Two-image resituation: practical algorithm

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ABSTRACT

Two-image resituation refers to the recovery of the geometric configuration of two stereo images. This involves determining three intrinsic parameters for each image and five relative orientation parameters. We show here that this can be achieved using only the image coordinates of homologous points, and needs no any other control information from object space. The approach is based on a thorough analysis of epipolar constraints. The explicit coplanarity equation defined by the intrinsic and relative orientation parameters is recast into a quadratic form whose parameters define a general coplanarity matrix. This matrix in turn can be written as the product of three matrices, two of which are defined by the intrinsic parameters, and one, called the special coplanarity matrix, is a function of the five relative orientation parameters. This paper presents a practical procedure for computing all these parameters from only image measurements. The basic strategy is first to find approximate values via closed-form solutions, and then to iteratively fine-tune them to precise values. The key steps are: (1) solving for the general coplanarity matrix via a nonlinear least-squares optimization; (2) solving for two focal lengths from the general coplanarity matrix via a closed-form algebraic solution; (3) determining the special coplanarity matrix from the general coplanarity matrix and the focal lengths; (4) determining the relative orientation parameters including three baseline components and three rotation angles via closed-form solutions; (5) fine-tuning all the explicit parameters via an iterative linearized least-squares solution. Original or improved solutions are developed for most stages of this procedure. Finally the computational theory is tested numerically.

Keywords: image resituation, interior and relative orientation, camera calibration, coplanarity equations, special and general coplanarity matrices, closed-form solution, iterative least-squares.

1 TWO-IMAGE RESITUATION

Image resituation refers to the process of placing images into their original relative geometric situation of imaging. This term was first coined as a functional component of a generic digital photogrammetric system. Pan et al. provide a general formulation of image resituation, a notion which could also be considered to be a generalization of aerial triangulation network adjustment in aerial photogrammetry to sciences of photogrammetry and computer vision. This paper addresses the computational aspects of image resituation in the case of a single stereo pair. We shall refer to this as two-image resituation.
1.1 Basic principle

Two-image resituation involves finding two sets of intrinsic parameters and the relative orientation parameters. This problem is nontrivial if no control information from object space is available and only the coplanarity equation can be used. Pan et al\(^{19}\) have proved in theory that for a restricted model of the camera intrinsic geometry, all the defined parameters in two-image resituation can be found from homologous image points only. This section summarizes the basic formulation of two-image resituation. Subsequent sections provide closed-form algebraic and iterative least-squares solutions for the geometric parameters in this formulation.

We define the origin of the image coordinate system to be at the centre of the physical image frame (e.g. as defined by fiducial marks on an optical metric camera or by the image frame centre of a CCD camera). Let \((x_c, y_c)\) be the coordinates of the principal point (the orthogonal projection of the camera perspective centre on the image plane) and \(f\) be the focal length of the left image (Fig.1). Similar tokens for the right image are denoted with a prime ' '. The two sets of intrinsic parameters are \((x_c, y_c, f)\) and \((x_c', y_c', f')\). Let \((x, y)\) and \((x', y')\) be a pair of homologous image points. In a global coordinate system, two images are oriented with rotation matrices \(R\) and \(R'\) respectively. In this coordinate system let \(b = (b_x \ b_y \ b_z)^T\) denote the baseline vector, and \(p\) and \(p'\) be a pair of homologous image point vectors. The coplanarity equation is then expressed as

\[ p \cdot (b \times p') = 0 \quad (1) \]

where

\[ p = R \begin{pmatrix} x - x_c \\ y - y_c \\ -f \end{pmatrix}, \quad p' = R' \begin{pmatrix} x' - x_c' \\ y' - y_c' \\ -f' \end{pmatrix} \quad (2) \]

Equation (1) can be reformulated to an implicit form,

\[ (x - x_c \ y - y_c \ -f) A \begin{pmatrix} x' - x_c' \\ y' - y_c' \\ -f' \end{pmatrix} = 0 \quad (3) \]

where \(A = (a_{ij}), i, j = 1, 2, 3,\) is a \(3 \times 3\) matrix, which we will denote the special coplanarity matrix. \(A\) can be expressed in terms of the explicit relative orientation parameters via

\[ A = R^T BR' \quad (4) \]

where

\[ B = \begin{pmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{pmatrix} \quad (5) \]

The implicit coplanarity equation (3) can be rewritten as

\[ (x \ y \ 1) D \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0 \quad (6) \]

where \(D = (d_{ij}), i, j = 1, 2, 3,\) is a \(3 \times 3\) matrix, called here the general coplanarity matrix. It is related to the intrinsic parameters and \(A\) by

\[ D = \Omega^T A \Omega' \quad (7) \]

where

\[ \Omega = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & -f \end{pmatrix}, \quad \Omega' = \begin{pmatrix} 1 & 0 & -x_c' \\ 0 & 1 & -y_c' \\ 0 & 0 & -f' \end{pmatrix} \quad (8) \]
A global coordinate system may be conveniently defined by taking as a reference frame either (1) the left (or right) image, or (2) the baseline and left image principal point. In this paper, we adopt the first reference frame. We now let \( R = (r_{ij}) \) denote the \( 3 \times 3 \) rotation matrix of the right image relative to the left image. Relation (4) then becomes

\[
A = B R
\]

(9)

\( R \) can be expressed as

\[
R = R_\alpha R_\beta R_\gamma
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\cos \beta & 0 & -\sin \beta \\
0 & 1 & 0 \\
\sin \beta & 0 & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(10)

where \( \alpha, \beta, \) and \( \gamma \) represent rotation angles around the \( C-x, C-y, \) and \( C-z \) axes respectively.

