Point regular groups of automorphisms of generalised quadrangles

Michael Giudici
joint work with John Bamberg

Centre for the Mathematics of Symmetry and Computation

The University of Western Australia
Achieve International Excellence

Finite Geometries, Third Irsee Conference 2011
A **generalised quadrangle** is a point-line incidence geometry $Q$ such that:

1. any two points lie on at most one line, and
2. given a line $\ell$ and a point $P$ not incident with $\ell$, $P$ is collinear with a unique point of $\ell$.

If each line is incident with $s + 1$ points and each point is incident with $t + 1$ lines we say that $Q$ has **order** $(s, t)$.

If $s, t \geq 2$ we say $Q$ is **thick**.
The Classical GQ’s

<table>
<thead>
<tr>
<th>Name</th>
<th>Order</th>
<th>Automorphism group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(3, q^2)$</td>
<td>$(q^2, q)$</td>
<td>$\text{PGU}(4, q)$</td>
</tr>
<tr>
<td>$H(4, q^2)$</td>
<td>$(q^2, q^3)$</td>
<td>$\text{PGU}(5, q)$</td>
</tr>
<tr>
<td>$W(3, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PGSp}(4, q)$</td>
</tr>
<tr>
<td>$Q(4, q)$</td>
<td>$(q, q)$</td>
<td>$\text{PGO}(5, q)$</td>
</tr>
<tr>
<td>$Q^-(5, q)$</td>
<td>$(q, q^2)$</td>
<td>$\text{PGO}^-(6, q)$</td>
</tr>
</tbody>
</table>

Take a sesquilinear or quadratic form on a vector space

- Points: totally isotropic 1-spaces
- Lines: totally isotropic 2-spaces
A group $G$ acts \textbf{regularly} on a set $\Omega$ if $G$ is transitive on $\Omega$ and $G_\omega = 1$ for all $\omega \in \Omega$.

The study of regular groups of automorphisms of combinatorial structures has a long history:

- A graph $\Gamma$ is a Cayley graph if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup.
- A group $G$ acts regularly on a symmetric block design if and only if $G$ has a difference set.
Groups acting regularly on GQ’s

Ghinelli (1992): A Frobenius group or a group with nontrivial centre cannot act regularly on a GQ of order \((s, s)\), \(s\) even.

De Winter, K. Thas (2006): A finite thick GQ with a regular abelian group of automorphisms is a Payne derivation of a TGQ of even order.

Yoshiara (2007): No GQ of order \((s^2, s)\) admits a regular group.
Known GQs with a point transitive group of automorphisms

- the classical GQs and their duals
- the Payne derivations of $W(3, q)$
- $T_2^*(O)$ for $O$ a hyperoval.
- the generalised quadrangle of order $(5, 3)$
- the generalised quadrangle of order $(17, 15)$ that is the dual of the GQ constructed from the Lunelli-Sce hyperoval.
A $p$-group $P$ is called special if $Z(P) = P' = \Phi(P)$.

A special $p$-group is called extraspecial if $|Z(P)| = p$.

Extraspecial $p$-groups have order $p^{1+2n}$, and there are two isomorphism classes for each order.

For $p$ odd, one has exponent $p$ and one has exponent $p^2$.

$Q_8$ and $D_8$ are extraspecial of order 8.
Regular groups and classical GQs

Theorem

If $Q$ is a finite classical GQ with a regular group $G$ of automorphisms then

- $Q = Q^-(5, 2)$, $G$ extraspecial of order 27 and exponent 3.
- $Q = Q^-(5, 2)$, $G$ extraspecial of order 27 and exponent 9.
- $Q = Q^-(5, 8)$, $G \cong GU(1, 2^9).9 \cong C_{513} \rtimes C_9$.

Use classification of all regular subgroups of primitive almost simple groups by Liebeck, Praeger and Saxl (2010)

Alternative approach independently done by De Winter, K. Thas and Shult.
Payne derived quadrangles

Begin with $W(3, q)$ defined by the alternating form

$$\beta(x, y) := x_1y_4 - y_1x_4 + x_2y_3 - y_2x_3$$

Let $x = \langle(1, 0, 0, 0)\rangle$.

Define a new GQ, $Q^x$, with

- points: the points of $W(3, q)$ not collinear with $x$, that is, all $\langle(a, b, c, 1)\rangle$
- lines:
  - (a) lines of $W(3, q)$ not containing $x$, and
  - (b) the hyperbolic 2-spaces containing $x$.

