \leq (G_e)_1 \) and so \( R = G_W \cap (G_e)_1 \). Thus by Lemma 4.1, \( (G_e)_1 \cap G_w \) is maximal in \( N \times ((G_e)_1 \cap G_w) \). Since \( G_w \) normalises \( N_v \) and \( N_w \), it follows that \( N_v = N_w = 1 \). Thus \( |\text{VT}| = |\text{N}| \text{ or } 2|\text{N}| \). However, \( |\text{ET}| = |\text{N}| \) and so \( |\text{VT}| = |\text{N}| \) and \( \Gamma \) is a cycle. This contradicts \( G \) being insoluble and so \( (G_e)_1 \leq G_w \).

Since \( (G_e)_1 \leq G_w \) it follows that \( G_w \) has two equal sized orbits on the set of \( k \) simple direct factors of \( N \). Without loss of generality we may suppose that these are \( \{ T_1, \ldots, T_k/2 \} \) and \( \{ T_{k/2+1}, \ldots, T_k \} \) and note that they are interchanged by elements of \( G_e \) not in \( G_w \). Moreover, \( (G_e)_1 \) normalises \( N_v \). Since \( \varphi ((G_e)_1) \) contains \( \text{Inn}(T) \) it follows that the projection of \( N_v \) onto the first simple direct factor of \( N \) is either trivial or equal to \( T \). Thus \( N_v \) is a subdirect product of either \( T_{k/2+1} \times \cdots \times T_k \) or \( N_{k/2} \leq T_{k/2} \times \cdots \times T_k / 2 \). Moreover, \( G_e \) normalises \( \{ N_v, N_w \} \) and so by the maximality of \( G_e \) in \( G \) we have \( \langle N_v, N_w \rangle = N \). Thus \( N_v = T_{k/2+1} \times \cdots \times T_k \), and so \( N \) has two orbits on vertices and is faithful on each. Hence by Lemma 3.6, \( \Gamma \) is a complete bipartite graph. Thus we are left to consider the case where \( N_v \) is a subdirect product of \( N \). Thus there exists a partition \( \mathcal{P} \) of \( \{ 1, \ldots, k \} \) such that \( N_v = \prod_{I \in \mathcal{P}} T_I \) where \( T_I \) is a diagonal subgroup of \( \prod_{i \in I} T_i \). Since \( (G_e)_1 \leq G_w \leq G_e \), it follows from Lemma 4.1 that \( G_w \) is a maximal subgroup of \( \langle T_1 \times \cdots \times T_k/k \rangle \times G_w \). Hence \( N_v \cap (T_1 \times \cdots \times T_k/k) = 1 \). Similarly, \( G_w \) is maximal in \( (T_{k/2+1} \times \cdots \times T_k) \times G_w \) and so \( N_v \cap (T_{k/2+1} \times \cdots \times T_k) = 1 \). It follows that each \( I \in \mathcal{P} \) is split equally between \( \{ 1, \ldots, k/2 \} \) and \( \{ k/2+1, \ldots, k \} \). However, since \( G_w \) normalises \( N_v \) and \( M = T_1 \times \cdots \times T_k \) it follows that \( G_w \) normalises the projection of \( N_v \) onto \( M \). Thus \( |I| = 2 \), as \( G_w \) normalises no proper nontrivial subgroup of \( M \). Hence \( |N_v| = |T|^{k/2} \) and \( |\text{VT}| = |T|^{k/2} \) or \( 2|T|^{k/2} \). The first case is not possible.
as $|VT|^2 = |ET|$, a contradiction. Hence we have the second. This implies that $\Gamma$ is complete bipartite and we are done.

We can also deduce more information when $X = PA$.

**Lemma 6.14.** Let $\Gamma$ be a $G$-edge-primitive graph such that $G$ is of type PA on edges and $\Gamma$ is not complete bipartite. Then one of the following holds:

1. $G$ is quasiprimitive on vertices of type PA;
2. $G$ is biquasiprimitive and $G^+$ is quasiprimitive of type PA on each bipartite half;
3. $G$ is biquasiprimitive and $G^+$ is not quasiprimitive on either bipartite half.

