Some locally 3–arc transitive graphs constructed from triality

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Abstract

Two new infinite families of locally 3–arc transitive graphs are constructed which have Aut(PΩ+(8, q)) as their automorphism group.

Key words: locally s–arc transitive graphs, orthogonal groups, triality.

1 Introduction

The hyperbolic quadric associated with an 8-dimensional vector space has a rich geometry which admits a triality between the totally singular 1–spaces and the two classes of maximal totally singular subspaces. This triality has been used by Tits to construct generalised hexagons [18] and is associated with the graph automorphism of order three of the orthogonal groups PΩ+(8, q). In turn, this graph automorphism gives rise to the simple groups 3D4(q) discovered by Steinberg [16]. We exploit this geometry to construct two new infinite families of locally 3–arc transitive graphs which have Aut(PΩ+(8, q)) as their automorphism group.

* This paper forms part of an Australian Research Council large grant project which supports the first author. The second author is supported by an ARC Fellowship.

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Preprint submitted to Elsevier Science 8 December 2004
A graph $\Gamma$ is **biregular of valency** $\{k_1,k_2\}$ if it is bipartite and vertices in the $i^{th}$ part of the bipartition have valency $k_i$, for $i = 1, 2$. Given a vertex $v$ in $\Gamma$, we denote the set of vertices adjacent to $v$ by $\Gamma(v)$. Then for a group $G$ of automorphisms of $\Gamma$, the permutation group induced by the vertex stabiliser $G_v$ on $\Gamma(v)$ is denoted by $G^{\Gamma(v)}_v$. Given an intransitive normal subgroup $N$ of $G$, the **quotient graph** $\Gamma_N$ is the graph whose vertices are the orbits of $N$ and two $N$–orbits $B_1$ and $B_2$ are adjacent if there exists a vertex in $B_1$ which is adjacent to a vertex in $B_2$. An $s$–arc in $\Gamma$ is an $(s + 1)$–tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that each $v_i$ is adjacent to $v_{i+1}$ and $v_{i-1} \neq v_{i+1}$. We call $\Gamma$ **locally $s$–arc transitive** if for each vertex $v$ of $\Gamma$, the stabiliser of $v$ in the automorphism group of $\Gamma$ induces a transitive action on the set of $s$–arcs beginning at $v$.

**Theorem 1.1** Let $T = \text{P}\Omega^+(8, q)$, where $q = p^f$ is a prime power, and let $G = \text{Aut}(T)$. Then there is a connected biregular graph $\Gamma = G(q)$, of valency $\{3, q^2 + q + 1\}$ and with automorphism group $G$ such that

1. $\Gamma$ is locally $3$–arc transitive but not locally $4$–arc transitive,
2. $\Gamma_T \cong K_{1,3}$,
3. given a vertex $v$ of valency 3 and an adjacent vertex $w$ of valency $q^2 + q + 1$, we have $G^{\Gamma(v)}_v \cong S_3$ and $G^{\Gamma(w)}_w \cong \text{P}GL(3, q)$, and
4. $G_v = G_1 : S_3$, $G_w = G_2 : C_2$ and $G_{vw} = G_1 : C_2$, where

$$G_1 \cong [q^{11}] : ((C_{q-1}^3 \circ \text{GL}(2, q)) : C_f),$$

$$G_2 \cong [q^6] : ((C_{q-1}^2 \circ \text{GL}(3, q)) : C_f).$$

By a group $A \circ B$, we mean the group $(A \times B)/H$ where $H$ is some normal subgroup contained in the centre of $A \times B$. For more precise information about the structure of the vertex stabilisers of these graphs, and in particular the subgroups $G_1$, $G_2$ of part (4), see Section 2 (especially the discussion preceding the displayed equations (2.1) and (2.2)).

**Theorem 1.2** Let $T = \text{P}\Omega^+(8, p)$, where $p$ is an odd prime, and let $G = \text{Aut}(T)$. Then there is a connected biregular graph $\Gamma = \mathcal{H}(p)$, of valency $\{4, 7\}$ and with automorphism group $G$ such that

1. $\Gamma$ is locally $3$–arc transitive but not locally $4$–arc transitive,
2. $\Gamma_T \cong K_{1,4}$,
3. given a vertex $v$ of valency 4 and an adjacent vertex $w$ of valency 7, we have $G^{\Gamma(v)}_v \cong S_4$ and $G^{\Gamma(w)}_w \cong \text{SL}(3, 2)$, and
4. $G_v \cong (2^{3+6} : S_4).S_4$, $G_w \cong 2^{3+6} : (\text{SL}(3, 2) \times S_3)$ and $G_{vw} \cong 2^{3+6} : (S_4 \times S_3)$.

Both $G(q)$ and $\mathcal{H}(p)$ have geometric interpretations, with $G(q)$ associated with the polar space of the hyperbolic quadric and $\mathcal{H}(p)$ associated with a geometry whose diagram is an extended $D_4$, that is, a central node with four neighbours, (see Remark 4.5). The construction for $\mathcal{H}(p)$ can also be applied to produce
a graph $\mathcal{H}(q)$ corresponding to $P\Omega^+(8, q)$ for any odd $q$, but if $q$ is not prime, then $\mathcal{H}(q)$ is not connected, see Remark 4.6. Our proofs to determine the full automorphism groups of $\mathcal{G}(q)$ and $\mathcal{H}(p)$ use various results that rely on the finite simple group classification.

Let $\Gamma$ be an undirected graph with vertex set $V\Gamma$. Given $G \leq \text{Aut}(\Gamma)$, we say that $\Gamma$ is \textit{locally} $(G, s)$–\textit{arc} transitive if $\Gamma$ contains an $s$–arc and, given any two $s$–arcs $\alpha$ and $\beta$ starting at the same vertex $v$, there is an element of $G_v$ which maps $\alpha$ to $\beta$. Thus $\Gamma$ is locally $s$–arc transitive if $\Gamma$ is locally $(G, s)$–arc transitive for some $G \leq \text{Aut}(\Gamma)$. If $G$ also acts transitively on $V\Gamma$ then $G$ acts transitively on the set of all $s$–arcs of $\Gamma$ and we say that $\Gamma$ is $(G, s)$–\textit{arc transitive}. The problem of finding locally $s$–arc transitive graphs with large values of $s$ has received much interest. Stellmacher [17], following earlier work of Tutte [19], [20] and Weiss [21] in the vertex transitive case, proved that, for a locally $(G, s)$–arc transitive graph with all vertices having valency at least three, $s \leq 9$. This bound is sharp as the maximum value is attained by the incidence graphs of the generalised octagons associated with the simple groups $^2F_4(2^n)$.

Suppose in this paragraph that all vertices in $\Gamma$ have valency at least two. Then if $\Gamma$ is locally $(G, s)$–arc transitive, it is also locally $(G, s - 1)$–arc transitive. Also $\Gamma$ is locally $(G, 2)$–arc transitive if and only if, for all vertices $v$, $G^{\Gamma(v)}_v$ is 2–transitive (see for example [6, Lemma 3.2]). If $G$ is vertex intransitive, then $\Gamma$ is a bipartite graph and the two parts $\Delta_1$ and $\Delta_2$ of the bipartition are orbits of $G$. Thus bipartite locally $(G, s)$–arc transitive graphs are biregular.

