Background

$G \leq \text{Sym}(\Omega)$

transitive if for all $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $\alpha^g = \beta$.

2-transitive if $G$ is transitive on the set of all distinct ordered pairs.

$\frac{1}{2}$-transitive if either $G$ is transitive or all orbits of $G$ have the same length $m > 1$.

$\frac{3}{2}$-transitive if it is transitive and $G_\alpha$ is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$.

For a transitive group $G$, we call the lengths of the orbits of $G_\alpha$ the subdegrees of $G$. 
Examples

- Any 2-transitive group.
- Any Frobenius group with finite point stabilisers.
- Any finite normal subgroup of a 2-transitive group.
- $\text{PSL}(2, q)$, for $q$ even, acting on $q(q - 1)/2$ points.
- $A_7$ or $S_7$ acting on 2-subsets.
Primitive groups

$G$ is called **primitive** if there is no nontrivial partition of $\Omega$ preserved by $G$.

The structure of finite primitive groups is given by the O’Nan-Scott Theorem which classifies them into eight types.

Two important types:

- **almost simple:** $T \leq G \leq \text{Aut}(T)$, for some finite nonabelian simple group $T$.
- **affine:** $G \leq \text{AGL}(d, p)$ and contains all translations.
Burnside’s Theorem

Every 2-transitive group is primitive.

**Burnside’s Theorem** A finite 2-transitive group is either almost simple or affine.
$\frac{3}{2}$-transitive groups

**Wielandt (1965):** A finite $\frac{3}{2}$-transitive permutation group is either primitive or Frobenius.

**Passman (1967-1969):** Classified all finite soluble $\frac{3}{2}$-transitive permutation groups.

**Theorem (BGLPS)**

A finite primitive $\frac{3}{2}$-transitive group is either almost simple or affine.
Characterising the $\text{PSL}(2, q)$ examples:

Suppose that $G$ is a finite $3_2$-transitive group that is neither 2-transitive nor of affine type.

**McDermott (1977):** If $G_{\alpha, \beta} = C_2$ for all $\alpha, \beta \in \Omega$ then $G = \text{PSL}(2, 2^f)$.

**Camina (1979):** If $G_{\alpha, \beta}$ is a cyclic 2-group then $\text{soc}(G) = \text{PSL}(2, 2^f)$.

**Zieschang (1992):** If all two point stabilisers are conjugate then $\text{soc}(G) = \text{PSL}(2, 2^f)$. 
An elementary observation

- $G$ a transitive permutation group on $\Omega$.
- $r$ a prime dividing $|\Omega|$.

If $G$ has a subdegree divisible by $r$ then $G$ is not $\frac{3}{2}$-transitive.

**Problem A**

Determine the groups of Lie type of characteristic $p$ which have a primitive action with no subdegrees divisible by $p$. 
A very useful lemma

Lemma

Let $G$ be a finite permutation group with point stabiliser $H$ and let $x \in H$ have order a power of $p$. If

$$|x^G| > |x^G \cap H|^2 - |x^G \cap H| + 1$$

then $G$ has a subdegree divisible by $p$. 
Solution to Problem A

Theorem (BGLPS)

Let $G$ be a finite almost simple group, with socle $L$ of Lie type of characteristic $p$, acting primitively on $\Omega$ such that $p$ divides the order of a point stabiliser.

Then one of the following holds:

1. $G$ has a subdegree divisible by $p$;
2. $G$ is 2-transitive;
3. $L = \text{PSL}(2, 2^f)$ acting on $2^{f-1}(2^f - 1)$ points, $|G : L|$ odd;
4. $L = \text{PSU}(3, 5)$ acting on 50 points (subdegrees 1, 7, 42);
5. $L = \text{PSp}(4, 3)$ acting on 27 points (subdegrees 1, 10, 16);
6. $G = G_2(2)'$ acting on 36 points (subdegrees 1, 7, 7, 21).
Classification of almost simple examples

Theorem (BGLPS)

Let $G$ be a finite almost simple $\frac{3}{2}$-transitive permutation group. Then one of the following holds:

1. $G$ is 2-transitive.
2. $\text{soc}(G) = \text{PSL}(2, 2^f)$ acting on $2^{f-1}(2^f - 1)$ points and either $G = \text{soc}(G)$ or $f$ is prime.
3. $G$ is $S_7$ or $A_7$ acting on 2-subsets.
Affine examples

Here $G = \mathbb{Z}_p^d \rtimes G_0$ where $G_0$ is an irreducible subgroup of $GL(d, p)$ acting $\frac{1}{2}$-transitively on the set of nonzero vectors.

Passman (1969): determined all possibilities when $G_0$ imprimitive on vector space. All soluble.

If $G_0$ has an orbit of length divisible by $p$ on vectors then it is not $\frac{1}{2}$-transitive.

GLPST: Determined all irreducible, primitive linear groups with no orbits of length divisible by $p$. 
Theorem (GLPST)

If $G \leq AGL(d, p)$ is a $\frac{3}{2}$-transitive affine permutation group with point-stabiliser $G_0$ having order divisible by $p$, then one of the following holds:

(i) $G$ is 2-transitive;

(ii) $G_0 \leq \Gamma L(1, p^d)$;

(iii) $SL(2, 5) \triangleleft G_0 \leq \Gamma L(2, 9) < GL(4, 3)$ and $G$ has rank 3 with subdegrees 1, 40, 40.
Let $G$ be an infinite $\frac{3}{2}$-transitive permutation group on $\Omega$ that is not 2-transitive.

All subdegrees are finite and the same size.

**Example** $G = \mathbb{R}^2 : \langle \tau \rangle$ where $\tau$ is a rotation of order $n$.

Wielandt’s proof that a finite imprimitive $\frac{3}{2}$-transitive group is Frobenius still holds if there is a system of imprimitivity with finite blocks.

**Question:** Is there an imprimitive $\frac{3}{2}$-transitive group that is not Frobenius?
Infinite primitive $\frac{3}{2}$-transitive groups

Primitive and all subdegrees finite implies $\Omega$ is countable.

Schlichting, Bergman-Lenstra: Primitive and list of subdegrees has finite upper bound implies finite stabilisers.

Simon Smith recently gave an O’Nan-Scott Theorem for primitive permutation groups whose set of subdegrees has a finite upper bound.
Infinite primitive $\frac{3}{2}$-transitive groups

Theorem

Let $G$ be an infinite primitive $\frac{3}{2}$-transitive permutation group. Then either

- $G$ is a countable almost simple group.
- $G = M : G_\alpha$ where $M = T^k$ for some countable simple group $T$, $M$ acts regularly on $\Omega$ and $G_\alpha$ normalises no proper nontrivial subgroup of $M$. 
Let $p > 10^{75}$ be a prime.

Then there exist simple groups $T_p$ such that all nontrivial proper subgroups have order $p$. (Tarski–Ol’Shanskiĭ Monsters)

$T_p$ acting on the set of right cosets of a subgroup of order $p$ is primitive with all subdegrees equal to $p$.

Can also choose $T_p$ such that it has an outer automorphism $\sigma$ of order two normalising no proper nontrivial subgroup.

Let $G = T_p : \langle \sigma \rangle$ acting on the cosets of $\langle \sigma \rangle$. This is primitive with all subdegrees equal to 2.

**Question**: Are there infinite $\frac{3}{2}$-transitive groups that are not Frobenius?