**Proof.** Let $N$ be the unique minimal normal subgroup of $G$. Then $N = T^k$ for some finite nonabelian simple group $T$ and $k \geq 2$. Also given an edge $e = \{v, w\}$ we have $N_v = R^k$ for some proper nontrivial subgroup $R$ of $T$. Since $G^{ET}$ is primitive, there exists an almost simple group $A$ with socle $T$ and maximal subgroup $H$ such that $H \cap T = R$. Suppose that $|R| = 2$. Then $H = C_A(z)$ and $R = C_T(z)$, where $z$ is the involution which generates $R$. However, 4 divides $|T|$ and so either $z$ is contained in a cyclic group of order 4 or an elementary abelian group of order 4, a contradiction. Thus $|R| > 2$. It follows that $N_v$ does not have an index 2 subgroup and so $N_v/\sim = N_v$. Hence $R^k \leq N_v$. Thus for each $i$ such that $\pi_i(N_v) = T$, we have that $N_v$ contains the $i^{th}$ factor of $N$. Since $\Gamma$ is not complete bipartite, Lemma 3.6 implies that $N$ is faithful on each of its orbits on $VT$, and so $N$ cannot contain any of its simple direct factors. Thus $\pi_i(N_v) \neq T$ for all $i$. Hence if $G$ is quasiprimitive on $VT$, this implies that $G$ is of type PA on vertices and we have case (1). If $G$ is biquasiprimitive on vertices and $G^+$ is transitive on the set of simple direct factors of $N$ then we have that $G^+$ is quasiprimitive of type PA on each of its orbits and we have case (2). If $G^+$ has two orbits on the set of simple direct factors of $N$ then $G^+$ has two minimal normal subgroups contained in $N$. Since $N_v$ does not project onto $T$ in any coordinate, it follows that $G^+$ is not quasiprimitive on either orbit and so case (3) holds.

Note that when $G$ is biprimitive on vertices, $G^+$ is primitive on each bipartite half. Hence Lemma 6.14 combined with Theorem 6.12, Corollaries 6.6 and 6.7, and Proposition 6.13 yields Theorem 1.2.

We complete this section by reducing the study of edge-primitive graphs of type PA to the study of edge-primitive graphs of type AS. Before doing so we need to establish some notation. Let $T$ be a nonabelian simple group and let $G$ be a subgroup of $\text{Aut}(T) \wr S_k$ for some $k \geq 2$, which contains $N = T_1 \times \cdots \times T_k$ where each $T_i \cong T$, and such that $G$ induces a transitive subgroup of $S_k$ on the set of $k$ simple direct factors of $N$. Let $G_1$ be the normaliser in $G$ of $T_1$. Then $G_1 = G \cap (\text{Aut}(T) \times (\text{Aut}(T) \wr S_{k-1}))$ and there exists a projection $\pi_1 : G_1 \to \text{Aut}(T)$. Let $B = \pi_1(G_1)$. By [12, (2.2)], conjugating by a suitable element of $\text{Aut}(T) \wr S_k$ we may have chosen $G$ such that $G \leq B \wr S_k$. We call $B$ the group induced by $G$.

**Proposition 6.15.** Suppose that $\Gamma$ is a $G$-edge-primitive graph such that $G^{ET}$ is of type PA, and let $e = \{v, w\}$ be an edge. Let $N = \text{soc}(G) \cong T^k$ for some finite nonabelian simple group $T$ and $k$ a positive integer at least two. Suppose that $G$ induces the primitive
almost simple group $B$ with socle $T$, and that $G_e$ and $G_v$ induce the subgroups of $B$ respectively. Then there exists a $B$-edge-primitive graph with edge-stabiliser $E$ and vertex-stabiliser $H$.

Proof. Since $G$ is a primitive group of type PA on $E T$, we have that $G \leq B \wr S_k$ and $G_e = G \cap (E \wr S_k)$ where $\pi_1((G_1)_e) = E$ is a maximal subgroup of $B$. Let $A = \pi_1((G_1)_v)$ and $H = \pi_1((G_1)_e)$. Note that $H \cap E = A$ and $H$ is a proper subgroup of $B$. Since $G$ is arc-transitive, $|G_e : G_{vw}| = 2$ and so $|(G_1)_e : (G_1)_{vw}| \leq 2$. Thus $|E : A| \leq 2$. If $E = A$ then $E \leq H$. However, by the maximality of $E$ this implies that $E = H$ and so $G_e$ is contained in some $G$-conjugate of $G_e$. This contradicts the fact that there are more edges than vertices and so $|E : A| = 2$. For the same reason $A < H$. Let $\sigma \in (G_1)_e \setminus (G_1)_{vw}$. Then $g = \pi_1(\sigma) \in E \setminus A$ and $\Cos(G, H, HgH)$ is a $G$-edge-primitive graph with edge stabiliser $E$ and vertex stabiliser $H$. \hfill $\Box$

7 Quasiprimitive examples

In this section we construct examples of edge quasiprimitive graphs where the types of actions do not occur in the edge primitive case.