In [6] a program for studying locally $(G, s)$–arc transitive graphs with $s \geq 2$ was initiated which focused on the ‘global’ action of $G$. Previously, the local action of $G_v$ on $\Gamma(v)$, and the structure of the subgroup $G_v$, had been the main theme of investigations. Let $\Gamma$ be a bipartite graph with bipartite halves $\Delta_1$ and $\Delta_2$ and having a group $G$ of automorphisms whose orbits are $\Delta_1$ and $\Delta_2$. If $G$ has a nontrivial normal subgroup $N$ intransitive on at least one of $\Delta_1$, $\Delta_2$, we can form the quotient graph $\Gamma_N$. It was shown in [6] that if $\Gamma$ is a locally $(G, s)$–arc transitive graph and $N$ is intransitive on both $\Delta_1$ and $\Delta_2$, then $\Gamma_N$ is locally $(G/N, s)$–arc transitive. If $N$ is intransitive on only one of the $G$–orbits, say on $\Delta_i$, then $\Gamma_N$ is the star $K_{1,n}$, where $n$ is the number of orbits of $N$ on $\Delta_i$. This suggests that to study locally $(G, s)$–arc transitive graphs attention should focus on the case where $G$ acts faithfully on both orbits and quasiprimively on at least one (where a permutation group is \textit{quasiprimitive} if all nontrivial normal subgroups are transitive).

The quasiprimitive permutation groups were classified in [14] in an ‘O’Nan–Scott like’ theorem, and were described in [15] as being of one of eight types. The possible quasiprimitive types for the action of a locally $(G, 2)$–arc transitive graph on one of its orbits were determined in [6]. If $G$ acts quasiprimively
on only $\Delta_1$ then there are five possible types for this quasiprimitive action. For four of them, a nice characterisation of the possible graphs was given in [7]. The fifth type is the almost simple case, that is, there exists some nonabelian simple group $T$ such that $T \leq G \leq \text{Aut}(T)$ and $T$ acts transitively on $\Delta_1$. As $G$ does not act quasiprimively on $\Delta_2$, $T$ acts intransitively on $\Delta_2$. It was shown in [7, Theorem 1.4] that for such a graph to exist $\text{Out}(T)$ must have a 2–transitive representation of degree equal to the valency of the vertices in $\Delta_1$. This result drew our attention to the groups $T = P\Omega^+(8, q)$ as here $\text{Out}(T)$ contains $S_3$ or $S_4$. Studying these groups led us to our constructions of two new families of locally 3–arc transitive graphs in Sections 3 and 4. The second construction is the only one we currently know of a locally $(G, 2)$–arc transitive graph where $G$ is an almost simple group whose socle $T$ acts transitively on $\Delta_1$ but intransitively on $\Delta_2$ such that the vertices in $\Delta_1$ have valency greater than 3.

**Problem 1.3** Classify the biregular, locally $(G, s)$–arc transitive graphs of valency $\{3, k\}$ or $\{4, k\}$, where $G$ is an almost simple group with socle $T = P\Omega^+(8, q)$ such that $T$ acts transitively on $\Delta_1$ and intransitively on $\Delta_2$.

Note that by [6, Lemma 5.6], such graphs are not locally $(G, 4)$–arc transitive.

Given two groups $G$ and $H$, we denote the split extension of $G$ by $H$, by $G : H$ while $G.H$ denotes some extension of $G$ by $H$, not necessarily split. Furthermore, $[n]$ denotes a group of order $n$, $p^d$ denotes an elementary abelian group of order $p^d$, while $p^{d+r}$ denotes a $p$–group $P$ with centre $Z = p^d$ such that $P/Z = p^r$. For a subgroup $L$ of $G$, we denote the normaliser of $L$ in $G$ by $N_G(L)$ and the centraliser of $L$ in $G$ by $C_G(L)$. Also $O_2(G)$ denotes the largest normal 2–subgroup of $G$.

**Acknowledgements.** The authors would like to thank the anonymous referee for their helpful comments and for bringing our attention to Kantor’s geometry.

## 2 Some geometry

In this section we provide a brief outline of the geometry associated with the 8–dimensional orthogonal groups and introduce some notation. For more details, the reader may refer to [1], [3], [10], or [11]. The second and third references have a wealth of information in the 8–dimensional case. Let $V$ be an 8–dimensional vector space over the field $F = GF(q)$ equipped with a nondegenerate quadratic form $Q$, that is, $Q : V \rightarrow F$ is a function such that

$$Q(\lambda v) = \lambda^2Q(v) \text{ for all } \lambda \in F \text{ and } v \in V$$
and

\[ B : V \times V \to F \]

\[ (v, w) \mapsto Q(v + w) - Q(v) - Q(w) \]

is a nondegenerate bilinear form. A subspace \( W \) of \( V \) is called \textit{totally singular} if \( Q(v) = 0 \) for all vectors \( v \in W \). We let \( Q \) have \textit{maximal Witt index}, that is, the maximal totally singular subspaces of \( V \) have dimension 4. Then \( V \) has a basis \( \{e_1, \ldots, e_4, f_1, \ldots, f_4\} \), known as the \textit{standard basis}, such that \( \langle e_1, \ldots, e_4 \rangle \) and \( \langle f_1, \ldots, f_4 \rangle \) are totally singular 4–spaces and \( B(e_i, f_j) = \delta_{ij} \) for all \( i, j \in \{1, \ldots, 4\} \).

The group \( \Gamma L(8, q) \) is the group of all invertible semilinear transformations of \( V \) under composition, that is, all \( g : V \mapsto V \) such that \( g \) preserves addition and there exists \( \alpha \in \text{Aut}(F) \) such that, for all \( \lambda \in F \) and \( v \in V \),

\[ (\lambda v)^g = \lambda^\alpha v^g. \]

The subgroup of all invertible linear transformations is denoted by \( \text{GL}(8, q) \). Let

\[ \Gamma O^+(8, q) = \{ g \in \Gamma L(8, q) : \exists \lambda \in F, \alpha \in \text{Aut}(F) \text{ s.t.} \quad Q(v^g) = \lambda Q(v)^\alpha \forall v \in V \}, \]

\[ \text{GO}^+(8, q) = \{ g \in \text{GL}(8, q) : \exists \lambda \in F \text{ s.t.} \quad Q(v^g) = \lambda Q(v) \forall v \in V \}, \]

\[ \text{O}^+(8, q) = \{ g \in \text{GL}(8, q) : Q(v^g) = Q(v) \forall v \in V \}, \]

\[ \text{SO}^+(8, q) = \{ g \in \text{O}^+(8, q) : \det(g) = 1 \} \]

and let \( \Omega^+(8, q) \) be the commutator subgroup of \( \text{SO}^+(8, q) \), that is the subgroup generated by all commutators \( h^{-1} g^{-1} h g \). Let \( Z \) be the group of all scalar transformations in \( \text{GL}(8, q) \) and for each group \( X \) in our sequence let \( PX = XZ / Z \cong X / (X \cap Z) \). Then \( T = P\Omega^+(8, q) \) is a finite nonabelian simple group and

\[ |T| = \frac{1}{(2, q - 1)^2} q^{12}(q^2 - 1)(q^4 - 1)^2(q^6 - 1). \]

Now \( Z \leq \text{GO}^+(8, q) \) and we have from [11, (2.7.2)], that if \( q \) is even, then \( \text{GO}^+(8, q) = \text{O}^+(8, q) \times Z \). If \( q \) is odd, let \( \mu \) be a primitive element of \( \text{GF}(q) \). Then the linear transformation

\[ \delta = \begin{pmatrix} \mu I & 0 \\ 0 & I \end{pmatrix} \]

written with respect to the standard basis, lies in \( \text{GO}^+(8, q) \). (Here \( I \) is the \( 4 \times 4 \) identity matrix.) Furthermore, by [11, (2.7.2)], \( \text{GO}^+(8, q) = \text{O}^+(8, q) : \langle \delta \rangle \).
when $q$ is odd. If $\text{GF}(q)$ has characteristic $p$, let $\phi$ be the semilinear transformation

$$\phi : \sum_{i=1}^{4} (\lambda_i e_i + \xi_i f_i) \mapsto \sum_{i=1}^{4} (\lambda_i^p e_i + \xi_i^p f_i).$$

Then by [11, (2.7.3)], $\Gamma O^+(8, q) = \Gamma O^+(8, q) : \langle \phi \rangle$.