![Figure 1: Analytical Geometry of Two-Image Restitution](image)

With these formulations, our problem becomes to determine: (1) the general coplanarity matrix \( D \) from \( n \) pairs of homologous image points \( \{(x_i, y_i, x'_i, y'_i)\} \) where \( n \geq 8 \); (2) two sets of intrinsic parameters \( (x_c, y_c, f) \) and \( (x'_c, y'_c, f') \), after an appropriate modelling of camera intrinsic geometry, and the special coplanarity matrix \( A \) from \( D \); (3) the relative orientation defined by the baseline vector \( b \) and the rotation matrix \( R \) from \( A \). Our basic strategy consists of two stages. In the first stage, we use closed-form solutions to determine the approximate values of the above parameters. In the second stage, all the explicit parameters are fine-tuned through a global iterative least-squares solution.

### 1.2 Related work

In the literature of computer vision, Longuet-Higgins\(^7\) reformulated the parameters in the coplanarity equation into a \( 3 \times 3 \) matrix which he called the *essential matrix*. This matrix is equivalent to the special coplanarity matrix \( A \) defined above. While Longuet-Higgins assumed that the image coordinates had been offset by \( (x_c, y_c) \) and \( (x'_c, y'_c) \) and had been normalized by \( f \) and \( f' \), our formulation is done from the explicit coplanarity equation and without assuming the use of any specific reference system, and is thus more general. Faugeras et al\(^3\) further generalized the essential matrix to a \( 3 \times 3 \) *fundamental matrix*, which is equivalent to the general coplanarity matrix \( D \). Their formulation, however, is based on the assumption that a ‘frozen’ mobile camera is used for
taking a stereo pair, hence \((x_c, y_c, f) = (x'_c, y'_c, f')\). This situation mainly applies to motion analysis. They also incorporated into the intrinsic camera parameters some other systematic camera distortions, such as the nonorthogonality of the \(x\)- and \(y\)-axes, and the unequal units in the \(x\) and \(y\) measurements. In our formulation, we strictly distinguish those factors that can be precalibrated in the laboratory from those that cannot be fixed in real applications, for instance the focal lengths \(f\) and \(f'\). Furthermore, the camera model we use can adapt to real situations by allowing the focal length to change. The principal point \((x_c, y_c)\) either is detectable using fiducial marks in the case of optical metric cameras, or varies proportionally with the change of focal length (refer to section 3). Huang and Faugeras\(^4\) and Faugeras et al\(^5\) discussed various properties of the essential matrix and fundamental matrix. In particular, an approach based on singular value decomposition (equation (32)) for solving the general coplanarity matrix \(D\) in the next section was mainly developed by these groups.\(^\text{12}\) Hartley\(^4\) investigated the amount of information recoverable from the fundamental matrix, and developed a non-iterative procedure for factorizing out the intrinsic and extrinsic parameter matrices.

In the literature of photogrammetry, the closed-form solution for relative orientation was first studied by Khelebnikova\(^8\) and followed by Chang\(^1\) and Shih.\(^11\) However, the latest work\(^11\) is still not sufficiently developed in depth in comparison with that of Huang and Faugeras. The approach described in the next section is built upon, and is a generalization of, Huang and Faugeras’ work.

Pan et al\(^\text{18}\) presented the general problem formulation of image resituation. In particular, they presented a closed-form algebraic solution for solving for the two different focal lengths from the general coplanarity matrix which in turn can be solved using only image measurements. In this solution, the baseline and relative rotation matrix are first eliminated and then recovered once the focal lengths have been found. This work is important because it paves the way to multiple image resituation, e.g. aerial triangulation network adjustment, without the use of control points in the object space.

Nevertheless, a complete solution for the joint interior and relative orientation problem is not available in the literature. This paper provides a complete computational algorithm starting from the measured homologous image points through the general coplanarity matrix to the explicit geometric parameters in two-image resituation.

### 2 FINDING THE GENERAL COPLANARITY MATRIX

We begin by looking at the problem of finding the general coplanarity matrix \(D\) from the original image measurements. For a pair of data points indexed by \(i\) with image coordinates \((x_i, y_i)\) and \((x'_i, y'_i)\) on the left and right images, let us rewrite the general implicit coplanarity equation (6) as

\[
(x \quad y \quad 1) D \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = M_i d = 0
\]  
(11)

where

\[
M_i = (x_i x'_i \quad x_i y'_i \quad x_i \quad y_i x'_i \quad y_i y'_i \quad y_i \quad x'_i \quad y'_i \quad 1)
\]  
(12)

\[
d = (d_{11} \quad d_{12} \quad d_{13} \quad d_{21} \quad d_{22} \quad d_{23} \quad d_{31} \quad d_{32} \quad d_{33})^T
\]  
(13)

With \(n \geq 8\) pairs of homologous image points, we obtain \(n\) equations, which can be written as:

\[
M d = 0
\]  
(14)

where

\[
M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix} = \begin{pmatrix} x_1 x'_1 \quad x_1 y'_1 \quad x_1 \quad y_1 x'_1 \quad y_1 y'_1 \quad y_1 \quad x'_1 \quad y'_1 \quad 1 \\ x_2 x'_2 \quad x_2 y'_2 \quad x_2 \quad y_2 x'_2 \quad y_2 y'_2 \quad y_2 \quad x'_2 \quad y'_2 \quad 1 \\ \vdots \\ x_n x'_n \quad x_n y'_n \quad x_n \quad y_n x'_n \quad y_n y'_n \quad y_n \quad x'_n \quad y'_n \quad 1 \end{pmatrix}
\]  
(15)
From the definition of the general coplanarity matrix $D$ in (6), we observe that $D$ has two properties: (1) $D$ can only be determined up to a scale factor, and (2) the determinant $|D| = 0$, since $|B| = 0$. Because the second property is a third-order polynomial nonlinear constraint on the elements of $D$, there are 8 linearly independent parameters in $D$; however, only 7 of these are generally independent. The following formulation of least-squares solution is based on the fact that there are 8 linearly independent parameters in $D$.

