A Logic of Belief and Omission

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Abstract

Here we present a modal logic that emerged when considering the interactions of awareness and knowledge. Whereas knowledge is described by those propositions that are true in every world the agent considers possible, awareness is described by the propositions that the agent perceives as either true or false, in any world. As such, we can model an agent who ignores the existence of a proposition in the context of some possible worlds. The resulting logic is able to capture reasoning free of logical omniscience, where an agent may, consciously or unconsciously, omit facts and deductive rules from their arguments.

1 Introduction

Despite the fact that the rules of propositional logic have been settled for centuries, people still persist in believing contradictory statements. It seems as though people either do not have the capacity of rational thought, or do not have the desire. For example, many people (blindly trusting science) hold the belief that cigarette smokers will die a premature death, yet continue to smoke even though they do not want to die prematurely. Issues such as addiction are complex psychological and sociological phenomena, and while we would never expect to model such intricacies using formal logic, it is important to have a model of reasoning that is at least tolerant of such contradictory conclusions. The only alternative would be to accept that human rationality is without any basis and sense we seem to make is purely accidental.

Epistemic logic is the logic of knowledge and is very much focussed on the element of certainty (or uncertainty). This uncertainty is modelled by a set of worlds, each presenting a possible configuration of propositions, and the agent is uncertain as to which world is the actual world. This is the possible worlds model [8]. Aside from the uncertainty that is modelled by having multiple possible worlds, the agents are considered to be perfect reasoners (or logically omniscient). There has been considerable interest in weakening this constraint to permit fallible reasoners. Human reasoners are notoriously fallible, and frequently entertain contradictions [12]. Being able to formally model fallible reasoning is an important capability in developing systems that are robust for fallible agents. Default logic [2], logics of awareness [10], societies of mind [5], non-monotonic reasoning [11], belief revision [1] and dynamic reasoning [13] have all presented models where an agent is not assumed to automatically know all tautologies, or immediately perform all valid deductions of a logical system.

We present a novel logic that further contributes to the goal of modelling fallible reasoners. As with models of agent uncertainty, we focus on a quite specific but widespread type of fallibility. We augment the model of agent uncertainty with contextual awareness. By contextual awareness we mean the result of ignoring facts, propositions or deductions in the context of one another within a system. Thus within each possible world, we suppose each agent has a “scope of vision”: a set of propositions the agent considers relevant in imagining this possible world. An agent’s scope of vision is then able to quarantine contradictory or uncomfortable facts in distinct worlds allowing the agent to proceed in a state of delusion.

2 Belief and omission

Here we will present the basic intuitions and semantics for a logic of belief and omission. We will suppose that there are a fixed set of agents, \( N = \{1, \ldots, n\} \), and a set of worlds \( S \). Each agent is unable to distinguish certain worlds from other worlds. This induces, for each agent, a partitioning of the set of worlds into equivalence classes. Now we add the assumption that for each agent in each world, every indistinguishable world is not a complete in-
terpretation of that agent’s universe, but is only considered possible in the context of a fixed set of propositions. A mathematician may consider worlds where a theorem is true, and other worlds where the proof may be flawed. He may also consider worlds where his wife is having an affair, and worlds where his wife is not having an affair. However, he does not consider worlds where the theorem is true and his wife is having an affair, simply because he does not have the same frame of mind. In each world an agent has a set of properties of which they are aware, and they omit the interpretation of all other properties.

Logics of awareness have been, and continue to be, investigated at some depth [5, 14]. The novelty in the approach here is to have an agent’s awareness dependent on the worlds they consider, rather than the world they are in. Thus although an agent may consider several worlds possible, the agent does not consider the same set of propositions in all worlds. As the example of the mathematician above shows, this allows agents to maintain different frames of minds for differences aspects of their interests. But these aspects need not be distinct, and this is where we are able to model fallible reasoning. An agent may believe smoking causes cancer because in all worlds he considers smoking and cancer, smoking leads to cancer. He may also believe cancer leads to death, as in all worlds he considers cancer and death, cancer leads to death. However the agent may still consider it certain that he will live a long life as his wife is having an affair, simply because he does not consider worlds where the theorem is true and other worlds where his wife is not having an affair. However, should the agent forsake one of these assertions.