We denote the set of totally singular 1–spaces of $V$ by $P$, the set of totally singular 2–spaces by $L$, the set of totally singular 3–spaces by $T$, and the set of totally singular 4–spaces by $S$. The group $T$ acts transitively on $P$, $L$ and $T$, and has two orbits $S_1$ and $S_2$ on $S$. Two totally singular 4–spaces lie in the same $T$–orbit if and only if their intersection has even dimension. The two $T$–orbits on 4–spaces are fused together under $PO^+(8, q)$. For a subspace $X$, $T_X$ denotes the setwise stabiliser of $X$ in $T$. Each $X \in T$ lies in precisely two totally singular 4–spaces, say $S$ and $R$, one lies in $S_1$ and the other in $S_2$. Thus $T_X = T_S \cap T_R$.

Let $S = \langle e_1, e_2, e_3, e_4 \rangle$, a totally singular 4–space. Then $S$ has $S' = \langle f_1, f_2, f_3, f_4 \rangle$ as a complementary totally singular 4–space and by [11, Lemma 4.1.9], the stabiliser, $L$, in $SO^+(8, q)$ of both $S$ and $S'$ is the group of all matrices

$$\begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix}$$

where $B$ is any matrix in $\text{GL}(4, q)$ and the matrices are written with respect to the standard basis. Such an element belongs to $\Omega^+(8, q)$ if and only if $\det(B)$ is a square ([11, Lemma 4.1.9]). Also let $M$ be the subgroup consisting of all matrices of the form

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$$

which lie in $SO^+(8, q)$. Then by [11, Lemmas 2.1.8 and 4.1.12], $C$ is any $4 \times 4$ matrix such that $C = -C^T$ and every entry of $C$ on the diagonal is 0. Thus $M$ is elementary abelian of order $q^6$ and by [11, Lemma 4.1.12], $M \leq \Omega^+(8, q)$. Also $M$ is normalised by $L$, and we have

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^T C B & I \end{pmatrix}$$

and $SO^+(8, q)_S = M : L$. Let $R$ be the totally singular subspace $\langle e_1, e_2, e_3, f_4 \rangle$. Then $\dim(S \cap R) = 3$ and so $S$ and $R$ lie in distinct $T$–orbits. Also $SO^+(8, q)_S \cap$
\[ \text{SO}^+(8,q)_R = M : (L_S \cap L_R), \text{ and } L_S \cap L_R \text{ consists of all matrices of the form} \]

\[
\begin{pmatrix}
B & 0_{3 \times 1} \\
v & \alpha \\
0_{4 \times 4} & (B^{-1})^T w^T \\
0_{1 \times 3} & \alpha^{-1}
\end{pmatrix}
\]

where \( B \in \text{GL}(3,q), v \in \text{GF}(q)^3, \alpha \in \text{GF}(q) \setminus \{0\} \) and \( w = -\alpha^{-1}vB^{-1} \). Thus

\[ \text{SO}^+(8,q)_S \cap \text{SO}^+(8,q)_R \cong [q^9] : (C_{q-1} \times \text{GL}(3,q)). \]

Now \( Z \) fixes both \( S \) and \( R \) and so when \( q \) is even

\[ \text{GO}^+(8,q)_S \cap \text{GO}^+(8,q)_R \cong [q^9] : (C_{q-1} \times \text{GL}(3,q) \times Z). \]

When \( q \) is odd, we see that \( \delta \) fixes both \( S \) and \( R \), normalises \( M \) and centralises \( L_S \cap L_R \). Thus in this case

\[ \text{GO}^+(8,q)_S \cap \text{GO}^+(8,q)_R \cong [q^9] : (C_{2(q-1)} \times \text{GL}(3,q)) : \langle \delta \rangle. \]

The element \( \phi \) also fixes \( S \) and \( R \), and so

\[ \Gamma \text{GO}^+(8,q)_S \cap \Gamma \text{GO}^+(8,q)_R = (\text{GO}^+(8,q)_S \cap \text{GO}^+(8,q)_R) : \langle \phi \rangle. \]

Hence letting \( H = \text{PGO}^+(8,q) \), we see that, when \( q \) is even,

\[ H_S \cap H_R \cong [q^9] : ((C_{q-1} \times \text{GL}(3,q)) : C_f) \]

while when \( q \) is odd,

\[ H_S \cap H_R \cong ([q^9] : ((C_{q-1}^{2} \times \text{GL}(3,q)) : C_f)) / Z. \]

We will denote the subgroup \( H_S \cap H_R \) in both cases by

\[ H_S \cap H_R \cong [q^9] : ((C_{q-1}^{2} \circ \text{GL}(3,q)) : C_f). \tag{2.1} \]

Note that the subgroup \( H_S \cap H_R \) is the group \( G_2 \) of Theorem 1.1(4). Furthermore, if we let \( U \) be a 1-space contained in \( S \cap R \) then we have

\[ H_S \cap H_R \cap H_U \cong [q^9] : ((C_{q-1}^{2} \circ ([q^2] : (C_{q-1} \times \text{GL}(2,q))) : C_f) \]

\[ \cong [q^{11}] : ((C_{q-1}^{3} \circ \text{GL}(2,q)) : C_f). \tag{2.2} \]

This is the subgroup \( G_2 \) of Theorem 1.1(4).

The following lemma follows from (2.1).
Lemma 2.1 Let $H = \text{PGO}^+(8, q)$ and let $S$ and $R$ be totally singular 4–spaces such that $\dim(S \cap R) = 3$. Then $H_S \cap H_R$ induces the semilinear group $\Gamma L(3, q)$ on $S \cap R$.

For our second construction we will need further information about the structure of $T_{S \cap R} = T_S \cap T_R$. Now $L_S \cap L_R$ has a normal subgroup $N$ of order $q^3$ consisting of all such matrices with $B = I$ and $\alpha = 1$, and so $\text{SO}^+(8, q)_S \cap \text{SO}^+(8, q)_R$ has a normal subgroup $Q = M : N$ of order $q^9$. The centre $P$ of $Q$ consists of all matrices of the form

$$
\begin{pmatrix}
I & 0 \\
C & I
\end{pmatrix}
$$

where

$$
C = \begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

The action of $\text{SO}^+(8, q)_S \cap \text{SO}^+(8, q)_R$ on $P$ is equivalent to the action of $\text{GL}(3, q)$ on the $3 \times 3$ matrices $C$ for which $C^T = -C$ and every diagonal entry of $C$ is 0, such that, for each $A \in \text{GL}(3, q)$,

$$
A : \begin{pmatrix}
0 & a & b \\
-a & 0 & d \\
-b & -d & 0
\end{pmatrix} \mapsto A^T \begin{pmatrix}
0 & a & b \\
-a & 0 & d \\
-b & -d & 0
\end{pmatrix} A.
$$

This action is dual to the action of $\text{SO}^+(8, q)_S \cap \text{SO}^+(8, q)_R$ on the 3–space $S \cap R$, that is, if $g \in \text{SO}^+(8, q)_S \cap \text{SO}^+(8, q)_R$ fixes a 1–space of $S \cap R$ then it normalises a subgroup of order $q^2$ in $P$. We have the following lemma in the case where $q = 2$, which we use in Section 4.

Lemma 2.2 Let $T = \text{PO}^+(8, 2)$, and $S$ and $R$ be totally singular 4–spaces which intersect in a 3–space. Then $T_{S \cap R} \cong 2^{3+6} : \text{SL}(3, 2)$ and has two conjugacy classes of subgroups of the form $2^{3+6} : S_4$. Furthermore, subgroups from different classes are not isomorphic.