We still assume all the 9 elements in $D$ to be unknowns. In order to constrain them into 8 linearly independent parameters, we add a normalization constraint
\[ d^T d = 1 \] (16)
Note that the normalization in (16) does not uniquely determine the sign of $d$, so both $d$ and $-d$ are valid solutions.

For each data point, we have an observation equation
\[ F_i(d) = M_i d = 0 \] (17)
In general, we assume the existence of noise in the image measurements $(x_i, y_i)$, so the observation equation can be linearized to
\[ F_{x,i} v_{x,i} + F_{y,i} v_{y,i} + F_{x',i} v_{x',i} + F_{y',i} v_{y',i} + M_i d = 0 \] (18)
where $v_{x,i}$ is the correction to $x_i$, and $F_{x,i}$ is the partial derivative of $F$ to $x$ computed by using the $i$-th data point.

Let $v$ be the vector of corrections to observations,
\[ v = (v_{x_1} \ v_{y_1} \ v_{x_1'} \ v_{y_1'} \ \ldots \ v_{x_n} \ v_{y_n} \ v_{x'_n} \ v_{y'_n})^T \] (19)
(20)
then the linearized observation equations can be written as
\[ G v + M d = 0 \] (21)
where
\[ G = -\begin{pmatrix}
F_{x,1} & F_{y,1} & F_{x,1}' & F_{y,1}' & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & F_{x,2} & F_{y,2} & F_{x,2}' & F_{y,2}' & \ldots & 0 & 0 & 0 & 0 \\
 & & & & \vdots & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & F_{x,n} & F_{y,n} & F_{x,n}' & F_{y,n}'
\end{pmatrix} \] (22)

The least-squares criterion requires minimizing the function
\[ \Phi = v^T W v - 2 \gamma^T (Gv + M d) - \lambda (d^T d - 1) \] (23)
where $W$ is the $4n \times 4n$ weight matrix of the $4n$ observations $x_i, y_i, x_i', y_i'$, $\gamma$ is a vector of coefficients, and $\lambda$ is a scalar coefficient. In general, we assume that there is no correlation between different measurements, so
\[ W = \begin{pmatrix}
w_{x_1} & 0 & \ldots & 0 \\
0 & w_{y_1} & \ldots & 0 \\
 & & \vdots & \ddots \\
0 & 0 & \ldots & w_{y_n}
\end{pmatrix} \] (24)
Taking the partial derivatives of $\Phi$ in (23), we obtain

$$\frac{\partial \Phi}{\partial \mathbf{v}} = 2\mathbf{v}^T \mathbf{W} - 2\gamma^T \mathbf{G} = 0$$

(25)

$$\frac{\partial \Phi}{\partial \mathbf{d}} = -2\gamma^T \mathbf{M} - 2\lambda \mathbf{d}^T = 0$$

(26)

From (25), we have

$$\mathbf{v} = \mathbf{W}^{-1} \mathbf{G}^T \gamma$$

(27)

Applying this expression to (21),

$$\gamma = - (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M} \mathbf{d}$$

(28)

Applying $\gamma$ to (26), we get

$$(\mathbf{M}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M}) \mathbf{d} = \lambda \mathbf{d}$$

(29)

From this equation, we see that $\lambda$ is an eigenvalue of the data matrix $\mathbf{M}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M}$, and $\mathbf{d}$ is an eigenvector corresponding to this eigenvalue.

Under the least-squares criterion, we seek

$$\min \mathbf{v}^T \mathbf{W} \mathbf{v} = \min (\mathbf{W}^{-1} \mathbf{G} \mathbf{v}^T \gamma)^T \mathbf{W} (\mathbf{W}^{-1} \mathbf{G} \mathbf{v}^T \gamma) = \min \mathbf{v}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T) \mathbf{v}$$

$$= \min \mathbf{d}^T \mathbf{M}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M} \mathbf{d}$$

$$= \min \mathbf{d}^T \mathbf{M}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M} \mathbf{d} = \min \lambda \mathbf{d}^T \mathbf{d} = \min \lambda$$

(30)

Therefore, we are looking for the smallest eigenvalue $\lambda$ of the data matrix $\mathbf{M}^T (\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T)^{-1} \mathbf{M}$.

Note that if $\mathbf{W}$ is diagonal as in (24), $\mathbf{G} \mathbf{W}^{-1} \mathbf{G}^T$ is also diagonal and its $i$-th diagonal element is computed as

$$\frac{1}{w_{x,i}^2} F_{x,i}^2 + \frac{1}{w_{y,i}^2} F_{y,i}^2 + \frac{1}{w_{x,i}^2} F_{x,i}^2 + \frac{1}{w_{y,i}^2} F_{y,i}^2$$

(31)

The solution to equation (29) requires an iterative nonlinear procedure because the computation of $\mathbf{G}$ involves an approximate value of $\mathbf{d}$. To begin the procedure, we first solve $\mathbf{d}$ from

$$(\mathbf{M}^T \mathbf{M}) \mathbf{d} = \lambda \mathbf{d}$$

(32)

This is equivalent to a least-squares solution where the observation equation (18) is simplified to

$$\mathbf{v}_i + \mathbf{M}_i \mathbf{d} = \mathbf{0}$$

(33)

where $\mathbf{v}_i$ is the correction to the fictitious observation $\mathbf{M}_i \mathbf{d}$ whose ideal value should be zero.