2.1 Language

We suppose that we are extending a multi-agent propositional epistemic logic, defined over a set of agents $N$ and set of atomic propositions, $P$. For each agent, $i$ we suppose that there is an operator $K_i \varphi$, to stand for agent $i$ knows $\varphi$ (or more precisely, in all worlds that agent $i$ does not ignore $\varphi$, $\varphi$ is true). We will retain the term know and the notation of epistemic logic, although in this case believes, concludes or reckons may be more appropriate.

Definition 1 (Language) Given are a countably infinite set of propositional variables (facts) $P$, and a (disjoint) countably infinite set of agents $N$. The language of belief and omission $\mathcal{L}$ is defined as

$$\varphi ::= \top | p | \varphi \land \varphi | \neg \varphi | K_i \varphi$$

where $i \in N$ and $p \in P$. Implication $\rightarrow$, disjunction $\lor$, and equivalence $\leftrightarrow$ are defined by abbreviation. We use the abbreviation $L_i \varphi$ for $\neg K_i \neg \varphi$ for agent $i$ suspects $\varphi$ may be true.

Note that $\top$ (true) is included explicitly in the language as using an abbreviation $p \lor \neg p$ requires the agent to be aware of the proposition $p \lor \neg p$, which cannot be taken for granted. Later, we will also introduce an awareness operator as an abbreviation. While awareness is not an atomic operator of the language, it is a crucial aspect of the semantics which are presented below.

Definition 2 (Contextual awareness model) A contextual awareness model for $N$ and $P$ is a tuple $M = (S, R, V)$ that consists of a domain $S$ of (factual) states (or ‘worlds’), an accessibility function $R : N \to \mathcal{L} \to \mathcal{P}(S \times S)$, and a valuation function $V : P \to \mathcal{P}(S)$. For $R(i)(\varphi)$ we write $R_i^\varphi$, and we denote the set $\{t | (s, t) \in R_i^\varphi\}$ by $sR_i^\varphi$. Given any agent $i$, $\varphi \in \mathcal{L}$, and state $s \in S$, we suppose for all $t \in sR_i^\varphi$, $tR_i^\varphi = sR_i^\varphi$. If we also require that $s \in R_i^\varphi$ then we have an epistemic contextual awareness model. A pointed contextual awareness model $(M, s)$ is a contextual awareness state (we will sometimes write $M_s$).

A contextual awareness model simply takes a doxastic (rational belief) model and refines the accessibility relation so that it is dependent on propositions. The epistemic (knowledge) awareness model includes the restraint that the accessibility relations be reflexive. Thus for an agent to know something, they must at least consider the true configuration of the universe (the actual world) possible, whereas a belief is not necessarily true, and may therefore be not upheld by the actual world. Naturally, models of failed reasoning and logical non-omniscience lend themselves more to the nature of belief than knowledge, but both give reasonable interpretations of omission. For the sake of generality, we will focus on doxastic models for now.

The other important question we must address is how an agent’s bias, focus or ignorance is formed. For example, should an agent who is aware of $\alpha \land \beta$ be necessarily aware of $\alpha$, and also aware of $\beta$? What about $\beta \land \alpha$. If we say no to all such questions we risk having agents who are incapable recognizing the construction of complex propositions, and thus being unable to reason (fallibly or not). However, if we assume strong closure properties for awareness we may demand to much from our agents as they are presumed to compute such closure functions. We will discuss various restrictions that may be placed on the awareness function after presenting the semantic interpretation of formulas.

Definition 3 (Semantics) Let $M = (S, R, V)$ be given,
As awareness is an important concept to be able to dis-
ϕ
under negation, agent
i
how this might be done. As the agent was unaware of
the notion of awareness. The third and fourth cases suggest
(what the agent knows and is also aware of) we need some
implicit knowledge of \[5\]. To capture
vacuously “knows” the proposition. This is akin to the
if an agent is totally unaware of a proposition, the agent
both cases the agent knew
(\[\varphi\])
and suppose \(s \in S\).

\[
\begin{align*}
(M, s) &\models \top \\
(M, s) &\models p \quad \text{iff} \quad s \in V(p) \\
(M, s) &\models \varphi \land \psi \quad \text{iff} \quad (M, s) \models \varphi \text{ and } (M, s) \models \psi \\
(M, s) &\models \neg \varphi \quad \text{iff} \quad (M, s) \not\equiv \varphi \\
(M, s) &\models K_i \varphi \quad \text{iff} \quad \forall t \in sR^i_t, (M, t) \models \varphi
\end{align*}
\]

The semantic interpretation of formulas simply guards the
accessibility relation of an agent with the formula in ques-
tion. This presents a number of scenarios:

- \(t \in sR^i_t \) and \((M, t) \models \varphi\): from \(s\) agent \(i\) considers
the proposition \(\varphi\) in the world \(t\) and it is true.