Proof. Since $T$ acts transitively on the set of totally singular 3–spaces, and $T_S \cap T_R = T_{S \cap R}$, we have already seen that $T_{S \cap R} \cong 2^{3+6} : \text{SL}(3, 2)$ and the largest normal 2–subgroup of $T_{S \cap R}$ is $Q$. Also the centre, $P$, of $Q$ has order $2^3$ and the action of $T_{S \cap R}$ on $P$ is the dual of the action of $T_{S \cap R}$ on $S \cap R$. The group $\text{SL}(3, 2)$ has two conjugacy classes of subgroups isomorphic to $S_4$, (the
stabilisers of 1–spaces and the stabilisers of hyperplanes), and hence $T_{37}\mathcal{R}$ has two conjugacy classes of subgroups containing $Q$ and of the form $2^{3+6} : S_4$. In one, the group $S_4$ centralises an involution of $P$ and so a group in this class has a normal subgroup of order 2. In the other conjugacy class, $S_4$ has orbits of length 3 and 4 on the involutions of $P$. Let $H$ be a subgroup in this second class and suppose that $Y$ is a normal subgroup of $H$ of order 2. Then $Y \leq O_2(H) = Q.2^2$. Since $H$ acts irreducibly on $O_2(H)/Q$ it follows that $Y \leq Q$. Thus $Y$ is a normal subgroup of $Q$ of order 2 and so lies in the centre, $P$, of $Q$. However, $H$ has orbits of lengths 3 and 4 on the involutions of $P$ and so no such $Y$ exists. Thus subgroups of the form $2^{3+6} : S_4$ in different $T$–conjugacy classes are not isomorphic.

The group $T$ can also be interpreted as a Chevalley group of type $D_4$ over $GF(q)$ (see [2]). Let $\{r_1, r_2, r_3, r_4\}$ be a fundamental system of roots with Dynkin diagram given in Figure 1.

![Fig. 1. Dynkin diagram of type $D_4$](image)

For any subset $\{i, j, k\}$ of $\{1, 2, 3, 4\}$, let $P_{i,j,k}$ be the parabolic subgroup of $T$ associated with $\{r_i, r_j, r_k\}$. The Dynkin diagram has symmetry group $S_3$ and by [8, p 78], there is a corresponding subgroup $A \leq \text{Aut}(T)$ such that $A = \langle \tau, \sigma \rangle \cong S_3$ where $\tau$ and $\sigma$ induce the symmetries $(r_1, r_3, r_4)$ and $(r_3, r_4)$ respectively, of the Dynkin diagram. Then $P_{1,2,3}^\tau = P_{2,3,4}^\tau = P_{1,2,4}^\tau = P_{1,2,3}$ and $P_{1,3,4}^\tau = P_{1,3,4}$ while $P_{1,2,3}^\sigma = P_{1,2,4}^\sigma = P_{1,3,4}^\sigma = P_{2,3,4}^\sigma = P_{2,3,4}$ and $P_{1,3,4}^\sigma = P_{1,3,4}$. Furthermore, (see for example [10]) $\text{Aut}(T) = \langle \text{PGO}^+(8, q), \tau \rangle$ and $|\text{Aut}(T) : \text{PGO}^+(8, q)| = 3$. Any automorphism of $T$ which induces a symmetry of order three of the Dynkin diagram is called a triality automorphism and does not lie in $\text{PGO}^+(8, q)$. Also

$$\text{Out}(T) = \frac{\text{Aut}(T)}{T} \cong \begin{cases} S_3 \times C_f & \text{for } q = 2^f \\ S_4 \times C_f & \text{for } q = p^f, p \text{ odd.} \end{cases}$$

We can interpret our parabolic subgroups geometrically in such a way that

$$P_{1,2,3} = T_{R_0}, P_{1,2,4} = T_{S_0}, P_{2,3,4} = T_{U_0} \text{ and } P_{1,3,4} = T_{W_0}, \quad (2.3)$$

where $S_0 \in \mathcal{S}_1$, $R_0 \in \mathcal{S}_2$, $U_0 \in \mathcal{P}$, and $W_0 \in \mathcal{L}$, such that $U_0 < W_0 < S_0 \cap R_0$. Following [10], we can then define an action of $\text{Aut}(T)$ on $\mathcal{C} = \mathcal{P} \cup \mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_2$.
such that, for each $X \in C$ and $a \in \text{Aut}(T)$, we define $X^a$ so that $T_{X^a} = (T_X)^a$. The action of $\text{PÔ}^+(8, q)$ is the natural action while $U_0^r = S_0$, $S_0^r = R_0$, $R_0^r = U_0$ and $W_0^r = W_0$, and so $\mathcal{P}^r = S_1$, $S_1^r = S_2$, $S_2^r = \mathcal{P}$ and $\mathcal{L}^r = \mathcal{L}$. Moreover, $S_0^r = R_0$ and $R_0^r = S_0$, and $\sigma$ fixes setwise each 1–dimensional subspace of $S_0 \cap R_0$. This action preserves incidence of elements of $C$, where we define two subspaces to be incident either if one is contained in the other, or if they are both totally singular 4–spaces which intersect in a 3–space.

3 The First Family

We can now define our first family of graphs.

**Construction 3.1** Let $V$ be an 8–dimensional vector space over $\text{GF}(q)$ equipped with a quadratic form $Q$ of maximal Witt index. Let $\mathcal{P}$ be the set of totally singular 1–spaces of $V$, and $S_1$ and $S_2$ be the two orbits of $\text{PÔ}^+(8, q)$ on the set of totally singular 4–spaces. Let

$$
\Delta_1 = \{\{U, S, R\}: U \in \mathcal{P}, S \in S_1, R \in S_2, \dim(S \cap R) = 3 \text{ and } U < S \cap R\},
$$

$$
B_1 = \{\{U, S\}: U \in \mathcal{P}, S \in S_1 \text{ and } U < S\},
$$

$$
B_2 = \{\{S, R\}: S \in S_1, R \in S_2 \text{ and } \dim(S \cap R) = 3\},
$$

$$
B_3 = \{\{U, R\}: U \in \mathcal{P}, R \in S_2 \text{ and } U < R\},
$$

and let $\Delta_2 = B_1 \cup B_2 \cup B_3$. Define $\mathcal{G}(q)$ to be the bipartite graph with vertex set $\Delta_1 \cup \Delta_2$, such that two vertices $\{U, S, R\}$ and $\{X, Y\}$ are adjacent if and only if $\{X, Y\} \subseteq \{U, S, R\}$.

From [1], we see that there are $2(1+q)(1+q^2)(1+q^3)$ totally singular 4–spaces in $V$ and that these are divided equally between $S_1$ and $S_2$. Each totally singular 3–space lies in a unique totally singular 4–space in $S_1$ and each totally singular 4–space contains $(q^4 - 1)/(q - 1)$ totally singular 3–spaces. Hence there are $(1 + q)(1 + q^2)(1 + q^3)(q^4 - 1)/(q - 1)$ totally singular 3–spaces in $V$. Thus

$$
|\Delta_1| = \frac{(q^3 - 1)(q^4 - 1)(1 + q)(1 + q^2)(1 + q^3)}{(q - 1)^2}
$$

and

$$
|\Delta_2| = \frac{3(q^4 - 1)(1 + q)(1 + q^2)(1 + q^3)}{q - 1}.
$$

Every vertex in $\Delta_1$ is adjacent to precisely three vertices in $\Delta_2$ while every vertex in $\Delta_2$ is adjacent to $(q^4 - 1)/(q - 1) = q^2 + q + 1$ vertices in $\Delta_1$. Thus $\mathcal{G}(q)$ is biregular of valency $\{3, q^2 + q + 1\}$.

Now $T$ acts transitively on $\Delta_1$ and by [10, see Table 1, line 4] the stabiliser of a vertex in $\Delta_1$ is a maximal subgroup of $\text{Aut}(T)$. Thus $\text{Aut}(T)$ acts primitively.
on $\Delta_1$. On the other hand, $T$ has three orbits $B_1, B_2$ and $B_3$ on $\Delta_2$. Recall from Section 2 that $\text{Aut}(T)$ has a subgroup $A = \langle \tau, \sigma \rangle \cong S_3$ that permutes transitively the three $T$–orbits $B_1, B_2$ and $B_3$. As $\text{Aut}(T)$ preserves incidence among $\mathcal{P} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{L}$, it follows that $G = \text{Aut}(T) \leqslant \text{Aut}(\mathcal{G}(q))$.