Example 7.1. (Quasiprimitive PA on edges and primitive SD on vertices) Let $G = T \wr S_2$ for some finite nonabelian simple group $T$ and let $H = \{(t, t) \mid t \in T\} \rtimes \langle \sigma \rangle$, where $\sigma$ interchanges the two simple direct factors of $N = T^2$. Let $x \in T$ be of order two and let $g = (1, x) \in G$. Then

$$H^g = \{(t, t^x) \mid t \in T\} \rtimes \langle (x, x) \sigma \rangle = \{(t, t^x) \mid t \in T\} \rtimes \langle \sigma \rangle$$

and $H \cap H^g = \{(t, t) \mid t \in C_T(x)\} \rtimes \langle \sigma \rangle$. Let $\Gamma = \Cos(G, H, HgH)$. Then $G$ is vertex-primitive of type SD. Let $e = \{H, Hg\}$, an edge of $\Gamma$. Then $G_e = \langle H \cap H^g, g \rangle$. Since $x \in C_T(x)$, it follows that $G_e = \{(x^t, x^d) \mid t \in C_T(x) ; i, j \in \{0, 1\}\} \rtimes \langle \sigma \rangle$ and so $G$ is quasiprimitive of type PA on edges.

Example 7.2. (Quasiprimitive PA on edges and primitive CD on vertices) Let $\sigma = (1, 2, 3, 4)$ and $G = T^4 \rtimes \langle \sigma \rangle$ for some finite nonabelian simple group $T$. Let $H = \{(t, s, t, s) \mid t, s \in T\} \rtimes \langle \sigma \rangle$ and $g = (x, x, 1, 1)$ where $x \in T$ has order two. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \Cos(G, H, HgH)$. Then $G$ is primitive of type CD on vertices. Let $v$ be the vertex corresponding to $H$ and $w$ the vertex corresponding to $Hg$. Then $G_w = H^g = \{(t^x, s, t, s) \mid t, s \in T\} \rtimes \langle \sigma \rangle$ and so for the edge $e = \{v, w\}$ we have $G_e = \{(tx^i, sx^j, tx^k, sx^l) \mid t, s \in C_T(x) ; i + j + k + l \equiv 0 \pmod{2}\} \rtimes \langle \sigma \rangle$. Thus $G$ is quasiprimitive of type PA on edges.

We now give examples in the bipartite case.

Example 7.3. (Quasiprimitive PA on edges and $G^+$ primitive HS on each vertex orbit) Let $T$ be a nonabelian simple group and let $x \in T$ have order 2. Let $G = T \wr S_2$ and $H = \{(t, t) \mid t \in T\}$. Let $g = (x, 1) \sigma$ where $\sigma$ interchanges the two simple direct factors of $N = T^2$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \Cos(G, H, HgH)$. Then $G^+ = N$ has two orbits on vertices. Let $v$ be the vertex corresponding to $H$ and
w be the vertex corresponding to $Hg$. Then $G_v = H$ and $G_w = H^g = \{(t, t^e) \mid t \in T\}$. Thus $G^+$ is primitive of type HS on each orbit. Moreover, $e = \{v, w\}$ is an edge and $G_e = \{(t, t) \mid t \in C_T(x)\} \times \langle g \rangle$. Thus $G$ is quasiprimitive of type PA on edges.

**Example 7.4.** (Quasiprimitive PA on edges and $G^+$ primitive HC on each vertex orbit) Let $\sigma = (1,2,3,4)$ and let $G = T \operatorname{wr}(\sigma)$ for some finite nonabelian simple group $T$. Let $H = \{(t, t, s, s) \mid t, s \in T\} \times \langle \sigma^2 \rangle$ and let $x \in T$ of order 2. Let $g = (1, x, 1, x)\sigma$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \operatorname{Cos}(G, H, HgH)$, let $v$ be the vertex corresponding to $H$ and $w$ the vertex corresponding to $Hg$. Then $\Gamma$ is bipartite with $G^+ = T^4 \rtimes \langle \sigma^2 \rangle$ and $e = \{v, w\}$ is an edge. Moreover, $G^+$ is primitive of type HC on each orbit. Now $G_v = H$ and $G_w = H^g = \{(t, t^e, s, s^e) \mid t, s \in T\} \rtimes \langle \sigma^2 \rangle$. Thus $G_e = \{(t, t, s, s) \mid t, s \in C_T(x)\} \rtimes \langle g \rangle$. Hence $G$ is quasiprimitive of type PA on edges.