This simplified one-step least-squares solution in (32) has been used by Huang and Faugeras and Faugeras et al. The iterative least-squares solution proposed in (29) is an improvement, as it takes into consideration the distribution structure of data points and the uncertainty in the image point measurement. On the other hand, from the coplanarity equations in (14), we can see that if $n = 8$ and if an element $d_{ij}$ in $\mathbf{D}$ is nonzero, the column of coefficients associated with this element in $\mathbf{M}$ can be shifted to the right side of the equation, and the remaining 8 elements of $\mathbf{d}$ are solvable via a direct linear solution. This idea was first mentioned by Longuet-Higgins and was also used by some photogrammetrists e.g. Shih. However, the well established least-squares solutions in (29) or (32) has better numerical stability than a direct linear solution in which a randomly chosen element $d_{ij}$ is set to one.
3 SOLVING THE INTRINSIC PARAMETERS AND THE SPECIAL COPLANARITY MATRIX

Having found $D$, we now need to determine the intrinsic and relative orientation parameters. To find two focal lengths $f$ and $f'$ from $D$, we simply adopt the closed-form solution of Pan et al.\textsuperscript{10} We first explain the camera model assumed by this solution and then list the main equations that we shall employ.

### 3.1 An adaptive camera model

In general, we assume cameras to be adaptive, such that each image is taken with different intrinsic parameters $(x_c, y_c, f)$. However, it is known that there are 5 independent parameters for the relative orientation, and there are only 7 generally independent parameters in the coplanarity constraint. Therefore, in addition to the 5 relative orientation parameters, we can only solve for 2 independent parameters from the 6 intrinsic parameters $(x_c, y_c, f)$ and $(x'_c, y'_c, f')$. For photogrammetric purposes, two types of cameras are commonly used: metric optical cameras with chemical film, or digital cameras. In the case of a metric camera, the principal point can be precisely detected via fiducial marks, and we then take the detected point as the new origin of the coordinate system. There may still be a small discrepancy from this origin to the true principal point; this discrepancy is now taken as $(x_c, y_c)$.

In the case of a digital camera, $(x_c, y_c)$ is also very small. We therefore choose two focal lengths $f$ and $f'$ as the two independent parameters. We will consider two alternative cases: (1) $(x_c, y_c)$ is constant with variable focal length $f$, and (2) $(x_c, y_c)$ varies in direct proportion to $f$. In the first case, $(x_c, y_c)$ can be precalibrated before applications, and taken as the image coordinate origin. We can therefore set $x_c = y_c = 0$ in the coplanarity equation. In the second case, we assume that nonzero $(x_c, y_c)$ is caused by the non-orthogonality of the optical axis with the image plane. When the focal length $f$ varies, $(x_c, y_c)$ changes proportionally because of the similarity of triangles formed by the optical centre, image coordinate origin, and principal point of the image.

From the viewpoint of mathematical modelling, two cases can be included in a more general formulation:

$$x_c = x_c(f) = \lambda_{x0} + \lambda_x f$$
$$y_c = y_c(f) = \lambda_{y0} + \lambda_y f$$

where $\lambda_{x0}$ and $\lambda_{y0}$ are two scalar constants, $\lambda_x$ and $\lambda_y$ are two scalar coefficients. Here, we assume general continuous dependence of the principal point on $f$. If the dependence functions are expressible in Taylor expansions, the first two terms can be taken as a reasonable approximation.

Without loss of generality, this general formulation can always be transformed to the equivalent form:

$$(x_c, y_c) = (\lambda_x f, \lambda_y f)$$

because $(\lambda_{x0}, \lambda_{y0})$ in (34) and (35), once precalibrated, can be taken as the new image coordinate origin. We therefore only need to consider the first-order terms as shown in (36).

### 3.2 A closed-form solution for solving focal lengths

Pan et al.\textsuperscript{10} showed that if $\lambda_x$ and $\lambda_y$ are nonzero, three 7-th order algebraic equations of two unknowns $f$ and $f'$ exist, in which the coefficients are made up of the elements of $D$. The direct solution of these equations is elusive. Under the adaptive camera model considered in section 3.1, we may assume

$$x_c = y_c = x'_c = y'_c = 0$$

(37)
to be a reasonable initial approximation. With this assumption, three third-order equations of an unknown \( f' \) are obtained, each in the form

\[
s_1 + s_2 f'^2 + s_3 f'^4 + s_4 f'^6 = 0,
\]

and we can then solve for \( f' \) via

\[
f' = -\frac{h_2 + h_5 f'^2 + h_6 f'^4}{h_1 + h_3 f'^2 + h_4 f'^4},
\]

where coefficients \( h_i \)'s are defined in terms of the elements of \( D \) and \( s_i \)'s are computed from coefficients \( h_i \)'s. Equations (38) and (39) are more direct reformulations of the relevant formulas developed by Pan et al.\textsuperscript{10}

If equation (37) is true or is a reasonable approximation, the true or approximate focal lengths \( f \) and \( f' \) can be solved with the procedure given above. The true or approximate values for the special coplanarity matrix \( A \) can also be solved via:

\[
a_{ij} = d_{ij}, \quad i, j = 1, 2
\]

\[
a_{13} = -d_{13} \frac{1}{f'} - d_{11} \lambda_x - d_{12} \lambda_y
\]

\[
a_{23} = -d_{23} \frac{1}{f'} - d_{21} \lambda_x - d_{22} \lambda_y
\]

\[
a_{31} = -d_{31} \frac{1}{f'} - d_{11} \lambda_x - d_{21} \lambda_y
\]

\[
a_{32} = -d_{32} \frac{1}{f'} - d_{12} \lambda_x - d_{22} \lambda_y
\]

\[
a_{33} = d_{33} \frac{1}{f'} + (d_{31} \lambda_x + d_{32} \lambda_y) \frac{1}{f'} + (d_{11} \lambda_x + d_{23} \lambda_y) \frac{1}{f'} + (d_{12} \lambda_y + d_{22} \lambda_y) \lambda_y
\]

\[
\textbf{4 SOLVING EXPLICIT RELATIVE ORIENTATION PARAMETERS}

\subsection{4.1 Two symmetric solutions for the baseline vector}

Expanding equation (9), we obtain

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} = \begin{pmatrix}
  b_y r_{31} - b_z r_{21} & b_y r_{32} - b_z r_{22} & b_y r_{33} - b_z r_{23} \\
  b_x r_{11} - b_y r_{13} & b_x r_{12} - b_y r_{13} & b_x r_{13} - b_y r_{13} \\
  b_x r_{21} - b_y r_{11} & b_x r_{22} - b_y r_{12} & b_x r_{23} - b_y r_{13}
\end{pmatrix}
\]