For a vertex $v$ of $\Gamma = \mathcal{G}(q)$, we denote the set of vertices adjacent to $v$ by $\Gamma(v)$. Let $U$, $S$ and $R$ be the subspaces $U_0$, $S_0$ and $R_0$ respectively given in (2.3). Then $v = \{ U, S, R \} \in \Delta_1$, and

$$\Gamma(v) = \{ \{ U, S \}, \{ U, R \}, \{ S, R \} \} \subseteq \Delta_2.$$ 

As $A$ induces $S_3$ on $\{ U, S, R \}$ it follows that $A \leqslant G_v$ and so $G_v^{\Gamma(v)} \cong S_3$. In fact, $G_v = (G_S \cap G_R \cap G_U) : A$ and $G_S \cap G_R \cap G_U \leqslant \text{PGO}^+(8, q)$. Since $G$ is transitive on pairs of incident totally singular 4–spaces we have from (2.2) that

$$G_S \cap G_R \cap G_U \cong [q^{11}] : ((C_{q-1} \circ \text{GL}(2, q)) : C_f)$$

and so $G_v$ is as given in Theorem 1.1(4).

Next let $w = \{ S, R \}$. Then

$$\Gamma(w) = \{ \{ U', S, R \} : U' \in \mathcal{P} \text{ and } U' < S \cap R \}.$$ 

Now $\sigma \in G_w$, as $\sigma$ interchanges $S$ and $R$. Thus $G_w = (G_S \cap G_R) : \langle \sigma \rangle$.

Furthermore, $G_S \cap G_R \leqslant \text{PGO}^+(8, q)$ and so by (2.1)

$$G_S \cap G_R \cong [q^9] : ((C_{q-1}^2 \circ \text{GL}(3, q)) : C_f).$$

Hence $G_w$ is as given in Theorem 1.1(4). Also by Lemma 2.1, $G_w^{\Gamma(w)} \cong \text{PGL}(3, q)$ which is a 2–transitive group. Thus $\mathcal{G}(q)$ is locally $(G, 2)$–arc transitive.

Now we consider 3–arcs. Let $u = \{ U_2, S, R \} \in \Delta_1$, where $U_2 \in \mathcal{P}\{ U \}$ and $U_2 < S \cap R$. Then $\Gamma(u) \setminus \{ w \} = \{ \{ U_2, R \}, \{ U_2, S \} \}$ where $w = \{ S, R \}$. Now $\sigma \in G_{vwu}$ interchanges $R$ and $S$ and hence $G_{vwu}$ acts transitively on $\Gamma(u) \setminus \{ w \}$. Hence $G_v$ acts transitively on the set of 3–arcs emerging from $v$. Since $G$ acts transitively on $\Delta_1$ it follows that $G$ acts transitively on the set of 3–arcs starting in $\Delta_1$. Moreover, each 3–arc starting in $\Delta_1$ ends in $\Delta_2$ and vice versa. Hence $G$ acts transitively on the set of 3–arcs starting in $\Delta_2$ and so $G_w$ acts transitively on the set of 3–arcs starting at $w$. Thus $\mathcal{G}(q)$ is locally $(G, 3)$–arc transitive.

Now $T \leqslant \text{Aut}(T)$, $T$ has three orbits on $\Delta_2$, and $T$ is transitive on $\Delta_1$. Hence [6, Lemma 5.5] implies that the quotient graph of $\mathcal{G}(q)$ with respect to the orbits of $T$ is $K_{1,3}$ and so, by [6, Lemma 5.6], $\mathcal{G}(q)$ is not locally $(\text{Aut}(T), 4)$–arc transitive. To complete the proof of Theorem 1.1, we show that there are no further automorphisms of $\mathcal{G}(q)$.

**Lemma 3.2** 
$\text{Aut}(\mathcal{G}(q)) = \text{Aut}(T)$. 

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PROOF. Now \( \Gamma = G(q) \) is locally \((\text{Aut}(\Gamma), 3)\)-arc transitive and is not a complete bipartite graph. Hence by \([6, \text{Lemma 5.3}]\), \(\text{Aut}(\Gamma)\) acts faithfully on both \(\Delta_1\) and \(\Delta_2\). Let \(n = |\Delta_1|\). As noted above, by \([10, \text{see Table 1, Line 4}]\), \(\text{Aut}(T)\) is primitive on \(\Delta_1\), and it follows from \([12]\) that the only group which may possibly lie between \(\text{Aut}(T)\) and \(S_n\) is \(A_n\). Since neither \(A_n-1\) nor \(S_n-1\) has a permutation representation of degree three, we deduce that \(\text{Aut}(\Gamma)^{\Delta_1} = \text{Aut}(T)\). The faithfulness of \(\text{Aut}(\Gamma)\) on \(\Delta_1\) then yields the result.

This completes the proof of Theorem 1.1.

4 The Second Family

For our second family of examples we first review the construction of an edge transitive graph from a given group and pair of subgroups. Let \(G\) be a group with subgroups \(L\) and \(R\) such that \(L \cap R\) is core free in \(G\). Let \(\Delta_1\) be the set \(\left[ G : L \right]\) of right cosets of \(L\) in \(G\), and \(\Delta_2\) the set \(\left[ G : R \right]\) of right cosets of \(R\) in \(G\). The graph \(\text{Cos}(G, L, R)\) is the bipartite graph with vertex set the disjoint union \(\Delta_1 \cup \Delta_2\) such that two vertices \(Lx\) and \(Ry\) are adjacent if and only if \(xy^{-1} \in LR\). Moreover \(G\) acts on the vertex set of \(\Gamma\) by right multiplication, \(G\) is edge transitive, and \(L\) and \(R\) are the stabilisers of the adjacent vertices \(L, R\) respectively. We collect the following results concerning coset graphs, see for example \([6, \text{Section 3.2}]\). A subgroup \(H\) of \(G\) is core free if \(\cap g \in G Hg = 1\).

Lemma 4.1 Let \(\Gamma = \text{Cos}(G, L, R)\) for some group \(G\) with subgroups \(L\) and \(R\) such that \(L \cap R\) is core free. Let \(\Delta_1 = \left[ G : L \right]\) and \(\Delta_2 = \left[ G : R \right]\). Then

(1) \(\Gamma\) is connected if and only if \(\langle L, R \rangle = G\),
(2) \(G \leq \text{Aut}(\Gamma)\), \(\Gamma\) is \(G\)-edge transitive, and \(\Delta_1\) and \(\Delta_2\) are \(G\)-orbits on vertices,
(3) \(G\) acts faithfully on \(\Delta_1\) and \(\Delta_2\) if and only if both \(L\) and \(R\) are core free,
(4) \(\Gamma\) is locally \((G, 2)\)-arc transitive if and only if \(L\) acts \(2\)-transitively on \([L : L \cap R]\) and \(R\) acts \(2\)-transitively on \([R : L \cap R]\).

Conversely, if \(\Gamma\) is a \(G\)-edge transitive but not \(G\)-vertex transitive graph, and \(v\) and \(w\) are adjacent vertices then, \(\Gamma \cong \text{Cos}(G, G_v, G_w)\).

We also collect together the following results from \([10, \text{Propositions 2.3.8, 3.4.2, 3.4.3 and Table 1}]\) concerning \(\text{P}\Omega^+(8, p)\).

Proposition 4.2 Let \(T = \text{P}\Omega^+(8, p)\) for some odd prime \(p\) and let \(G = \text{Aut}(T)\). Then the following hold.

(1) \(T\) has \(4\) conjugacy classes of maximal subgroups isomorphic to \(\Omega^+(8, 2)\)
permutated naturally by $G/T \cong S_4$. Furthermore, for such a subgroup $H$, $N_G(H) \cong \text{Aut}(H)$.