- \(t \in sR^i_t \) and \((M, t) \not\equiv \varphi\): from \(s\) agent \(i\) considers
the proposition \(\varphi\) in the world \(t\) and it is not true.

- \(t \notin sR^i_t \) and \((M, t) \models \varphi\): from \(s\) agent \(i\) does not
consider the proposition \(\varphi\) in the world \(t\) even though
it is true.

- \(t \notin sR^i_t \) and \((M, t) \not\equiv \varphi\): from \(s\) agent \(i\) does not
consider the proposition \(\varphi\) in the world \(t\) and it is not true.

If \(t\) was the only world in \(sR^i_t\), then in all cases above ex-
cept the second, we would have \((M, s) \models K_i \varphi\). That is,
if an agent is totally unaware of a proposition, the agent
vacuously “knows” the proposition. This is akin to the
implicit knowledge of \([5]\). To capture explicit knowledge
(what the agent knows and is also aware of) we need some
notion of awareness. The third and fourth cases suggest
how this might be done. As the agent was unaware of \(\varphi\), in
both cases the agent knew \(\varphi\). But furthermore, assuming an
awareness of \(\varphi\) implies an awareness of \(\neg \varphi\), in both cases
the agent also knew \(\neg \varphi\). Thus provided awareness is closed
under negation, agent \(i\) being unaware of \(\varphi\) (not consider
\(\varphi\) in any accessible world) is equivalent to \(K_i \neg \varphi\) \& \(K_i \neg \varphi\).

As awareness is an important concept to be able to dis-
uss, we will use this abbreviation: \(A_i \varphi \equiv L_i \varphi \lor L_i \neg \varphi\).
The requires the constraint that awareness is closed under
negation (that is for all agents \(i\), states \(s\) and formulas \(\varphi\),
\(sR_i^t = sR_i^t\neg\varphi\)). This is a reasonable restriction as all propo-
sitions are Boolean, so the existence of their negation is in-
herent in the fact that they are considered at all. However
for some purposes this restriction may be too much: one
could imagine a religious zealot who considers the non-
existence of a deity such a blasphemous concept that it may
not even be entertained for the purposes of refuting it. We
present some other closure properties of an awareness func-
tion below:

- **closure under true/false**: We say and agent \(i\)’s aware-
ness is closed under true/false if for all worlds \(s\),
\(sR^i_t = sR^i_t\top = S\). That is, the agent is aware of
the concepts of truth (\(\top\)) and falsity (\(\bot\)) in all worlds.

- **closure under negation**: We say an agent \(i\)’s aware-
ness is closed under negation if for all formulas \(\varphi\), for
all worlds \(s\), \(sR^i_t = sR^i_t\neg\varphi\). Thus if an agent con-
siders the interpretation of \(\varphi\) along any accessibility rela-
tion, the agent also considers the negation of \(\varphi\) along
that relation. As such to “consider \(\varphi\)” is to “consider
whether \(\varphi\) or \(\neg \varphi\) is true”.

- **closure under sub-formulas**: We say an agent \(i\)’s awareness is closed under sub-formulas if for all for-
formulas \(\varphi\), for subformulas \(\psi \subset \varphi\), for all worlds \(s\),
\(sR^i_t \subseteq sR^i_t\psi\). Thus if an agent considers the inter-
pretation of \(\varphi\) along any accessibility relation, the agent
also considers the interpretation of all subformulas of
\(\varphi\) along that relation. That is, the agent is aware of the
syntactic construction of \(\varphi\), and the meaning inherent
in that construction.

- **closure under Boolean combinations**: We say an agent
\(i\)’s awareness is closed under Boolean combinations if
for all worlds \(s \in S\), for all formulas \(\varphi, \psi \in \mathcal{L}\),
\(sR^i_t \subseteq sR^i_t\psi\) and \(sR^i_t \cap sR^i_t \psi \subseteq sR^i_t\psi\). If an
agent considers a set formulas in the context of a given
world, then the agent also considers all Boolean combin-
ations of those formulas in the context of that given
world. This is beginning to approach logical omniscience,
in the sense that this will require that the
agent knows all propositional tautologies, given that
the agent is aware of all propositions in that tautol-
ysimultaneously. However, the agent may still fail
to make valid propositional deductions if he does not
consider the relevant propositions in concert.