**Example 7.5.** (Quasiprimitive PA on edges and $G^+$ primitive SD on each vertex orbit) Let $T$ be a nonabelian simple group with outer automorphism $\tau$ of order two. Let $G = (T \times T) \rtimes \langle (1, \tau), \sigma \rangle$ where $\sigma$ interchanges the two minimal normal subgroups of $N = T^2$. Let $H = \{(t, t) \mid t \in (T, \tau)\} \times \langle \sigma \rangle$ and let $g = (1, \tau)$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Thus we can define the graph $\Gamma = \operatorname{Cos}(G, H, HgH)$. Let $v$ be the vertex corresponding to $H$ and $w$ the adjacent vertex corresponding to $Hg$. Then $G_v = H$ and $G_w = H^g = \{(t, t^e) \mid t \in (T, \tau)\} \rtimes \langle \sigma \rangle$. Hence $G^+ = T^2 \rtimes \langle (\tau, \tau), \sigma \rangle$ acts primitively of type SD on each orbit. Let $e = \{v, w\}$. Then $G_e = \{(t, t) \mid t \in C_T(\tau)\} \rtimes \langle (1, \tau), \sigma \rangle$ and so $G$ is quasiprimitive of type PA on edges.

**Example 7.6.** (Quasiprimitive PA on edges and $G^+$ primitive CD on each vertex orbit) Let $\sigma = (1,2,3,4)$, $T$ be a finite nonabelian simple group and $\tau$ an outer automorphism of $T$ of order two. Let $G = T^4 \rtimes \langle (\tau, \tau, 1, 1), \sigma \rangle$ and $H = \{(t, s, t, s) \mid t, s \in T\} \rtimes \langle (1, \tau, 1, \tau), \sigma \rangle$. Then letting $g = (\tau, \tau, 1, 1)$ we see that $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Thus we can define the graph $\operatorname{Cos}(G, H, HgH)$. Then $G^+ = T^4 \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$ acts primitively of type CD on each vertex orbit. Let $v$ be the vertex corresponding to $H$ and $w$ be the adjacent vertex corresponding to $Hg$. Then $G_v = H$ and $G_w = H^g = \{(t^e, s^e, t, s) \mid t, s \in T\} \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$. Thus $G_e = \{(t, s, t, s) \mid t, s \in C_T(\tau)\} \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$ and so $G$ acts quasiprimitively of type PA on edge.

**Construction 7.7.** (Quasiprimitive of type TW on edges and $G^+$ primitive of type PA on both orbits.) Let $T$ be a finite nonabelian simple group with maximal subgroup $R$ and suppose that there exists and outer automorphism $\tau$ or order two such that $R \cap R^\tau = 1$. A suitable choice of $T$ and $R$ is $\operatorname{PSL}(2,29)$ and $A_5$ respectively. Let $G = \langle T^k, (\tau, \ldots, \tau) \rangle \rtimes S_k$ and $H = R^k \rtimes S_k$. Then if $g = (\tau, \ldots, \tau)$ we have $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Hence $\Gamma = \operatorname{Cos}(G, H, HgH)$ is a $G$-arc-transitive connected graph. Moreover, $\langle g \rangle \rtimes S_k$ is the stabiliser of an edge. Thus letting $N = \operatorname{soc}(G) = T^k$ we have that $N$ acts regularly on $ET$ and so $G$ is quasiprimitive of type TW on edges. Note that $G^{ET}$ is not primitive as an edge stabiliser is not maximal. Furthermore, $\Gamma$ is bipartite with $G^+ = T^k \rtimes S_k$ acting primitively of type PA on both orbits.

### 8 Edge-primitive groups with socle $\operatorname{PSL}(2, q)$

The following theorem of Dickson [4] determines the maximal subgroups of $\operatorname{PSL}(2, q)$.
Theorem 8.1. Let $p$ be a prime, $f$ a positive integer and $q = p^f$. Then the conjugacy classes of maximal subgroups of $PSL(2, q)$ are as follows:

