$S_3$-involution graphs

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on joint work with Alice Devillers
An $M_{11}$-graph

Witt design $S(4, 5, 11)$: collection of 5-subsets (pentads) of an 11-set such that any 4-subset is contained in a unique pentad.

Automorphism group is the Mathieu group $M_{11}$ which acts 4-transitively on the 11-set.
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Automorphism group is the Mathieu group $M_{11}$ which acts 4-transitively on the 11-set.

Define a graph $\Sigma$

- vertices: 3-subsets of an 11-set
- adjacency: complement of union is a pentad.
$M_{11}$ also has a 3-transitive action on a set of size 12.

Has an orbit $\mathcal{O}$ of size 165 on the set of 4-subsets forming a $3 - (12, 4, 3)$ design.
Another definition

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**Theorem (Devillers, MG, Li, Praeger)**

$\Sigma$ is isomorphic to the graph defined as follows:

- **vertices:** elements of $\mathcal{O}$
- **adjacency:** intersection is a 3-subset

$J(12, 4)$ can be decomposed into 12 copies of $\Sigma$ with the 12 copies transitively permuted by $M_{12}$. 
A $\text{PSL}(2, 11)$-graph

$\text{PSL}(2, 11)$ has a 2-transitive action on 11 points.

Has two orbits on 3-subsets and these have lengths 55 and 110.

The orbit of length 55 forms a $2 - (11, 3, 3)$ design (Petersen design).
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Has two orbits on 3-subsets and these have lengths 55 and 110.

The orbit of length 55 forms a $2 - (11, 3, 3)$ design (Petersen design).

Define a graph

- vertices: blocks of Petersen design
- adjacency: intersection is a 2-subset
An alternative definition

$M_{11}$ has a unique conjugacy class of involutions.

An involution has 3 fixed points in action on 11-set and 4 fixed points in action on 12-set.
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The fixed point sets of two involutions are adjacent in $\Sigma$ if and only if generate an $S_3$ with normaliser $S_3 \times S_3$. 

Equivalent definition of $\Sigma$ is:

- vertices: involutions of $M_{11}$
- adjacency: generate an $S_3$ with normaliser $S_3 \times S_3$.

The $PSL(2,11)$ graph has a similar definition.
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Using this definition get a tower of graphs for the groups

$$A_5 < PSL(2, 11) < M_{11} < M_{12}$$

The graph for $A_5$ is the line graph of the Petersen graph.

The graph for $M_{12}$ is the Johnson graph $J(12, 4)$. 
In general

Given

- \( G \) a group
- \( X \) a set of involutions closed under conjugation
- \( S \) a set of \( S_3 \)-subgroups closed under conjugation
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Define the $S_3$-involution graph $\Gamma(G, X, S)$

- vertices the elements of $X$
- $x \sim y$ if $\langle x, y \rangle \in S$
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NB: The product of adjacent involutions has order three.
Reminiscent of:

- Fischer’s 3-transposition groups
- Coxeter graphs
- commuting involution graphs
Automorphisms

$G$ acts by conjugation as a group of automorphisms

$G$-vertex-transitive if and only if $X$ is single conjugacy class

$G$-arc-transitive if and only if $S$ is a single conjugacy class

If $g \in \text{Aut}(G)$ fixes $X$ and $S$ then $g$ induces automorphism of $\Gamma(G, X, S)$.

Full automorphism group can be much larger than $G$, eg $G = M_{12}$ and $\Gamma(G, X, S) = J(12, 4)$
Symmetric groups

\[ G = S_n, \ X \text{ the class of transpositions} \]

\[ S \text{ the class of } S_3\text{-subgroups with } n - 3 \text{ fixed points.} \]
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Given \( x = (a, b) \) and \( y = (c, d) \),

\[ \langle x, y \rangle \in S \text{ if and only if } \left| \{a, b\} \cap \{c, d\} \right| = 1 \]
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Thus \( \Gamma(G, X, S) \cong J(n, 2) \).
$M_{11}$ has unique class of involutions and they correspond to the 3-subsets of an 11-set.

Has two conjugacy classes of $S_3$-subgroups.

Already seen the graph obtained if we use one of the classes for adjacency.
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The other class yields the Johnson graph $J(11, 3)$. 
Complete graphs

\[ G = \operatorname{AGL}(1, 3^n) = \{ t_{a,b} : x \mapsto ax + b \mid a, b \in \operatorname{GF}(3^n), a \neq 0 \} \]

Unique class of involutions \( X = \{ t_{-1,b} \mid b \in \operatorname{GF}(3^n) \} \).

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\[ \Gamma(G, X, S) \cong K_{3^n} \]
Theorem

If $\Gamma(G, X, S)$ is the complete graph on $X$ for some group $G$ then $|X| = 3^n$ for some positive integer $n$. 
Paley graphs

$n$ even

\[ G = \{ t_{a,b} : x \mapsto ax + b \mid a, b \in \text{GF}(3^n), a = \square \neq 0 \} \]
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Two classes \( S_1, S_2 \) of \( S_3 \)-subgroups

\[ \langle t_{-1,b}, t_{-1,c} \rangle \in S_1 \text{ iff } c - b = \square \]
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\( \Gamma(G, X, S_1) \) is the Paley graph for \( \text{GF}(3^n) \).
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Theorem

If no $S \in S$ is contained in a subgroup of $G$ of the form $C_3^2 \rtimes C_2$ or $C_n^2 \rtimes S_3$, then the only triangles of $\Gamma(G, X, S)$ are those given by the subgroups of $S$. 
$G = \text{PSL}(2, q)$

Unique conjugacy class of involutions

One or two conjugacy classes of $S_3$-subgroups but if two then fused in $\text{PGL}(2, q)$.

| $q \pmod{12}$ | $|X|$ | $|S|$ | valency |
|---------------|------|-----|--------|
| 4, 8          | $q^2 - 1$ | $|G|/6$ | $q$ |
| 1             | $q(q+1)/2$ | $|G|/12$ | $(q-1)/2$ |
| 3             | $q(q-1)/2$ | 0     |        |
| 5             | $q(q+1)/2$ | $|G|/6$ | $q-1$ |
| 7             | $q(q-1)/2$ | $|G|/6$ | $q+1$ |
| 9             | $q(q+1)/2$ | $|G|/6$ | $q-1$ |
| 11            | $q(q-1)/2$ | $|G|/12$ | $(q+1)/2$ |
Theorem

\[ G = \text{PSL}(2, q) \text{ for } q \geq 4, \]
\[ X \text{ the unique conjugacy class of involutions,} \]
\[ S \text{ a conjugacy class of } S_3 \text{-subgroups.} \]

The size of the largest clique is

- \(3^e\) if \(q = 9^e\),
- \(4\) if \(q = 25^e\),
- \(3\) otherwise.
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- 4 if \( q = 25^e \),
- 3 otherwise.

For \( q = 9^e \) the subgraphs induced on the parabolics \( C_3^{2e} \times C_{(9^e-1)/2} \) are Paley graphs.
Duality

The **dual graph** of $\Gamma(G, X, S)$ is the graph with

- vertices: $S_3$-triangles of $\Gamma(G, X, S)$
- adjacency: if share a vertex

Theorem

$\Gamma(PSL(2, q), X, S)$ is isomorphic to its dual graph if and only if $q = 11$ or $13$. 
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