- **closure under introspection**: We say an agent \(i\)’s aware-
ness is closed under introspection if for all for-
formulas \(\varphi\), for all worlds \(s\), \(sR^i_t = sR^i_tK_i\varphi\). If an agent
is aware of a formula in the context of a given world,
then the agent is also aware of the concept of himself
knowing the formula in the context of the given world.
This is akin to awareness introspection, which is suit-
able for some systems, but not for others. An agent
could know \(\varphi\), but not know that they know \(\varphi\), if they
lack self-awareness or simply consider themselves to
external to the system under investigation.

- **closure under agents**: We say an agent \(i\)’s awareness is closed under a set of agents \(A\) if for all formulas \(\varphi\),
for all worlds \(s\), for all \(j \in A\), \(sR^j_t = sR^jK_i\varphi\). If an
agent is aware of a formula in the context of a given
world, then the agent is also aware of the concept of
any agent in \(A\) knowing the formula in the context of the
given world.

For this paper we will suppose that awareness is closed un-
der negation, true/false and subformulas, as this quanta of
\footnote{If we want to ensure allow/ensure that agents’ awareness is finite, we may add the restriction that \(\varphi\) is not of the form \(\neg \psi\).}
We may now define explicit knowledge as $\text{K}_i\phi \equiv \text{K}_i\phi \land \text{L}_i\phi$. Note that this interpretation of explicit knowledge varies from others found in the literature [10, 5]. For example, it still does not satisfy the axiom $K \vdash K(\alpha \rightarrow \beta) \rightarrow K\alpha \rightarrow K\beta$.

## 3 Examples and motivation

Typical approaches to logics of awareness (such as [5, 10]) have worked with the assumption that an awareness function will, at any given world, tell whether an agent is aware of a given proposition (i.e., if $\phi \in A_i(s)$). Implicit in this is the assumption that an agent’s awareness of a proposition is independent of the interpretation of that proposition. This is a reasonable assumption for perfect reasoners, but given that awareness is intended to model imperfect reasoning, there is an interesting avenue of investigation in considering what might arise if we relaxed this assumption.

- Consider a devout scientist, who believes that Darwin’s theory of evolution is supported by all scientific evidence and is true, but also maintains a literal belief in all that is stated in the Bible. While these are clearly contradictory beliefs, the contradiction only exists for the scientist if he is aware of both the Bible and the theory of evolution at the same time. The scientist (perhaps through design, perhaps through denial) could entertain a state of belief such that in every world in which he was aware of the Bible, was unaware of Darwin, and in every world in which he was aware of Darwin, was unaware of the Bible.

- A more common example is the addicted smoker mentioned earlier. The smoker knows smoking increases the risk of dying, and the smoker knows that he does not want to die. But the smoker is also adamant that he wants to continue smoking. Such a set of beliefs can only be facilitated by a state of partial awareness where the smoker does not maintain an awareness of all facts simultaneously, but is only aware of facts in certain contexts. In this case, the smoker knows he wants to smoke and he entertains this belief in all worlds where he lives a full and long life. He also knows that smoking leads to premature death, and considers this fact in all worlds where he has the future courage to quit smoking. Thus with his awareness tailored to his preferred outcomes the smoker is able to continue in a state of denial, without having to confront the incoherence of his beliefs. We may realize this scenario with a simple model with one agent and two states, $q$ and $s$ (for quit and smoke). In the world $q$, smoking ($S$) causes premature death ($D$), but our agent quits smoking. In the world $s$, smoking does not cause premature death, and our agent lives a long smoky life. The awareness function assigns awareness of the propositions $S$, $\neg S$, $D$, $\neg D$ to all four pairs of worlds, but only assigns the awareness of the propositions $S \rightarrow D$, ($S$ and $\neg D$) to the pair $(q, q)$.

The situation is depicted in Figure 1 (assuming negation and subformula closure). We can see that in world $q$, the agent knows smoking causes premature death ($K(S \rightarrow D)$), because in all worlds that the agent considers smoking implies death (relative to $q$), we have the proposition $S \rightarrow D$ being true. However the agent also considers it possible that he continues to smoke and not die ($L(S \land \neg D)$), as he also considers the propositions $S$ and $D$ in the context of the world $s$, where he supposes that he smokes without suffering a premature death.