$Q^x$ is a GQ of order $(q - 1, q + 1)$ known as the Payne derivation of $W(3, q)$. 
Automorphisms

- $\text{PGammaSp}(4, q)_x \leq \text{Aut}(Q^x)$
- For $q = 3$, $Q^x \cong Q^-(5, 2)$.
- Note $\text{Sp}(4, q)_x$ consists of all matrices

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
\mathbf{u}^T & A & 0 \\
z & \mathbf{v} & \lambda^{-1}
\end{pmatrix}
\]

with

- $A \in \text{GL}(2, q)$ such that $AJA^T = J$,
- $z \in \text{GF}(q)$ and $\mathbf{u}, \mathbf{v} \in \text{GF}(q)^2$ such that $\mathbf{u} = \lambda \mathbf{v}JA^T$,

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. 
An obvious regular subgroup

Let

\[ E = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -c & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \right| a, b, c \in \text{GF}(q) \} \triangleleft \Gamma \text{Sp}(4, q)_x \]

- \(|E| = q^3\),
- acts regularly on the points of \( Q^x \),
- elementary abelian for \( q \) even,
- special of exponent \( p \) for \( q \) odd (Heisenberg group).
But there are more . . .

<table>
<thead>
<tr>
<th>$q$</th>
<th># conjugacy classes</th>
<th># isomorphism classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2 (this is $Q^{-}(5, 2)$)</td>
</tr>
<tr>
<td>4</td>
<td>58</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>231</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>23</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>25</td>
<td>7</td>
<td>-</td>
</tr>
</tbody>
</table>
For $\alpha \in GF(q)$, let

$$\theta_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ -\alpha^2 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{pmatrix} \in Sp(4, q)_x.$$
Construction 1

Let \( \{\alpha_1, \ldots, \alpha_f\} \) be a basis for \( \text{GF}(q) \) over \( \text{GF}(p) \).

\[
P = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & 0 & 1 \end{pmatrix}, \theta_{\alpha_1}, \ldots, \theta_{\alpha_f} \right\rangle
\]

- \( P \) acts regularly on points and is not normal in \( \text{Aut}(Q^x) \)
- nonabelian for \( q > 2 \)
- not special for \( q \) even, special for \( q \) odd
- exponent 9 for \( p = 3 \)
- \( P \cong E \) for \( p \geq 5 \)
- for \( p \geq 5 \), any regular subgroup on \( W(3, p) \) must be isomorphic to \( E \).
Construction 2

Let $U \oplus W$ be a decomposition of $\text{GF}(q)$, $q = p^f$, $f \geq 2$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for $U$ over $\text{GF}(p)$.

$$S_{U,W} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ a & b & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ -w & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & w & 1 \end{pmatrix}, \theta_{\alpha_1}, \ldots, \theta_{\alpha_k} \right\rangle$$

- $S_{U,W}$ acts regularly on points,
- for $U$ a 1-space:
  - $E \not\cong S_{U,W} \not\cong P$
  - $S_{U,W}$ is not special
## GQ of order $(5, 3)$

<table>
<thead>
<tr>
<th>Group</th>
<th>Shape</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>$C_2^4 \rtimes S_3$</td>
<td>$Z(H_1) = 1$, $H'_1 = C_2^4 \rtimes C_3$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$2^{2+3} \rtimes C_3$</td>
<td>$Z(H_2) = 1$, $H'_2 = C_2^4$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$2^{2+3} \rtimes C_3$</td>
<td>$Z(H_3) = 1$, $H'_3 = C_4^2$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$C_4^2 \rtimes S_3$</td>
<td>$Z(H_4) = 1$, $H'_4 = C_4^2 \rtimes C_3$</td>
</tr>
<tr>
<td>$H_5$</td>
<td>$C_2^4 \rtimes S_3$</td>
<td>$</td>
</tr>
<tr>
<td>$H_6$</td>
<td>$2^{2+2} \rtimes S_3$</td>
<td>$</td>
</tr>
</tbody>
</table>

De Winter, K. Thas and Shult: The GQ of order $(17, 15)$ has no regular groups of automorphisms.
Summing up

The class of groups that can act regularly on the set of points of a GQ is richer/wilder than previously thought.

Such a group can be

- a nonabelian 2-group,
- a 2-group of nilpotency class 7,
- \( p \)-groups that are not Heisenberg groups,
- \( p \)-groups that are not special.