As \( R \) is orthonormal, it follows that

\[
a_{11}^2 + a_{12}^2 + a_{13}^2 = b_y^2 + b_z^2
\]

\[
a_{21}^2 + a_{22}^2 + a_{23}^2 = b_x^2 + b_z^2
\]

\[
a_{31}^2 + a_{32}^2 + a_{33}^2 = b_x^2 + b_y^2
\]

\[
a_{11} a_{21} + a_{12} a_{22} + a_{13} a_{23} = -b_x b_y
\]

\[
a_{11} a_{31} + a_{12} a_{32} + a_{13} a_{33} = -b_x b_z
\]

\[
a_{21} a_{31} + a_{22} a_{32} + a_{23} a_{33} = -b_y b_z
\]
Once the intrinsic parameters and $a_{ij}$'s are solved, $b_x^2$, $b_y^2$, and $b_z^2$ can also be solved via equations (47)-(49) viz

$$\begin{pmatrix} b_x^2 \\ b_y^2 \\ b_z^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 \end{pmatrix} \quad (53)$$

The signs of $b_x$, $b_y$, and $b_z$ can then be determined by using equations (50)-(52). Two symmetric solutions $b_1 = (b_{x1}, b_{y1}, b_{z1})$ and $b_2 = (b_{x2}, b_{y2}, b_{z2})$ will be obtained, i.e.

$$b_1 = -b_2 \quad (54)$$

We may resolve the sign ambiguity as follows. Define

$$\begin{align*}
  b_{xy} &= -(a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23}) \\
  b_{xz} &= -(a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33}) \\
  b_{yz} &= -(a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33})
\end{align*} \quad (55-57)$$

From equations (50)-(52), we know

$$\begin{pmatrix} b_{xy} \\ b_{xz} \\ b_{yz} \end{pmatrix} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

All combinations of the signs of $b_x$, $b_y$, $b_z$, $b_{xy}$, $b_{xz}$, and $b_{yz}$ can be tabulated as

<table>
<thead>
<tr>
<th>$b_x$</th>
<th>$b_y$</th>
<th>$b_z$</th>
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From this table, for any combination of the known signs of $b_{xy}$, $b_{xz}$, and $b_{yz}$, we can select a unique symmetric pair of baseline vectors. At this stage we can not determine which of the two solutions for the baseline vector is uniquely valid in practice. However, with the two baseline solutions and the given $A$ matrix, we can solve for two different rotation matrices $R_1$ and $R_2$, and the corresponding sets of rotation angles $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. This can be done in the following way.

### 4.2 Two symmetric solutions for the rotation matrix and angles

Given the special coplanarity matrix $A$ and a candidate baseline solution, the rotation matrix $R$ can be solved via equation (9). As $|B| = 0$, however, $R$ cannot be solved directly from this relation. The orthonormality of $R$ needs to be exploited, and there are two possible approaches. The first approach is to represent $R$ by using an anti-symmetric matrix in terms of three independent elements. Although this approach has the advantage that a minimum number of unknowns is involved, it leads to highly nonlinear equations which are difficult to solve. The second approach is to use 6 unknowns, e.g. $r_{11}$, $r_{21}$, $r_{31}$, $r_{22}$, $r_{23}$, $r_{23}$. As $R$ is orthonormal, and $|R| = 1$, any element $r_{ij}$ can be represented by its cofactor, e.g.,

$$r_{13} = \frac{r_{21}}{r_{31}} \quad r_{23} = \frac{r_{11}}{r_{32}} \quad r_{33} = \frac{r_{11}}{r_{22}}$$

$$r_{11} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix} \quad r_{12} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix} \quad r_{12} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{13} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix} \quad r_{23} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix} \quad r_{33} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{11} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{12} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{13} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{23} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$

$$r_{33} = \begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$$
By using this property of \( R \), from (46) we obtain

\[
\begin{pmatrix}
    b_x^2 & b_x b_y & b_x b_z \\
    b_x b_y & b_y^2 & b_y b_z \\
    b_x b_z & b_y b_z & b_z^2
\end{pmatrix}
\begin{pmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22} \\
    r_{31} & r_{32}
\end{pmatrix}
= \begin{pmatrix}
    a_{22} & a_{23} \\
    a_{32} & a_{33}
\end{pmatrix}
- \begin{pmatrix}
    a_{21} & a_{23} \\
    a_{31} & a_{33}
\end{pmatrix}
\begin{pmatrix}
    a_{12} & a_{13} \\
    a_{22} & a_{23}
\end{pmatrix} 
\]  

(60)

Directly from relation (46) we obtain

\[
\begin{pmatrix}
    0 & -b_z & b_y \\
    b_z & 0 & -b_x \\
    -b_y & b_x & 0
\end{pmatrix}
\begin{pmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22} \\
    r_{31} & r_{32}
\end{pmatrix}
= \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22} \\
    a_{31} & a_{32}
\end{pmatrix}
\]  

(61)

Although each of the coefficient matrices in (60) and (61) is singular, particular combinations of their equations can lead to nonsingular matrices.

If \( b_x \neq 0 \), we can use

\[
\begin{pmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22} \\
    r_{31} & r_{32}
\end{pmatrix}
= \begin{pmatrix}
    b_z & 0 & -b_y \\
    -b_y & b_x & 0 \\
    b_y b_z & b_x b_y & b_x b_z
\end{pmatrix}^{-1}
\begin{pmatrix}
    a_{21} & a_{22} \\
    a_{31} & a_{32} \\
    a_{22} & a_{23} \\
    a_{32} & a_{33}
\end{pmatrix}
\]  

(62)

The determinant of the coefficient matrix to be inverted is \( b_x^2 (b_y^2 + b_z^2 + b_x^2) \), so if \( b_x \neq 0 \), the inverse is guaranteed to exist.