1. one class of subgroups isomorphic to $[q] \rtimes C_{(q-1)/(2,q-1)}$,
2. one class of subgroups isomorphic to $D_{2(q-1)/(2,q-1)}$, if $q \notin \{5, 7, 9, 11\}$,
3. one class of subgroups isomorphic to $D_{2(q+1)/(2,q-1)}$, if $q \notin \{7, 9\}$,
4. two classes of subgroups isomorphic to $A_5$, if $q \equiv \pm 1 \pmod{10}$, and $\mathbb{F}_q = \mathbb{F}_p[\sqrt{5}]$,
5. two classes of subgroups isomorphic to $S_4$, if $q = p \equiv \pm 1 \pmod{8}$,
6. one class of subgroups isomorphic to $A_4$, if $q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$,
7. two classes of subgroups isomorphic to $PGL(2, p^{f/2})$ when $p$ odd,
8. one class of subgroups isomorphic to $PSL(2, p^m)$ where $f/m$ an odd prime or $p = 2$ and $m \geq 2$.

We also have the following theorem about maximal subgroups of almost simple groups with socle $PSL(2, q)$.

Theorem 8.2. [6, Theorem 1.1] Let $T = PSL(2, q) \leq G \leq PGL(2, q)$ and let $E$ be a maximal subgroup of $G$ which does not contain $T$. Then either $E \cap T$ is maximal in $T$, or we have one of the following cases.

1. $G = PGL(2, 7)$ and $E = N_G(D_6) = D_{12}$.
2. $G = PGL(2, 7)$ and $E = N_G(D_8) = D_{16}$.
3. $G = PGL(2, 9)$, $M_{10}$ or $PGL(2, 9)$ and $E = N_G(D_{10})$.
4. $G = PGL(2, 9)$, $M_{10}$ or $PGL(2, 9)$ and $E = N_G(D_8)$.
5. $G = PGL(2, 11)$ and $E = N_G(D_{10}) = D_{20}$.
6. $G = PGL(2, q)$, $q = p \equiv \pm 11, \pm 19 \pmod{40}$ and $E = N_G(A_4) = S_4$.

The following lemma shows that except for the 8 exceptions in Theorem 8.2, we can restrict our attention to searching for $G$-arc-transitive $G$-edge primitive graphs with $G = PSL(2, q)$.

Lemma 8.3. Let $\Gamma$ be a nontrivial $G$-edge-primitive connected graph with $T = PSL(2, q) \vartriangleleft G \leq PGL(2, q)$. Let $E$ be the stabiliser in $G$ of an edge of $\Gamma$. If $E \cap T$ is maximal in $T$ then $T$ is arc-transitive and edge-primitive.

Proof. Since $G$ is edge-primitive and $T \vartriangleleft G$ it follows that $T$ acts transitively on the set of edges with edge stabiliser $E \cap T$. Hence $T$ is edge-primitive. Since $\Gamma$ is $G$-arc-transitive, $\Gamma$ is not a star and so by Lemma 3.4, $\Gamma$ is also $T$-arc-transitive. 

We have the following proposition.
Proposition 8.4. Let $G = \text{PSL}(2, q)$, where $q = p^f$, $E$ be a maximal subgroup of $G$ and $H$ be a subgroup of $G$ such that $A = H \cap E$ is an index two subgroup of $E$ and a proper subgroup of $H$. Suppose $G$ is not 2-transitive on the set of cosets of $H$. Then one of the following holds.

1. $q = p \equiv \pm 1, \pm 9 \pmod{40}$, $E = S_4$, $H = A_5$ and $A = A_4$.
2. $q = 17$, $E = D_{16}$, $H = S_4$ and $A = D_8$.
3. $q = 19$, $E = D_{20}$, $H = A_5$ and $A = D_{10}$.
4. $q = 25$, $E = D_{24}$, $H = \text{PGL}(2, 5)$ and $A = D_{12}$.

In the first case, given $E$ there are two choices for $H$ and these are conjugate in $T$. In the last three cases, given $E$ there are four choices for $H$ and these come in conjugate pairs.

Proof. We work our way through the list of maximal subgroups of $G$ given in Theorem 8.1. We note first that $E$ cannot be $A_5$, $A_4$, $\text{PSL}(2, p^m)$ for $p^m \neq 2$, or $[q] \rtimes C_{q-1}$ for $q$ even, as these groups do not have an index 2 subgroup. Furthermore, $E \neq [q] \rtimes C_{(q-1)/2}$ for $q$ odd as the only possible index 2 subgroup is $[q] \rtimes C_{(q-1)/4}$ which is only contained in $E$.