One may note that the knowledge operator presented does actually satisfy the necessity rule, and thus for all agents $i$, for all validities $\theta$, we have $K_i\theta$ is valid. This seems counter-intuitive to our intent to model fallible reasoners, who certainly should not be able to access every validity of the logic. However, the fallibility is in the agents explicit knowledge, $K_i$: what the agent both knows and is aware off. The knowledge operator $K_i$ describes what the agent is able to synthesize. Our model of reasoning is one where the agent imagines various configurations of the universe. The agent is limited so its imaginings are restricted to subsets of propositions, and not all possible worlds consider the same propositions. However all worlds are consistent, not because the agent is an expert in logic, but simply because it is impossible to synthesize a world which contains a contradiction. Contradictory statements are necessarily quarantined in separate possible worlds. Thus one agent may profess to (explicitly) know the three propositions $A \rightarrow B$, $A$, and $\neg B$ are all true. However the agent cannot possibly imagine a world where all three propositions are true. Rather the agents flawed knowledge comes from the agent ignoring propositions where they might refute his knowledge. Thus this agent would only consider $A \rightarrow B$ in the context of worlds where both $A$ and $B$ are
false, and would find \( A \) true only in worlds where he did not consider \( A \rightarrow B \). Thus he never has to synthesized a world containing both \( A \) and \( A \rightarrow B \), which is necessary to conclude that \( B \) is at least possible.

4 Related work

Levesque’s response to the problem of logical omniscience was to separate implicit belief from explicit belief. From [10]: “...a sentence is explicitly believed when it is actively held to be true by an agent and implicitly believed when it follows from what is believed”. The semantic formulation for this logic is given through \textit{situations} rather than possible worlds, and for each each formula a situation may support the truth of that formula, the falsity of that formula or neither. These situations were potentially \textit{incoherent} in that a situation may support the truth of both \( \varphi \) and \( \neg \varphi \) for some formula \( \varphi \). This gave a very general approach for agents without logical omniscience. The semantics structures were also kept as general as possible. In fact, to avoid concerns about how implicit and explicit belief should interact with one another, Levesque simply proposed that formulas with nested belief operators not be permitted\(^2\).

This approach combines elements of model logic and situation calculus to reason about the knowledge of imperfect reasoners. A subsequent, and more involved, approach a long these lines is given by Cadoli and Schaerf [2], which separate interpretations for situations which are coherent, but incomplete, and those which are complete, but incoherent. See [9] for further discussion and generalizations.

Fagin’s and Halpern’s seminal 1988 paper [5] was also concerned with treating the problem of logical omniscience in agents. Their approach was to generalize epistemic logic [6] to handle logically non-omniscient agents, rather than to invent a wholly new approach. As with the approach described here, the intent was to permit logically non-omniscient agents, but with having any incoherent situations. The paper [5] referenced the earlier work of Levesque, but highlighted several important criticisms:

1. the lack of nesting for the modalities is clearly a significant limitation on the logic.

2. the motivation of modelling logical non-omniscience was only partially met. Particularly, it noted that while it was not necessary that agents believed all propositional validities, they would believe all validities of relevance logic, and there is no good reason to suppose that this is any more realistic than having logically omniscient agents.

In fact, three separate approaches were described. The first two logics use Levesque’s concept of implicit and explicit belief, modelled by including an explicit awareness function. The awareness function defines for each agent and for each world the set of propositions of which the agent is aware of in the given world. (In the first logic the propositions are assumed to be generated from a set of atomic propositions, and in the second more general sets of propositions are considered). This allows us to syntactically restrict an agent’s explicit belief to only apply to formulas of which the agent is aware. It is a very similar approach to the one we have taken, except instead of awareness being a function of pairs of worlds, it is simply a function of worlds. However, this makes a significant difference in the interpretation of knowledge. In the Fagin and Halpern logics, we first determine whether an agent is aware of a formula, and then determine their state of uncertainty with respect to that formula. Conversely, in our approach we first determine whether the agent agent may distinguish two worlds, and then determine whether the agent considers the formula relevant for the given pair of worlds.

The third approach mentioned in Fagin’s and Halpern’s paper was quite different and was built on the concept of a “society of minds”. The logic of local reasoning is a generalization of the previous logics, where an agent’s knowledge or belief is not determined by a set of worlds the agent cannot distinguish, but rather is determined by a set of \textit{sets of worlds}, each corresponding to a set of worlds the agent may not distinguish in a given frame of mind. The agent is said to know a proposition if, for at least one frame of mind, every world supports that proposition. As such an agent is able to believe \( \alpha \) and \( \neg \alpha \) at the same time, as long as they are support in separate worlds. An earlier version of the paper included a strong belief operator. It also mentioned the possibility to include awareness in the semantics although the semantics were not given.