Similarly, if \( b_y \neq 0 \), we can use

\[
\begin{pmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22} \\
    r_{31} & r_{32}
\end{pmatrix}
= \begin{pmatrix}
    0 & -b_z & b_y \\
    -b_y & 0 & b_x \\
    b_y b_z & b_x b_y & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22} \\
    a_{31} & a_{32}
\end{pmatrix}
\]  

(63)

If \( b_z \neq 0 \), we can use

\[
\begin{pmatrix}
    r_{11} & r_{12} \\
    r_{21} & r_{22} \\
    r_{31} & r_{32}
\end{pmatrix}
= \begin{pmatrix}
    0 & -b_z & b_y \\
    b_z & 0 & -b_y \\
    b_y b_z & b_x b_y & b_x b_z
\end{pmatrix}^{-1}
\begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22} \\
    a_{21} & a_{23}
\end{pmatrix}
\]  

(64)

The problem of solving \( R \) from a given \( A \) and known baseline components is equivalent to a problem treated by Shih\textsuperscript{11} where the solution requires inversion of a \( 9 \times 9 \) coefficient matrix. The invertibility of that \( 9 \times 9 \) matrix is not guaranteed. Our solution presented above has the advantage of requiring inversion of a \( 3 \times 3 \) matrix whose inverse is guaranteed to exist.

Once \( r_{ij} \)'s, \( i = 1, 2, 3; j = 1, 2 \), are solved, the remaining three elements \( r_{13}, r_{23}, \) and \( r_{33} \) can be computed via (59). Once \( R \) is solved, the three explicit rotation angles about the three principal axes can be solved via trivial trigonometric functions.
4.3 Determining the unique baseline and rotation

Given the special coplanarity matrix \( A \), two symmetric solutions of the baseline vector \( b_1, b_2 \) and the corresponding rotation matrices \( R_1(\alpha_1, \beta_1, \gamma_1) \) and \( R_2(\alpha_2, \beta_2, \gamma_2) \) are solved in the way described above. Because the sign of \( A \) is arbitrary, there are four combinations of baseline and rotation which are valid with respect to the original coplanarity equation: \( (b_1, R_1), (b_2, R_2), (b_1, R_2), \) and \( (b_2, R_1) \). However, only one of the four will be valid in practice. In order to determine this unique combination, we impose two constraints.

The first constraint is on the rotation angles. Because the two rotation matrices are symmetric, one of them corresponds to physically impractical rotation angles. In particular, \( \alpha \) and \( \beta \), as rotation angles about the \( x \) and \( y \) axis respectively, need to be in the range \([-\frac{\pi}{2}, \frac{\pi}{2}]\) for a reasonable stereo overlapping or vergence. However, to avoid using any subjective threshold, we can simply select one set of angles with

\[
\min(\max(|\alpha_1|, |\beta_1|), \max(|\alpha_2|, |\beta_2|))
\] (65)

After finding the unique rotation matrix \( R \), we can now determine the unique baseline by using the second constraint, which requires the imaged objects and the image plane to be on the same side of the perspective centre. The unique set of baseline components can be determined in the following manner. For any pair of homologous image points \((x, y)\) and \((x', y')\), let

\[
\begin{pmatrix}
u \\ v \\ w \\
\end{pmatrix} = \begin{pmatrix}
x - x_c \\ y - y_c \\ -f \\
\end{pmatrix}, \quad \begin{pmatrix}
u' \\ v' \\ w' \\
\end{pmatrix} = R \begin{pmatrix}
x' - x'_c \\ y' - y'_c \\ -f' \\
\end{pmatrix}
\] (66)

Let \((X \, Y \, Z)^T\) denote the corresponding object point represented in the left camera coordinate system, then

\[
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \kappa \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} + \kappa' \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}
\] (67)

where \( \kappa \) and \( \kappa' \) are two scale factors, which can be solved as

\[
\begin{pmatrix} \kappa \\ \kappa' \end{pmatrix} = (U^T U)^{-1} U^T \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} u \\ v \\ w \\
\end{pmatrix}
\] (68)

With \( \kappa \) and \( \kappa' \) solved as above, we can now consider the \( Z \) coordinate of the object point. We require

\[
Z = -\kappa f < 0, \quad \text{therefore} \quad \kappa > 0
\] (69)

With the unique rotation matrix \( R \) determined by using the first constraint, two symmetric baseline solutions lead to two \( \kappa \)'s that are opposite in sign. The baseline with positive \( \kappa \) is then selected as the final correct solution.

5 Fine-Tuning all explicit parameters

At this stage, we have obtained an approximate value for each of the explicit parameters \( f, f', b_x, b_y, b_z, \alpha, \beta, \gamma \). The approximate values for the principal points may now be computed via formulas (36) using the precalibrated \( \lambda_x \) and \( \lambda_y \) and the approximate \( f \).

As we already have an approximate value for each baseline component, we can select the maximum absolute value of these components and then normalize them by this selected value. Without loss of generality, we consider
the case where $|b_x| = \max(|b_x|, |b_y|, |b_z|)$, where $|.|$ denotes the absolute value. We may then initialize the baseline components to $b_x, b_y, b_z$. In all the subsequent iterations, we will then keep $b_x$ fixed to $\pm 1$ (depending on the sign of $b_x$), and only update $b_y$ and $b_z$. Note that this initialization of baseline components is essentially different from arbitrarily selecting one of three totally unknown components and fixing it to 1, and taking the other two as variables.