Suppose next that $E = D_{2(q-1)/(2q-1)}$ and note that $q \notin \{5, 7, 9, 11\}$. Then $A = C_{(q-1)/(2q-1)}$, the stabiliser of two points of the projective line, is an index two subgroup of $E$. The only possibility for $H$ is a subgroup isomorphic to $[q] \rtimes C_{(q-1)/(2q-1)}$, but in this case the action of $G$ is 2-transitive. If $(q-1)/(2q-1)$ is even then $E$ also contains two subgroups isomorphic to $D_{(q-1)/2}$ which are conjugate in $\text{PGL}(2, q)$ but not $\text{PSL}(2, q)$. The restrictions on $q$ imply that $(q-1)/(2q-1) \geq 6$ and so if $A \cong D_{(q-1)/2}$ then $A$ is not contained in a $D_{q+1}$. Furthermore, $A$ is not contained in an $A_4$. If $A$ is contained in an $A_5$ then $(q-1)/2 = 6$ or 10. The first implies that $q = 13$, but $\text{PSL}(2, 13)$ does not contain an $A_5$ while the second implies that $q = 21$, a contradiction. Thus $A$ is not contained in an $A_5$. If $A$ is contained in an $S_4$ then $(q-1)/2 = 6$ or 8. Again the first is not possible as $\text{PSL}(2, 13)$ does not contain an $S_4$ and so $q = 17$. Since $D_8$ is maximal in $S_4$, it follows that in this case we have $H \cong S_4$. Counting again shows that given $A$ there are two choices for $H$ and these are conjugate in $T$. The two nonconjugate choices for $A$ give us two nonconjugate pairs of choices for $H$. Thus we are in case (2). If $A \leq \text{PGL}(2, p^{f/2})$ then $(q-1)/2$ divides either $2(p^{f/2} - 1)$ or $2(p^{f/2} + 1)$. Since $q - 1 = (p^{f/2} - 1)(p^{f/2} + 1)$ either $p^{f/2} - 1$ or $p^{f/2} + 1$ divides 4. Thus $p^{f/2} = 3$ or 5. Since $q \neq 9$ this gives us $D_{12} \leq \text{PGL}(2, 5) \leq \text{PSL}(2, 25)$. Counting again gives that there are two choices for $H$ and these are conjugate in $T$. Again the two nonconjugate choices for $A$ give nonconjugate pairs of choices for $H$ and we have case (4). If $A \leq \text{PSL}(2, p^{f/r})$ for $r \geq 3$ then $(q-1)/2$ divides either $p^{f/r} - 1$ or $p^{f/r} + 1$. Since $r \geq 3$ we have $p^{f/r} - 1 > 2(p^{f/r} + 1)$ and so this is not possible.

Next let $E = D_{2(q+1)/(2q-1)}$ with $q \notin \{7, 9\}$. One choice for $A$ is $C_{(q+1)/(2q-1)}$. If $q = 5$ then $A = C_3$ and so $H \cong A_4$. However, in this case $G$ is 2-transitive on the cosets of $H$. Thus $(q+1)/(2q-1) \geq 6$ and so there is no possibility for $H$. If $(q+1)/2$ is even then $A$ can also be one of the two choices of $D_{(q+1)/(2q-1)}$ which are conjugate in $\text{PGL}(2, q)$ but not $\text{PSL}(2, q)$. Note then that $q \geq 11$ and so $(q+1)/(2q-1) \geq 6$. 

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Thus $A$ is not contained in $A_4$. Since $(q + 1)/2 \geq 6$ does not divide $q - 1$ it follows that $A$ is not contained in $D_{q-1}$. Now $A \leq A_5$, if and only if $(q + 1)/2 = 10$ or $6$. For $A_5$ to be a subgroup of $G$ we require that $q = 11$ or $19$. We do not have the first case as this yields a 2-transitive group. There are then two choices for $H$ and these are conjugate in $T$. Moreover the two nonconjugate choices for $A$ gives nonconjugate pairs of choices for $H$ and we have case (3). To have $A \leq S_4$ we require $(q + 1)/2 = 8$ or $6$. The first is not possible while the second has $q = 11$ in which case there is no $S_4$. To have $A \leq \mathrm{PGL}(2, p^{f/2})$ we require that $(q + 1)/2$ divides either $2(p^{f/2} - 1)$ or $2(p^{f/2} + 1)$. Hence $p^f + 1$ divides either $4(p^{f/2} - 1)$ or $4(p^{f/2} + 1)$ and so $p^{f/2} - 1 \leq 4$. This implies that $p^{f/2} = 3$ or $5$. However, we then have $q = 9$ or $25$, and in both cases $(q + 1)/2$ is odd. Hence $A$ is not contained in $\mathrm{PGL}(2, p^{f/2})$. For $A \leq \mathrm{PSL}(2, p^m)$, for some $m < f/2$, we need $(q + 1)/2$ to divides either $p^m - 1$ or $p^m + 1$. Neither of these are possible and so $A$ and $H$ are one of the groups listed.