The logic of local reasoning has many similarities with the logic we present. Particularly the agent’s “frame of mind” in the logic of local reasoning is similar to \( sR^\varphi \), the set of world’s in which an agent considers the interpretation of \( \varphi \). There is an important difference in that \( \varphi \) is only considered in the context of of the set \( sR^\varphi \), whereas in the logic of local reasoning it is considered in the context of all frames of mind. We will note in the discussion of axioms below that this leads to some notable discrepancies between the logics.

A novel approach to distinguishing what an imperfect agent might as compared to a logically omniscient agent can be found in [15, 13]. Here a dynamic epistemic logic is used to model deductive step an agent makes, so we might find an agent who is able to deduce a fact and may indeed be in the process of deducing a fact, but has not yet come to the final realization that the fact is indeed true. This model goes deeper than just awareness and the difference between
implicit and explicit knowledge: it requires us to model the processes through which an agent acquires knowledge. Part of this is becoming aware of new facts, but agents might also execute deductive steps to known facts, or they may even become aware of new deductive steps, that may then be applied to known facts. However, we are still required to model the difference between what the agents knows (explicitly) and what the agent is able to know.

This logic now allows deductions to occur incrementally by building up the set of formulas that an agent may be aware of. For example, an agent may explicitly know \( P \); on being informed of \( P \to Q \), the agent implicitly knows \( Q \), and on becoming aware of \( Q \) the agent then explicitly knows \( Q \). This captures the process by which knowledge is acquired and promoted from implicit knowledge to explicit knowledge.

In [7] Grossi and Velazquez expand on these ideas using the ideas of multi-valued logic. They differentiate between formulas agents are aware of and formulas agents have access to, such that access is stronger than awareness and captures the idea that a deduction has occurred that has brought the truth of a formula to the attention of an agent, and thus the agent may proceed to use this formula in future deductions. Understandably the semantics are quite complex and include the awareness of rules that might be used to acquire access to new formulas. Despite this complexity, the logic does give an agreeable solution to the issue of logical omniscience, where semantic entailment and awareness alone is not sufficient for an agent to infer the truth of a proposition. Instead we are able to model the process used by the agent to apply awareness of propositions and rules to acquire explicit access to the truth of a proposition. This is also useful for modelling resource bounded reasoning, where an agent may have limits on the types, size or number of formulas they have access to at any one point in time.

Finally, we will mention some other approaches to nonlogical agents, and fallible reasoning that are not based on auto-epistemic logics. These approaches are generally grouped together as nonmonotonic logics [11] and include default reasoning and belief revision [1]. The logics are referred to as nonmonotonic as their consequence relation is not monotonic, so you might be able to be able to infer \( \alpha \) from the propositions in \( \Gamma \), but not be able to infer \( \alpha \) from a superset of \( \Gamma \). This suggests that the initial inference of \( \alpha \) from \( \Gamma \) was in some way flawed, or was based on an incorrect assumption (a default). The logic we present is monotonic in the strict sense (with respect to logical deduction), but agent reasoning may be non-monotonic, in the sense that knowing additional facts may change an agents scope of vision. For example,

\[
K_i(\alpha \to \beta) \land \neg K_i((\alpha \land \gamma) \to \beta)
\]

could be satisfied if every world the agent considered \( \alpha \) implying \( \beta (\alpha \to \beta) \), both \( \alpha \) and \( \beta \) were true, but for some  

world where the agent considered \( \alpha \land \gamma \) implying \( \beta \) (but not just \( \alpha \) implying \( \beta \)), \( \alpha \) and \( \gamma \) were true and \( \beta \) was false. Note that this example would not apply if awareness was closed under subformulas and Boolean combinations.

5 Axiomatization

Here we present an axiomatization, LBO, for the logic of belief and omission. This is not a normal modal logic as we find that the axiom schema \( K_i(\alpha \to \beta) \to K_i\alpha \to K_i\beta \) is not valid. To see this suppose that the agent \( i \) does not consider \( \alpha \) and \( \beta \) together in any world. Then \( K_i(\alpha \to \beta) \) is vacuously true. However the agent may also know \( \alpha \) in the worlds in which he considers \( \alpha \), but not believe \( \beta \) certain in the worlds where he considers \( \beta \).