(2) $T$ has 4 conjugacy classes of subgroups isomorphic to $2^{3+6} : \text{SL}(3,2)$ permutated naturally by $G/T \cong S_4$. Furthermore, given such a subgroup $K$,

(a) $K$ has a unique minimal normal subgroup $P$ of order $2^3$ whose involutions belong to the only $T$–conjugacy class of involutions which is fixed by a triality automorphism of $T$,

(b) $K$ acts irreducibly on $P$ inducing $\text{SL}(3,2)$,

(c) $N_G(K) \cong K.S_3$, and

(d) $N_G(K)$ is a maximal subgroup of $T.S_3$.

Before we give our second construction we need the following group theoretic lemma.

**Lemma 4.3** Let $T = \text{PΩ}^+(8,p)$ for some odd prime $p$ and $G = \text{Aut}(T)$. Then $T$ contains subgroups $K_1, K_2, K_3$ and $K_4$ of the form $2^{3+6} : \text{SL}(3,2)$ and $N_G(K_1) = K_1.S_3 \cong 2^{3+6} : (\text{SL}(3,2) \times S_3)$ for each $i = 1, 2, 3, 4$. Furthermore, if $L_0 = K_1 \cap K_2 \cap K_3 \cap K_4$, then

(1) $L_0 \cong 2^{3+6} : S_4$, and

(2) $N_G(L_0) = L_0.S_4$.

**PROOF.** Let $H$ be a maximal subgroup of $T$ isomorphic to $\text{Ω}^+(8,2)$. Then by Proposition 4.2.1, $N_G(H) = \text{Aut}(H)$ and so $N_G(H) = H : A$ where $A = \langle \tau, \sigma \rangle \cong S_3$ is a group of graph isomorphisms of $H$ as defined in Section 2 (note here $\tau$ and $\sigma$ are defined for $H$, not for $T$). As $H$ is maximal in $T$, $A \cap T = 1$, $T : A \cong T : S_3$ and $\tau$ is also a triality automorphism of $T$.

Let $V$ be an 8–dimensional vector space over $\text{GF}(2)$ upon which $H$ acts naturally, and let $U$, $S$ and $R$ be the totally singular subspaces $U_0$, $S_0$ and $R_0$ of $V$ given in (2.3), (again for $H$, not $T$). Let $K_1 = H_S \cap H_R$, $K_2 = H_R \cap H_U$ and $K_3 = H_U \cap H_S$. Then $A$ permutes the set $\{K_1, K_2, K_3\}$ inducing the group $S_3$ and by Lemma 2.2, each $K_i \cong 2^{3+6} : \text{SL}(3,2)$. Let $Q = O_2(K_1) = 2^{3+6}$ and $P$ be the centre of $Q$. By Lemma 2.2, the action of $K_1$ on $P$ is the dual of the action of $K_1$ on $S \cap R$. For any distinct $i, j \in \{1, 2, 3\}$, we have

$$K_1 \cap K_2 \cap K_3 = K_i \cap K_j = H_S \cap H_U \cap H_R.$$ 

Then as $H_S \cap H_U \cap H_R$ is the stabiliser in $K_1$ of a 1–space in $S \cap R$ we have $Q \trianglelefteq K_1 \cap K_2 \cap K_3 \cong 2^{3+6} : S_4$ such that the $S_4$ normalises a subgroup $Z$ of $P$ of order 4. Furthermore, since $O_2(S_4)$ centralises each of the 3 involutions of $Z$ and is regular on $P \setminus Z$, it follows that $Z$ is equal to the centre of $O_2(K_1 \cap K_2 \cap K_3) = Q : O_2(S_4)$. 

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Now $K_1$ acts irreducibly on $P$ inducing the group $\text{SL}(3,2)$ and so all involutions in $P$ lie in the same $T$–conjugacy class $\mathcal{C}$. Furthermore, $Z$ is characteristic in $K_1 \cap K_2 \cap K_3$ and $A$ normalises $K_1 \cap K_2 \cap K_3$. As $A$ contains a triality automorphism of $T$, it follows that the class $\mathcal{C}$ is fixed by some triality automorphism of $T$. Thus by [10, Proposition 3.4.2], each $K_i$ is one of the subgroups described in Proposition 4.2.2.

By Proposition 4.2.2, $N_G(K_i) = K_i \cdot S_3$. We claim that in fact, $N_G(K_i) = 2^{3+6} : (\text{SL}(3,2) \times S_3)$. Since all the $K_i$ are conjugate in $G$ it suffices to consider $K_1$. Since $\sigma \in A$ interchanges $S$ and $R$ we have $\sigma \in N_G(K_1)$. Thus it follows that $N_G(K_1) = 2^{3+6} : (\text{SL}(3,2), S_3)$. Now the outer automorphism group of $\text{SL}(3,2)$ is $C_2$ and any outer automorphism interchanges the stabilisers of 1–spaces with the stabilisers of 2–spaces. It follows that the $S_3$ must centralise $\text{SL}(3,2)$, as otherwise it would interchange the centraliser of an involution of $P$ with the normaliser of a subgroup of order $2^2$. The two conjugacy classes of subgroups isomorphic to $K_i$ in $G$ are isomorphic to $K_i$. Furthermore, by adjusting $\gamma$ by some element of $T$ if necessary, we may assume that $K_i = K_1$. Thus

$$K_1 \cap K_2 \cap K_3 \cong (K_1 \cap K_2 \cap K_3)^\rho = K_1 \cap K_2^\rho \cap K_4 \leq K_1.$$
However, by Lemma 2.2, $K_1$ contains only one conjugacy class of subgroups isomorphic to $K_1 \cap K_2 \cap K_3$ and so we may have chosen $\rho$, adjusting by some element of $K_1$ if necessary, such that

$$K_1 \cap K_2 \cap K_3 = (K_1 \cap K_2 \cap K_3)\rho = K_1 \cap K_2^\rho \cap K_4.$$ 

Hence $K_1 \cap K_2 \cap K_3 \cap K_4 = K_1 \cap K_2 \cap K_3$ and is normalised by $\rho$. It is also normalised by $A$ and so, letting $L_0 = K_1 \cap K_2 \cap K_3 \cap K_4$, we see that $N_G(L_0)/L_0 \cong S_4$.

We are now in a position to give our second construction.

**Construction 4.4** Let $H(p) = Cos(G, L, R)$ where $G = Aut(T)$, $R = N_G(K_4)$ and $L = N_G(L_0)$, where $K_4$ and $L_0$ are as given in Lemma 4.3.

Now $H(p)$ is a bipartite graph with bipartite halves $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$, and we have

$$|\Delta_1| = \frac{p^{12}(p^2 - 1)(p^4 - 1)^2(p^6 - 1)}{2^{14}3}$$

and

$$|\Delta_2| = \frac{p^{12}(p^2 - 1)(p^4 - 1)^2(p^6 - 1)}{2^{14}3 \cdot 7}.$$ 

By Lemma 4.1, $G \leq Aut(\Gamma)$. From Proposition 4.2, $N_G(K_4)$ is a maximal subgroup of $T.S_3$ and so $\langle L, R \rangle = G$. Hence Lemma 4.1 implies that $H(p)$ is connected. From Lemma 4.3, we have

$$R = 2^{3+6} : (SL(3, 2) \times S_3)$$

and so $R \cap L = R \cap N_G(L_0) = 2^{3+6} : (S_4 \times S_3)$. Hence $H(p)$ is biregular of valency $\{4, 7\}$. The action of $L$ on $[L : L \cap R]$ is equivalent to the action of $S_4$ on four points while the action of $R$ on $[R : L \cap R]$ is equivalent to the action of $SL(3, 2)$ on the seven 1–spaces of a 3–dimensional vector space over $GF(2)$. As both of these actions are 2–transitive, Lemma 4.1 implies that $H(p)$ is locally $(G, 2)$–arc transitive.

