Wallpaper and symmetry

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How many types of wallpaper are there?

Will study wallpaper according to the symmetry.

If have the same symmetries then say of the same type.
What do we mean by a symmetry?

By a symmetry, we mean a transformation of the object which leaves it unchanged.

For example, a reflection about the vertical axis is a symmetry of the triangle below.
In fact the triangle has six symmetries:
Order and composition

Note that any symmetry followed by a symmetry is still a symmetry.

Doing nothing is always a symmetry.

The order of a symmetry is the smallest number of repetitions such that the result is the same as doing nothing.

For the triangle, the reflections had order 2, the two rotations had order 3 and “doing nothing” has order 1.
Another example

The following shape also has six symmetries:

This is a different collection of 6 symmetries to the triangle’s:

- It has a symmetry of order 6 but the triangle does not.
- The order of composition matters for the triangle but not here.
Groups

So it is not enough to just count symmetries.

We also need to see how they interact with each other.

The set of all symmetries of an object forms a group.
Our goal

We will classify types of wallpaper according to their group of symmetries.

A different group means different wallpaper.

Same group means same type of wallpaper.
Our assumptions

• Assume we are wallpapering the whole plane.
• The pattern is invariant under translations in two different directions.
• There is a translation of minimal length (no stripes).
• There is some finite region which generates the whole plane (the pieces of wallpaper).
• Our symmetries preserve distance (no shrinking/expanding).
Types of symmetries

There are only four possible types of symmetries

- Translations
- Rotations
- Reflections
- Glide reflections
Glide reflections

A glide reflection is a translation along a line composed with a reflection about the line.
Orders of rotations

Theorem (The Crystallographic Restriction)

*Our assumptions imply that rotations can only have order 2, 3, 4 or 6.*

That is, by angles $180^\circ$, $120^\circ$, $90^\circ$ or $60^\circ$, respectively.
Proof

Let $O$ be center of a rotation $s$ of order $n$.

Let $A$ be image of $O$ under a translation $t$ of minimum distance.

Spinning $A$ under $s$ gives a regular $n$-gon.

The translation $t' = t^{-1}s^{-1}ts$ maps $A$ to $A^s$.

For $n \geq 7$, a side of a regular $n$-gon is less than the radius and so $t'$ moves points a smaller distance than $t$.

Thus $n \leq 6$.

Can also show $n \neq 5$. 
Penrose tiling on floor of chemistry building

Has fivefold rotational symmetry and mirror symmetry but no translational symmetry.
The possible symmetry groups

Theorem (Barlow, Fedorov, Schönsflies 1890’s)

*There are only 17 possible symmetry groups.*

These are known as the **wallpaper groups** or the **plane crystallographic groups**.

Only 17 discrete subgroups of the Euclidean group $\mathbb{Z}^2 \rtimes O(2)$ which contain translations in two linearly independent directions.
Conway’s orbifold notation

Each group is represented by a sequence of numbers 2, 3, 4, 6 and symbols $\circ, \ast, \times$.

$\ast$ represents a mirror.

Numbers after a $\ast$ are orders of rotations about points on a mirror.

Numbers before a $\ast$ are orders of rotations about points not on a mirror (gyrations).

$\times$ denotes a glide reflection with axis not a mirror.
6-fold rotations

Only 2 groups contain a 6-fold rotation: *632 and 632.

*632 (p6m):
6-fold rotations

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Only 2 groups contain a 6-fold rotation: \( \ast 632 \) and 632.

\( \ast 632 \) (p6m):

![Hexagonal lattice with 6-fold rotation symmetry](image)
6-fold rotations

Only 2 groups contain a 6-fold rotation: *632 and 632.

*632 (p6m):

```
/0
/0
/0
/1
/1
/1
/0
/0
/1
/1
/0
/0
/1
/1
```
6-fold rotations

Only 2 groups contain a 6-fold rotation: *632 and 632.

*632 (p6m):
Need to remove all reflections.
4-fold rotations

There are 3 groups containing 4-fold rotations: *422, 4*2 and 442.

*442 (p4m): The group of the regular tiling into squares.
A variation
Another variation
Note rotation of order 4 is not on a mirror.
Again need to remove all reflections.
3-fold rotations

There are 3 groups where maximal rotation order is 3: \(*333\), \(3 \ast 3\), \(333\).

\(*333\) (p3m1):
3-fold rotations

There are 3 groups where maximal rotation order is 3: \(*333, 3 \ast 3, 333.\)

\(*333 \text{(p3m1)}:\)
3-fold rotations

There are 3 groups where maximal rotation order is 3: *333, 3 * 3, 333.

*333 (p3m1):
$3 \times 3$ (p31m)
2-fold rotations

There are 5 groups where maximal rotation order is 2: *2222, 2*22, 22*, 22×, 2222.

*222 (pmm)
$2 \times 22 \text{ (cmm)}$
2 * 22 (cmm)
22* (pmg)
No rotations

There are 4 types of groups with no rotations: **, **, × ×, o.

** (pm)
One glide reflection axis and one mirror.
Only contains glide reflections and translations.
Only translations.
Other dimensions?

• If the symmetry group only contains translations in one direction we obtain what are known as the 7 Frieze groups.
• In three dimensions there are 219 crystallographic groups (230 if distinguish mirror images) (Barlow, Fedorov, Schönflies)
• In arbitrary dimension there are a finite number of symmetry groups. (Bieberbach 1910)
• 4783 in dimension 4 (Brown, Bülow, Neubüser, Wondratschek, Zassenhaus 1978)
• 222018 in dim 5, 28927915 in dim 6 (Plesken, Schulz 2000)
Some useful references