We therefore have a total of 7 parameters $f, f', b_y, b_z, \alpha, \beta, \gamma$, together with their initial values. The unknowns are now the corrections to these initial values, denoted by $k$

$$k = (\Delta f \quad \Delta f' \quad \Delta b_y \quad \Delta b_z \quad \Delta \alpha \quad \Delta \beta \quad \Delta \gamma)^T$$

For each $i$-th pair of homologous image points, we have four measurements $x_i, y_i, x'_i, y'_i$ with corresponding corrections $(\Delta x_i, \Delta y_i, \Delta x'_i, \Delta y'_i)$. For $n$ pairs of homologous image points, we have the vector of corrections to the image coordinates

$$\mathbf{v} = (\Delta x_1, \Delta y_1, \Delta x'_1, \Delta y'_1, \ldots, \Delta x_n, \Delta y_n, \Delta x'_n, \Delta y'_n)^T$$

As the special coplanarity matrix $A$ is a function of five parameters $b_y, b_z, \alpha, \beta, \gamma$ ($b_x$ is set to $\pm 1$), we can write the special coplanarity equation (3) as

$$F(f, f', b_y, b_z, \alpha, \beta, \gamma) = \left( x - \lambda_x f \quad y - \lambda_y f \quad -f \right)^T A(b_y, b_z, \alpha, \beta, \gamma) \left( x' - \lambda_x f' \quad y' - \lambda_y f' \quad -f' \right) = 0$$

(72)

Linearizing this function, we obtain the observation equations for $n$ pairs of data points

$$G \mathbf{v} = H \mathbf{k} - L$$

(73)

where

$$G = - \begin{pmatrix} F_{x,1} & F_{y,1} & F_{x',1} & F_{y',1} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{x,2} & F_{y,2} & F_{x',2} & F_{y',2} & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & F_{x,n} & F_{y,n} & F_{x',n} & F_{y',n} \end{pmatrix}$$

(4)

$$H = \begin{pmatrix} F_{f,1} & F_{f',1} & F_{f,1} & F_{f',1} & F_{f,1} & F_{f',1} & F_{f,1} & F_{f',1} & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{f,2} & F_{f',2} & F_{f,2} & F_{f',2} & F_{f,2} & F_{f',2} & F_{f,2} & F_{f',2} & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{f,n} & F_{f',n} & F_{f,n} & F_{f',n} & F_{f,n} & F_{f',n} & F_{f,n} & F_{f',n} & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

(5)

$$L = -(F_1 \quad F_2 \quad \ldots \quad F_n)^T$$

(6)

where $F_{f,i}$ denotes the partial derivative $F_f = \frac{\partial F}{\partial f}$ of the function $F$ in (72) with respect to variable $f$, computed with the $i$-th pair of data points $(x_i, y_i, x'_i, y'_i)$; $F_i$ denotes the value of function $F$ computed with the $i$-th pair of data points.

Under the least-squares criterion, we seek

$$\min_\mathbf{v} \mathbf{v}^T W \mathbf{v}$$

(77)

where $W$ is the $4n \times 4n$ weight matrix of the $4n$ measurements. The final solution can be derived as

$$k = (H^T (GW^{-1} G^T)^{-1} H)^{-1} H^T (GW^{-1} G^T)^{-1} L$$

(78)
After the correction vector $k$ is solved, the approximate values of unknowns are updated and then used for the next iteration.

Given good approximate values obtained from the closed-form algebraic solutions presented in the previous sections, this iterative linearized least-squares solution for the 7 parameters will converge to precise values. The numerical behavior of these solutions is studied in the next section.

6 NUMERICAL EXPERIMENTS

The computational theory presented above has been tested numerically. Unlike image matching and image understanding, the problem domain of image resituation is closed, meaning that given the homologous image points, we have a limited and clearly defined objective of solving 7 geometric parameters. The theory can therefore be extensively tested with synthetic data. However, real data testing is always necessary for any given camera for real applications to ensure that the camera fits our model.

We first assign arbitrary but physically valid values to the 9 parameters, including:

- Two focal lengths: $f$, $f'$. These must be in the same unit as the image measurements $(x, y, x', y')$.
- Principal point constants: $\lambda_x, \lambda_y, \lambda'_x, \lambda'_y$. We may alternatively supply values for $x_c, y_c, x'_c, y'_c$, and compute the $\lambda$'s from these parameters and the two focal lengths $f$ and $f'$. In any case, we require $x_c, y_c, x'_c, y'_c$ to be very small compared with $f$ and $f'$. This relation can be defined by unitless parameters
  \[
  \max(|\lambda_x|, |\lambda_y|, |\lambda'_x|, |\lambda'_y|) < 1
  \]  
  (79)

- Baseline: $b_x, b_y, b_z$. Here we do not restrict $b_x$ to be $+1$. The three components can be set to any real values. To avoid unnecessary overflow in implementation, they are set to
  \[
  \max(|b_x|, |b_y|, |b_z|) \leq 1
  \]  
  (80)

- Three rotation angles: $\alpha, \beta, \gamma$. For practicality, we set them to
  \[
  \max(|\alpha|, |\beta|, |\gamma|) < \frac{\pi}{2}
  \]  
  (81)

With these assigned parameters, we can compute the ideal general coplanarity matrix $D$ with equation (7), and then normalize it according to equation (16). For ease of later least-squares solution, we can fix the sign of $D$ by setting $d_{11}$ to be positive (if $d_{11} = 0$, we may consider $d_{12}$, and so on). We denote this unique general coplanarity matrix by $D_0$. This matrix may then be taken as a reference for the estimated $D$ matrix that is solved numerically from synthesized homologous image points and normalized in the same way.

Synthetic homologous image points can be generated either by using synthetic 3D object points or by using $D_0$. The actual procedure for this synthesis is not important to this discussion. The test data should be generated such that the image points are distributed randomly in the whole image. After the image points are generated, random noise can be added to the image coordinates. Let $pp$ denote the measurement precision of image coordinates in pixel units for a reference image size of $1000 \times 1000$, i.e., the error bound for $x$ and $y$ of any image point is $\pm pp$ pixels. The error bound for different sized images is scaled accordingly.