We will restrict our attention only to the case of belief (transitive, Euclidean models) and where the awareness function is closed under sub-formulas and negation.

For Fagin’s and Halpern’s logic of local reasoning [5] it is claimed that the axiom system EMNP [3] provides a complete axiomatization. (The system EMNP is designed for a generalizations of modal logic based on minimal models where the axiom K does not hold). However we find that this also is not valid: the axiom \( M : K_i(\alpha \land \beta) \to K_i\alpha \land K_i\beta \) fails for the same reason K does. However the axiom schema \( C : K_i\alpha \land K_i\beta \to K_i(\alpha \land \beta) \) is valid. (This schema is also presented in the context of minimal models [3]).

The axiomatization LBO consists of the axioms:

\[
P \quad \text{all propositional tautologies}
\]

\[
4 \quad K_i\alpha \to K_iK_i\alpha
\]

\[
5 \quad L_i\alpha \to K_iL_i\alpha
\]

\[
N \quad K_i\top
\]

\[
C \quad K_i\alpha \land K_i\beta \to K_i(\alpha \land \beta)
\]

and the rules:

\[
\text{MP} \quad \text{From } \vdash \alpha \to \beta, \vdash \alpha \text{ infer } \vdash \beta
\]

\[
\text{Subst} \quad \text{From } \vdash \alpha \text{ infer } \vdash \alpha[x\beta]
\]

\[
\text{Nec}^+ \quad \text{From } \vdash \bigwedge_{k=0}^{m} \beta_k \to \alpha \text{ infer } \vdash \bigwedge_{k=0}^{m} K_i\beta \to K_i\alpha
\]

where in Subst, \( x \) is free for \( \beta \) in \( \alpha \), meaning that no atomic proposition of \( \beta \) appears in \( \alpha \) (where \( \alpha[x\beta] \) means \( \alpha \) with every occurrence of \( x \) replaced by \( \beta \)).

We note that \( P, 4 \) and 5 are standard axioms for a logic of rational belief, and the rules are typical of all modal logics. The axiom C serves as a weak variation of the K axiom in the situation of limited awareness. The rule Nec+ is a
generalization of the \textbf{Nec} rule (the \textbf{Nec} rules comes from noting that the empty conjunct is equivalent to $\top$). An observant reader will note that as we have the axiom $\textbf{P}$ (all tautologies of propositional logic) and the rule \textbf{Nec} (all tautologies are believed), it appears we assume a high degree of logical omniscience from our agents. However recall that the base knowledge (or belief) operator includes vacuous knowledge, and it is feasible that the agent does not explicitly know many tautologies (where explicit knowledge is $K_o \land L_i o\alpha$). Rather the agent implicitly believes all tautologies because he cannot conceive a world where they do not hold. Note also that the seriality axiom \textbf{D} is replaced by \textbf{N} which states that truth is believed, but seriality cannot be assumed in the context of an arbitrary formula (thus permitting vacuous knowledge).

\textbf{Lemma 4} The axiom system \textbf{LBO} is sound for the logic of belief and omission.

\textbf{Proof} We show that every axiom and every rule is valid for all models:

$$\textbf{P}$$ Clearly every propositional tautology is evaluated with respect to a single world and according to the rules of propositional logic. Therefore every propositional tautology remains valid.

$$4$$ Suppose that $M_s \models K_o \alpha$, and let $t \in sR_i^{K_o \alpha}$. As awareness is closed under subformulas $t \in sR_i^\alpha$, and as $R_i^\alpha$ is transitive for all $u \in tR_i^\alpha$, we have $u \in sR_i^\alpha$. Since $M_s \models K_o \alpha$, we have $M_u \models \alpha$, and hence $M_t \models K_o \alpha$. Therefore $M_s \models K_o K_i \alpha$.

$$5$$ Suppose that $M_s \models L_i \alpha$, and let $t \in sR_i^{L_i \alpha}$ and $u \in sR_i^\alpha$ such that $M_u \models \alpha$. As the awareness function is Euclidean we have $u \in tR_i^\alpha$ so $M_t \models L_i \alpha$, and hence $M_s \models K_o L_i \alpha$.

$$\textbf{N}$$ This is trivially sound since $R_i^\top = S \times S$ and for all $s$, $M, s \models \top$.