Let $\Gamma = H(p)$. We now consider 3–arcs. Let $v$ be the vertex of $\Delta_1$ given by $L$ and $w$ the vertex of $\Delta_2$ given by $R$. For $w_2 \in \Gamma(v) \setminus \{w\}$ we have that $G_{w_2w} = T_{v}S_2 \cong 2^{3+6} : (S_4 \times S_2)$ which still acts transitively on the six vertices of $\Gamma(w) \setminus \{v\}$ inducing the group $S_4$. Hence $G_{w_2}$ acts transitively on the set of 3–arcs starting at $w_2$. Using the fact that every 3–arc starting in $\Delta_2$ ends in $\Delta_1$ and vice versa, it follows that $\Gamma$ is locally $(G, 3)$–arc transitive.

Now $T$ acts transitively on $\Delta_1$ and so the action of $G$ on $\Delta_1$ is quasiprimitive. However, $T$ has four orbits on $\Delta_2$ and so by [6, Lemma 5.5], $\Gamma_T = K_{1,4}$. 

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Furthermore, [6, Lemma 5.6] implies that $\Gamma$ is not locally $(G,4)$–arc transitive. All that is left to complete the proof of Theorem 1.2 is to prove that $\text{Aut}(\Gamma) = G$. We do this in the next section, see Proposition 5.1.

**Remark 4.5** In [9], Kantor constructed a geometry which is almost a building whose diagram is the extended $D_4$, that is, a middle node with four neighbours. The maximal parabolic subgroups corresponding to the four outer nodes are isomorphic to $\text{PΩ}^+(8,2)$, and the $K_i$ in Lemma 4.3 are the intersection of any three. Thus the graphs $\mathcal{H}(p)$ can be constructed in a similar way to the graphs $\mathcal{G}(q)$.

**Remark 4.6** Note that for $T = \text{PΩ}^+(8, p^f)$, where $f > 1$, the subgroup $H$ is no longer maximal in $T$ as it is contained in the subgroup $\text{PΩ}^+(8, p)$. If we choose $L_0$ and $K_4$ inside $H$ as before and take $L$ and $R$ to be their normalisers in $G = \text{Aut}(T)$, we get $\langle L, R \rangle = \text{Aut}(\text{PΩ}^+(8, p)) \times \langle \phi \rangle$ where $\phi$ is a field automorphism of $T$ of order $f$ which centralises $\text{PΩ}^+(8, p)$. Hence in this case, Lemma 4.1 implies that the graph $\text{Cos}(G, L, R)$ is not connected.

5 The full automorphism group of $\mathcal{H}(p)$

In this section we complete the proof of Theorem 1.2 by proving the following. Let $p$ be an odd prime and $T = \text{PΩ}^+(8, p)$.

**Proposition 5.1** $\text{Aut}(\mathcal{H}(p)) = \text{Aut}(T)$.

Let $\Gamma = \mathcal{H}(p)$, $G = \text{Aut}(T)$, $A = \text{Aut}(\Gamma)$ and let $v \in \Delta_1$ correspond to the subgroup $L$. For a finite group $H$, let $\Pi(H)$ be the set of primes dividing $|H|$. First we prove the following.

**Lemma 5.2** $\{2, 3\} \subseteq \Pi(A_v) \subseteq \{2, 3, 5\}$.

**PROOF.** Since $G_v \leq A_v$ it follows from Lemma 4.3 that 2 and 3 divide $|A_v|$. Suppose that $A_v$ contains an element $g$ of prime order greater than 5. Then as $|\Gamma(v)| = 4$, $g$ fixes $\Gamma(v)$ pointwise. Next let $w \in \Gamma(v)$. Then $g$ fixes $v \in \Gamma(w)$ and so as $|\Gamma(w)| = 7$, $g$ also fixes $\Gamma(w)$ pointwise. Proceeding in this manner through the vertices in $V\Gamma$, we deduce from the connectivity of $\Gamma$ that $g$ fixes $V\Gamma$ pointwise, contradicting $g \neq 1$. Thus the result holds.

Now as $\Gamma$ is not a complete bipartite graph, [6, Lemma 5.2], implies that $A$ acts faithfully on each of the two bipartite halves $\Delta_1$ and $\Delta_2$ of $\Gamma$. This enables us to prove the following.
Lemma 5.3 $C_A(T) = 1$ and $N_A(T) = G$.

**PROOF.** Now $G_v = N_G(L_0)$ with $L_0 \leq T$ as in Lemma 4.3. Furthermore, $N_T(L_0) = L_0 = T_v$. Thus $|\text{fix}_{\Delta_1}(L_0)| = |N_T(L_0) : L_0| = 1$, and so by [5, Theorem 4.2A], $C_{\text{Sym}(\Delta_1)}(T) = 1$. As $A$ acts faithfully on $\Delta_1$, the result follows.

Now suppose that $A = \text{Aut}(\Gamma) \neq G$. Then $A$ contains a subgroup $X$ which properly contains $G$ as a maximal subgroup. Let $N$ be a minimal normal subgroup of $X$. Then $N = S_1 \times \cdots \times S_k$ where each $S_i \cong S$ for some finite simple group $S$.

**Lemma 5.4** $N \cap G = 1$, and $X = N : G$.

**PROOF.** Since $N \triangleleft X$, we have $N \cap G \triangleleft G$ and so either $N \cap G = 1$ or $T \leq N \cap G$. Suppose $T \leq N \cap G$. Since $T \neq 1$, there exists $i$ such that $\pi_i(T) \neq 1$, where $\pi_i : N \to S_i$ is the $i^{th}$ projection map. Since $T$ is simple, $\pi_i(T) \cong T$ and so $S_i$ has a subgroup isomorphic to $T$, whence $N$ has a subgroup isomorphic to $T^k$. In particular $S$ is a nonabelian simple group. Let $r$ be a prime greater that 5 which divides $|T|$ and let $r^s$ be the largest power of $r$ dividing $|T|$. By Lemma 5.2, $\Pi(T_v) \subseteq \{2, 3, 5\}$, and so $r^s$ is the largest power of $r$ dividing $|\Delta_1|$. However $r^s$ divides $|N|$, and $r^s | N_v$. Therefore $r^{sk}$ divides $|N : N_v|$ which in turn divides $|\Delta_1|$. Thus $k = 1$ and so $T \leq N = S$. Since $|N| = |\Delta_1| |N_v|$ and $\Pi(N_v) \subseteq \{2, 3, 5\}$, it follows that the only primes dividing $|N|$ are those dividing $|T|$. Hence, we see from [13, Corollary 5] that $N = T$. Thus $X \leq N_A(T) = G$, contradicting the fact that $G$ is a proper subgroup of $X$. Hence $N \cap G = 1$. Since $G$ is maximal in $X$ it follows that $X = N : G$.

For an integer $n$ and a prime $r$, the $r$–part of $n$ is the highest power $r^s$ dividing $n$.

**Lemma 5.5** $\Pi(N) \subseteq \{2, 3, 5\}$.

**PROOF.** By Lemma 5.2, if $r$ is a prime dividing $|X|$ and $r > 5$, then $r$ does not divide $|X_v|$. Hence the $r$–part of $|X|$ divides $|\Delta_1|$. However, the $r$–part of $|T|$ divides $|X|$ and $|\Delta_1|$ divides $|T|$. Hence the $r$–part of $|X|$ equals the $r$–part of $|T|$. By Lemma 5.4, $N : T \leq X$ and so it follows that $r$ does not divide $|N|$. Thus $\Pi(N) \subseteq \{2, 3, 5\}$.

**Lemma 5.6** $\Pi(N_v) \subseteq \{2, 3\}$ and, for $w \in \Delta_2$, $N^\Gamma(w) = 1$. 
PROOF. Let \( w \) be a vertex in \( \Delta_2 \). Then \( \Gamma^{(w)}_w \triangleleft \Gamma^{(w)}_w \). Now \( |\Gamma(w)| = 7 \) and \( \Gamma^{(w)}_w \) contains \( G^{(w)}_w = SL(3,2) \). Thus \( \Gamma^{(w)}_w = SL(3,2), A_7 \) or \( S_7 \). Then as 7 does not divide \( |N| \) it follows that \( \Gamma^{(w)}_w = 1 \).