Suppose next that $E = S_4$ and $q = p \equiv \pm 1 \pmod{8}$. Then $A = A_4$. Since $q = p$, the only other subgroup of $G$ containing $A$ is $H \cong A_5$ when $q \equiv \pm 1 \pmod{10}$. Since each $A_5$ contains 5 copies of $A_4$ and the normaliser in $G$ of $A_4$ is $S_4$ it follows that there are two choices for $H$. This gives case (1).

Finally, if $E = \mathrm{PGL}(2, p^{f/2})$ with $p$ odd then $A = \mathrm{PSL}(2, p^{f/2})$. The only way that $A$ can be contained in another maximal subgroup of $G$ is if $A$ is soluble. Hence $q = 9$, $A = \mathrm{PSL}(2, 3) \cong A_4$. Looking at the maximal subgroups of $G$ it follows that $H \cong A_5$. However, in this case $G$ is 2-transitive on the cosets of $H$, a contradiction.

We also need the following proposition concerning the exceptional cases in Theorem 8.2.

**Proposition 8.5.** Let $T = \mathrm{PSL}(2, q) \vartriangleleft G \leq \mathrm{PGL}(2, q)$ and suppose that $E$ is a maximal subgroup of $G$ not containing $T = \mathrm{PSL}(2, q)$ such that $E \cap T$ is not maximal in $T$. Suppose that $G$ has a subgroup $H$ such that $A = H \cap E$ is a proper subgroup of $H$ and has index two in $E$, and that $G$ is not 2-transitive on the set of cosets of $H$. Then one of the following holds.

1. $G = \mathrm{PGL}(2, 7)$, $E = D_{12}$, $H = S_4$ and $A = E \cap T = D_6$.

2. $G = \mathrm{PGL}(2, 7)$, $E = D_{16}$, $H = S_4$ and $A = E \cap T = D_8$.

3. $G = \mathrm{PGL}(2, 9)$, $M_10$, or $\mathrm{PGL}(2, 9)$, $E = N_G(D_8)$, $H = N_G(\mathrm{PGL}(2, 3))$ and $A = E \cap \mathrm{PGL}(2, 9)$.

4. $G = \mathrm{PGL}(2, 9)$, $M_10$, $\mathrm{PGL}(2, 9)$, $E = N_G(D_{10})$, $H = N_G(A_5)$ and $A = E \cap \mathrm{PGL}(2, 9)$.

5. $G = \mathrm{PGL}(2, 11)$, $E = D_{20}$, $H = C_{11} \rtimes C_{10}$ and $A = C_{10}$.

6. $G = \mathrm{PGL}(2, 11)$, $E = D_{20}$, $H = A_5$ and $A = E \cap T = D_{10}$.

7. $G = \mathrm{PGL}(2, q)$, $q = p \equiv \pm 11, \pm 19 \pmod{40}$, $E = S_4$, $H = A_5$ and $A = A_4$.

In each case there are two conjugate choices for $H$. 

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Proof. Note that $G$ and $E$ are given by Theorem 8.2. The first 6 cases can all be dealt with by looking at the list of maximal subgroups in [3]. If $G = \text{PGL}(2, q)$ for $q = p \equiv 11, 19, 21, 29 \pmod{40}$ and $E = S_4$ then the only possibility for $A$ is $A_4$. There are then two choices for $H$ being $A_5$ and these are the only possibilities. \hfill $\Box$

We can now determine all $G$-edge-primitive graphs with $\text{soc}(G) = \text{PSL}(2, q)$.

Proof. (of Theorem 1.3) Let $\Gamma$ be a $G$-edge-primitive graph such that $T = \text{soc}(G) = \text{PSL}(2, q)$ with $q > 3$. Then by Proposition 2.5 there exists a maximal subgroup $E$ of $G$ with an index 2 subgroup $A$ also contained in a proper corefree subgroup $H$ of $G$ such that $\Gamma \cong \text{Cos}(G, H, HgH)$ for some $g \in E \setminus A$. If $G$ is 2-transitive on the set of cosets of $H$ then $\Gamma$ is a complete graph and $G$ is primitive on 2-subsets. By Theorem 2.1, $G$ appears in Table 2. Thus we can assume that $G$ is not 2-transitive on vertices. Then by Proposition 8.5 either $\Gamma$ is $T$-edge-primitive with $E \cap T$, $A \cap T$ and $H \cap T$ given by Proposition 8.4, or $G$, $E$, $A$ and $H$ are given by Proposition 8.5.