With the parameters defined above, the following is a test data set. All the distance measurements are in the same unit. The left image size is defined to be $2f \times 2f$, and the right image size is $2f' \times 2f'$. The angles are in
The parameters \( x_c, y_c, x_c', \) and \( y_c' \) correspond to
\[
(\lambda_x, \lambda_y, \lambda'_x, \lambda'_y) = (0.060600, 0.027000, 0.022567, 0.014700)
\]
With the convergence measured as
\[
C_2 = 0.00016376 \\
C_3 = 0.00000016
\] (92, 93)

The correctness of convergence for the \(i\)-th iteration can be measured by the relative distance \(CC_i\) of the solved \(D^{(i)}\) to the ideal \(D_0\), where
\[
CC_i = \frac{\|D_0 - D^{(i)}\|}{\|D_0\|}  \\
(94)
\]
The correctness of convergence for the three iterations is
\[
CC_1 = 0.00053587  \\
CC_2 = 0.00042682  \\
CC_3 = 0.00042675
\] (95, 96, 97)

Three remarks can be made regarding these results: (1) Note that the first iteration is in fact the approach based on singular value decomposition proposed in computer vision literature.\(^{12,3}\) The iterative nonlinear least-squares solution developed in section 2 is an improvement as \(CC_3\) in (96) differs from \(CC_1\) in (95) at the first significant digit. (2) The iterative solution converges very fast. One more iteration after the singular value decomposition has in fact reached the convergence point as \(CC_3\) in (97) has no significant difference from \(CC_2\). (3) The iterative solution converges in the correct direction as
\[
CC_1 > CC_2 > CC_3
\] (98)

The two focal lengths \(f\) and \(f'\) are solved from \(D\) via the closed-form solution described in section 3
\[
f = 4.939286 \quad (99)
\]
\[
f' = 9.104688 \quad (100)
\]
The special coplanarity matrix \(A\) is then solved using formulas (40)-(45)
\[
A = \begin{pmatrix}
0.012205 & -0.015148 & 0.017009 \\
-0.006469 & 0.010754 & -0.032953 \\
-0.026085 & 0.025793 & 0.020615
\end{pmatrix}
\] (101)
The rotation matrix \(R\) and angles \(\{\alpha, \beta, \gamma\}\), and the baseline \(b\) are solved from \(A\) using the closed-form solutions described in section 4
\[
R = \begin{pmatrix}
0.899032 & 0.245857 & -0.349179 \\
-0.119970 & 0.928498 & 0.328696 \\
0.408862 & -0.253800 & 0.864245
\end{pmatrix}
\] (102)
\[
(\alpha, \beta, \gamma) = (0.363433, \ 0.356695, \ 0.266942)
\] (103)
\[
b = (0.034252, \ 0.024444, \ 0.008381)^T
\] (104)
\[
b \text { is then normalized to} \quad b = (1.0 \ 0.713657 \ 0.244688)^T
\] (105)

The following is the result of the iterative linearized least-squares solution for the 7 explicit independent parameters. The convergence is measured in terms of the absolute difference of each parameter in any two successive iterations. Because the 7 parameters are explicit, and some of them may take absolute zero as their true values, the relative difference may not be generally defined. The solution converges after three iterations with threshold being \(10^{-3}\) for maximum absolute difference of the 7 parameters in successive iterations.
where iteration 0 refers to the initial values solved via the closed-form solutions. The differences between the solved and correct values of these parameters are

\[
\begin{pmatrix}
\Delta f \\
\Delta f' \\
\Delta b_y \\
\Delta b_z \\
\Delta a \\
\Delta \beta \\
\Delta \gamma
\end{pmatrix} = \begin{pmatrix}
0.005669 \\
0.010565 \\
0.000301 \\
-0.000575 \\
-0.001303 \\
0.000716 \\
-0.001390
\end{pmatrix}
\]

From these results, we see that the various closed-form solutions provide good approximate values to the unknowns, and the iterative linearized least-squares solution does converge, and converges to correct values. This also confirms that there are indeed only 7 independent unknowns in two-image resituation. Simple tests have been made, which show that with more than 7 independent unknowns assumed, the least-squares solution does not converge. Many other tests have been done by varying the initial parameter settings, for example by setting one or more of \(b_x, b_y, b_z, \alpha, \beta, \) and \(\gamma\) to negative; the algorithm still gives the correct result. Experiments show that better precision is reached if: (1) more data points (large \(n\)) are used, and/or (2) more precise homologous image points are measured (smaller \(pp\)), and/or (3) wider and less correlated is the data point distribution.

7 CONCLUSIONS

This paper provides a complete computational procedure for two-image resituation. It provides a solution for the intrinsic parameters, (in particular the focal length of each image), and five relative orientation parameters, using only homologous image points. The overall procedure consists of first solving the general coplanarity matrix, the two focal lengths, the special coplanarity matrix, the baseline vector and rotation matrix and angles, and finally fine-tuning all the 7 explicit independent parameters via an iterative linearized least-squares solution. In each step, the approach presented is either novel or an improvement over existing techniques. In particular, the iterative nonlinear least-squares solution for the general coplanarity matrix is a generalization and improvement on the approach based on singular value decomposition that has been used by the computer vision community; the closed-form solution for the baseline vector requires the inversion of only a \(3 \times 3\) data matrix, which is guaranteed to be nonsingular; the rotation matrix and the sign of the baseline vector are uniquely determined by practical constraints; and the final least-squares solution for the 7 independent parameters from image measurements also confirms that there are only 7 degrees of freedom in two-image resituation. The complete computational theory as a whole, for jointly solving interior and relative orientation from only image measurements, is novel to photogrammetry and computer vision.
M. Brooks and G. Newsam, as the group leaders, provided helpful suggestions.

9 REFERENCES