Now let \( g \) be an element of \( N_v \) of order 5. Then \( g \) fixes \( \Gamma(v) \) pointwise. Let \( w \in \Gamma(v) \). Then \( g \in N_w \) and so also fixes \( \Gamma(w) \) pointwise. It then follows from the connectivity of \( \Gamma \) that \( g \) fixes \( V\Gamma \) pointwise, contradicting \( g \neq 1 \). Thus \( \Pi(N_v) \subseteq \{2,3\} \).

Lemma 5.7 \( N \) is abelian.

PROOF. Suppose that \( N \) is nonabelian. Then by Lemma 5.5, \( S \) is a nonabelian simple group such that \( \Pi(S) \subseteq \{2,3,5\} \). By Burnside’s ‘\( p^aq^b \) Theorem’, \( \Pi(S) = \{2,3,5\} \). As \( X = N : G \) and \( N \) is a minimal normal subgroup of \( X \), the action by conjugation of \( G \) on the \( k \) simple direct factors of \( N \) is transitive. As \( T \) is normal in \( G \), all \( T \)-orbits on the \( S_i \) have the same length. Suppose that \( T \) fixes some \( S_i \) in this action, that is \( T \leq N_X(S_i) \). Since \( \text{Out}(S_i) \) is soluble and \( \Pi(S_i) = \{2,3,5\} \), it follows that \( \text{Aut}(S_i) \) has no subgroup isomorphic to \( T \). Hence \( T \leq C_X(S_i) \). However, this contradicts Lemma 5.3 and so \( T \) does not fix any simple direct factor of \( N \). Hence \( T \) has a transitive action of some degree \( t > 1 \), with \( t \) dividing \( k \). By [4], \( t \) is at least \( \frac{(p^a-1)(p^b+1)}{p-1} > p^6 \) and so \( k > p^6 \).

Now \( 5^k \) divides \( |N| \) and by Lemma 5.6, \( \Pi(N_v) \subseteq \{2,3\} \). Thus \( 5^k \) divides \( |\Delta_1| \) and so \( 5^{p^6} \) divides \( |\Delta_1| \). Hence \( 5^{p^6} \) divides \( p^{12}(p^4-1)^2(p^2-1)(p^6-1) \). This is not possible and so \( N \) is abelian.

We can now complete the proof of Proposition 5.1.

PROOF. [Proof of Proposition 5.1] Suppose that \( G \neq \text{Aut}(\Gamma) \). Let \( X \) be a subgroup of \( \text{Aut}(\Gamma) \) which properly contains \( G \) as a maximal subgroup. Let \( N \) be a minimal normal subgroup of \( X \). By Lemma 5.4, \( N \cap G = 1 \) and \( X = N : G \). By Lemmas 5.5 and 5.7, \( N = C_k^r \) where \( r = 2,3 \) or 5, and \( G \) acts irreducibly on \( N \). By Lemma 5.3, \( C_A(T) = 1 \), so \( T \) does not centralise \( N \) and in particular \( k > 1 \). Let \( v \in \Delta_1 \). Then \( |\Gamma(v)| = 4 \) and \( G^{\Gamma(v)}_v = S_4 \), so we have \( N^{\Gamma(v)}_v \triangleleft X^{\Gamma(v)}_v = S_4 \). Thus \( N^{\Gamma(v)}_v = 1 \) or \( C_2^2 \). By Lemma 5.6, \( N^{\Gamma(w)}_w = 1 \) for all \( w \in \Delta_2 \).

We claim that either \( |N| \) divides \( |\Delta_1| \), or \( r = 2 \) and \( |N|/4 \) divides \( |\Delta_1| \). Suppose first that \( N^{\Gamma(v)}_v = 1 \). Since \( N^{\Gamma(w)}_w = 1 \) for all \( w \in \Delta_2 \), the connectivity of \( \Gamma \) implies that \( N_v = 1 \). Thus \( |N| \) divides \( |\Delta_1| \). Suppose now that \( N^{\Gamma(v)}_v = C_2^2 \). Then \( N^{\Gamma(v)}_v \) is regular. If \( g \in N_v \) fixes \( \Gamma(v) \) pointwise then \( g \) fixes \( \Gamma(w) \)
pointwise for each \( w \in \Gamma(v) \). Then for \( u \in \Gamma(w) \), since \( g \) fixes \( w \in \Gamma(u) \) and \( N_u^{\Gamma(u)} \) is regular, it follows that \( g \) fixes \( \Gamma(u) \) pointwise. Thus by the connectivity of \( \Gamma \), \( g \) fixes \( VT \) pointwise and so \( g = 1 \). Thus \( N_v \cong N_v^{\Gamma(v)} = C_2^2 \). Hence \( |N|/4 = |N:N_v| \) divides \( |\Delta_1| \). Thus the claim is proved, so we have that \( r^k \) divides \( |\Delta_1| \) if \( r = 3 \) or \( 5 \), and \( 2^{k-2} \) divides \( |\Delta_1| \) if \( r = 2 \).

By Lemma 5.3, \( T \) does not centralise \( N \), and hence \( G \) acts faithfully on \( N \) and we have \( G \leq \text{GL}(k,r) \). The smallest degree of an irreducible representation of \( T \) over a field of characteristic \( r = 2, 3 \) or \( 5 \) is \( p^2(p^2 - 1) > p^4 \) if \( p \neq r \) and \( 8 \) if \( p = r \) (see for example [11, Theorem 5.3.9 and Proposition 5.4.13]). Thus \( k > p^4 \) if \( r \neq p \) and \( k \geq 8 \) if \( r = p \). Since none of \( 5p^4 \), \( 3p^4 \) or \( 2p^4 - 2 \) divide \( |\Delta_1| \) it follows that \( r = p = 3 \) or \( 5 \) and \( k \geq 8 \) (recall that \( p \) is odd). When \( p = 3 \) the largest power of \( 3 \) dividing \( |\Delta_1| \) is \( 3^{11} \) while when \( p = 5 \) the largest power of \( 5 \) dividing \( |\Delta_1| \) is \( 5^{12} \). Hence \( 8 \leq k \leq 12 \) and \( G \) is an irreducible subgroup of \( \text{GL}(k,p) \). Also, by Lemma 5.3, \( C_N(T) = 1 \). If \( T \) were reducible, then it would normalise and act irreducibly on some proper subgroup \( M \) of \( N \) of order \( p^t \), for some \( t \) such that \( 8 \leq t < k \). By the irreducibility of \( G \), there exists \( g \in G \) such that \( M^g \neq M \). Since \( T \not\leq G \), \( T \) also normalises \( M^g \) and hence normalises \( M \cap M^g \). As \( T \) is irreducible on \( M \), we have \( M \cap M^g = 1 \). Hence \( t \geq 2k \), contradicting \( 8 \leq k \leq 12 \). Thus \( T \) is also an irreducible subgroup of \( \text{GL}(k,p) \). Hence by [11, Theorem 5.4.11] and using its terminology, \( k = 8 \) and \( N \) is either the natural module or one of the two spin modules for \( T \). However, a triality automorphism for \( T \) permutes these three \( T \)-modules (see [3, IV.2.4 and IV.3.1]) and so none of these three \( T \)-modules is also a \( G \)-module. Hence \( X = G \) and so \( G = \text{Aut}(\Gamma) \).

This completes the proof of Theorem 1.2.

References


[10] P. B. Kleidman, The maximal subgroups of the finite 8–dimensional orthogonal groups $\mathrm{P\Omega}_8^+(q)$ and of their automorphism groups, J. Algebra 110 (1987), 173–242.