$$\textbf{C}$$ Suppose that $M_s \models K_o \alpha \land K_i \beta$ and also suppose that $t \in sR_i^{\alpha \land \beta}$. As awareness is closed under subformulas, $t \in sR_i^\alpha$ and $t \in sR_i^\beta$. Therefore $M_t \models \alpha \land \beta$ so it follows that $M_s \models K_i (\alpha \land \beta)$.

$$\text{MP}$$ Follows from the semantic description for propositional logic.

$$\text{Subst}$$ Follows from the recursive description of the semantics.

$$\text{Nec}^+$$ Suppose that $\bigwedge_{k=0}^n \beta_k \rightarrow \alpha$ is valid, where each $\beta_k$ is a subformula of $\alpha$ and $t \in sR_i^\alpha$. As the awareness function is closed under subformulas, we have $t \in sR_i^\beta_k$ for all $k$. If $M_s \models \bigwedge_{k=0}^n K_i \beta_k$, then $M_t \models \bigwedge_{k=0}^n \beta_k$, and thus $M_t \models \alpha$.

\textbf{Lemma 5} The axiom system \textbf{LBO} is complete for the logic of belief and omission.

\textbf{Proof (Sketch)} We present a sketch of a proof showing that every consistent formula of the logic of belief and omission has a canonical model. We proceed by construction. Suppose that $\Sigma$ is the set of all maximal consistent subsets of the language $L$. We take these to be the sets of our canonical model and provide the valuation $V$ such that $M \models \Sigma$ and the accessibility relation $R$ where for all $\Delta \subseteq \Sigma$, $\Gamma \in \Delta R_i^\alpha$ if and only if for all subformulas $\beta$ of $\alpha$, $K_i \beta \in \Delta$ if and only if $\beta, K_i \beta$ are in $\Gamma$. It is straightforward to check that $M = (\Sigma, R, V)$ is a contextual awareness model. This follows from the axioms 4 and 5. We provide a truth lemma stating that for all formulas $\alpha \in L$, for all $\Delta \subseteq \Sigma$, if $\alpha \in \Delta$, then $M_\Delta \models \alpha$.

\textbf{Proof (Sketch)} We proceed by induction over the complexity of formulas. The case for propositions, conjunction and negation are trivial, so we are left with the knowledge operator $K_i$. Suppose that $K_o \alpha \in \Delta$. By the definition of $R$, for all $\Gamma \in \mathcal{R}_i^\alpha$ we have $\alpha \in \Gamma$ so by induction, $M_\Gamma \models \alpha$, and thus $M_\Delta \models K_o \alpha$.

$\iff$ Again, we proceed by induction and again the cases for propositions, negation and conjunction are trivial. We deal with the belief operator through contrapositive. That is we assume $L_i \alpha \in \Delta$ and seek to show that $M_\Delta \models L_i \alpha$. We let $\Lambda = \{ \beta \in d(\alpha) | K_i \beta \in \Delta \}$. If it is the case that $\vdash \bigwedge_{\beta \in \Lambda} \beta \rightarrow \neg \alpha$, then by \textbf{Nec} we have $K_i \neg \alpha \in \Delta$, contradicting $\Delta$’s consistency. Therefore $\bigwedge_{\beta \in \Lambda} \land \alpha$ is consistent and hence belongs to some maximal consistent set $\Gamma$. By induction we have $M_\Gamma \models \alpha$, and by definition we have $\Gamma \in \bigwedge_{\beta \in \Lambda} \land \alpha$.

\textbf{Proof (Sketch)} We note that decidability of the satisfiability problem for logic of knowledge and omission may be shown by encoding the contextual awareness function in \textbf{KD45}$_m$ using reserved propositional atoms, and a translation over the set of formulas. It can be shown that the translated formula is satisfied by a \textbf{KD45}$_m$ model if and only if the original formula was satisfied in the translated model. A similar technique is used in [14]

\section{Future work}

We have present a logic for belief and omission that captures a very human form of fallible reasoning: omission of deductive steps by failing to be synchronously of two related propositions at the same time. As we reason, we must
seek facts from our knowledge base and combine them according to logical laws to derive new knowledge. However, it is difficult to know what is a relevant fact and through incompetence or intent we may fail to associate known facts. This is just one aspect of non-logically omniscient agents, but an important one. We have given a sound and complete axiomatization. In future work we will seek to apply the logic to practical scenarios (particularly in an economic context), and seek to generalize aspects of fallible reasoning.

References