Next let $q = p \equiv \pm 1, \pm 9 \pmod{40}$, $E \cap T = S_4$, $A \cap T = A_4$ and $H \cap T = A_5$. Since there are two conjugacy classes of $A_5$ subgroups in $\text{PSL}(2, q)$ and these are fused in $\text{PGL}(2, q)$ it follows that $\text{PGL}(2, q)$ is not an automorphism group of this graph and so have row 9 of Table 1.

The remaining cases from Proposition 8.4 are

1. $q = 17$, $E \cap T = D_{16}$, $A \cap T = D_8$ and $H \cap T = S_4$
2. $q = 19$, $E \cap T = D_{20}$, $A \cap T = D_{10}$ and $H \cap T = A_5$
3. $q = 25$, $E \cap T = D_{24}$, $A \cap T = D_{12}$ and $H \cap T = \text{PGL}(2, 5)$.

In all cases there are two $T$-conjugacy classes of subgroups $H \cap T$, and these are fused in $\text{PGL}(2, q)$. Hence we get isomorphic graphs. Also the only possibilities for $G$ are then $\text{PSL}(2, 17), \text{PSL}(2, 19), \text{PSL}(2, 25)$ and $\text{PSL}(2, 25)$. These give us rows 6–8 of Table 1.

It remains to deal with the groups left from Proposition 8.5.

If $G = \text{PGL}(2, 7)$, $E = D_{12}$, $A = D_6$ and $H = S_4$ then since $H \subseteq \text{PSL}(2, 7)$ it follows that $\Gamma$ is bipartite. Note that $G \cong \text{Aut}(\text{PSL}(3, 2))$, $H$ is the stabiliser in $\text{PSL}(3, 2)$ of a 1-space $U$ and $A$ is the stabiliser in $H$ of a 2-space which is complementary to $U$. Thus we have row 2.

Next let $G = \text{PGL}(2, 7)$, $E = D_{16}$, $A = D_8$ and $H = S_4$. Again we have that $\Gamma$ is bipartite, and $H$ is the stabiliser in $\text{PSL}(3, 2)$ of a 1-space $U$. However, this time $A$ is the stabiliser in $H$ of a 2-space containing $U$ and so $\Gamma$ is the Heawood graph, so we have row 1.

Now let $G = \text{PGL}(2, 9)$, $M_{10}$, or $\text{PTL}(2, 9)$, $E = N_G(D_8)$, $A = E \cap \text{PSL}(2, 9)$ and $H = N_G(\text{PSL}(2, 3))$. Note that $\text{PTL}(2, 9) \cong \langle \text{PSp}(4, 2), \tau \rangle$ where $\tau$ is a duality of the associated polar space. Moreover, $H$ is the stabiliser of a totally isotropic 1-space and $A$ is the stabiliser in $H$ of an incident totally isotropic 2-space. Thus $\Gamma$ is the Tutte–Coxeter graph and we have row 4.

When $G = \text{PGL}(2, 9)$, $M_{10}$ or $\text{PTL}(2, 9)$, $E = N_G(D_{10})$, $A = E \cap \text{PSL}(2, 9)$ and $H = N_G(A_5)$, we have that $H \leq (G \cap \text{PSL}(2, 9))$ and $G \cap \text{PSL}(2, 9)$ is an index two subgroup of $G$. Thus $\Gamma$ is bipartite. The vertices of $\Gamma$ are two sets of size 6 with $\text{PSL}(2, 9) \cong A_6$ acting on each with two different actions. Since the stabiliser in $A_6$ of a
point in one action is still transitive in the other action it follows that $\Gamma \cong K_{6,6}$ and we have row 3 of Table 1.

When $G = \text{PGL}(2, 11)$, $E = D_{20}$, $A = E \cap T = D_{10}$ and $H = A_5$ we have that $H \leq \text{PSL}(2, 11)$ and so we get a bipartite graph on 22 vertices with valency 6. Thus we have row 5.

Finally, let $G = \text{PGL}(2, q)$, $q = p \equiv 11, 19, 21, 29 \mod 40$, $E = S_4$, $A = A_4$ and $H = A_5$. Then we get the bipartite graph in row 10 of Table 1. □

References


