$p$-groups with few characteristic subgroups and ‘interesting’ automorphism groups

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July 2011
Overview

1. What makes a $p$-group ‘interesting’?

2. UCS $p$-groups: definition and properties

3. Exponent-$p^2$ case

4. Exponent-$p$ case
What makes a \( p \)-group ‘interesting’?

An interesting \( p \)-group \( G \) should:

(1) have interesting (e.g. symmetric/simple) internal structure, and
(2) have an interesting (e.g. \(|\text{Aut}(G)|_{p'} \), large) action on itself, and
(3) lie at the extreme of possibilities.

Classifying ‘interesting’ \( p \)-groups appears to require more representation theory and geometry than group theory.

\[ g(n) = \# \text{groups order } n; \quad s(n) = \# \text{semigroups order } n. \]

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<th>( g(n) )</th>
<th>( s(n) )</th>
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<tr>
<td>8</td>
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<td>( 1.8 \times 10^9 )</td>
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<td>256</td>
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<tr>
<td>257</td>
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Literature Review

Higman (1960) and Sims (1965) \( g(p^k) = p^{\frac{2k^3}{27}} + O(k^{\frac{8}{3}}) \).

Pyber (1993) \( g(n) \leq n^{\left(\frac{2}{27} + o(1)\right)\mu^2} (n = p_1^{k_1} \cdots p_r^{k_r}, \mu := \max\{k_1, \ldots, k_r\}) \).

Pyber (1993) \( \lim_{n \to \infty} \frac{\log(\# \text{ nilpotent gps order } \leq n)}{\log(\# \text{ gps order } \leq n)} = 1 \)

Kleitman, Rothschild, and Spenser (1976) \( \left(\frac{n}{2}\right)^\frac{n}{2} \leq s(n) \leq n^{n^2} \)

Large number of \( p \)-groups: ‘interesting’ \( p \)-groups on ‘boundary’ (extremal)

Higman (1960) and Sims (1965) Almost all \( p \)-groups \( G \) have a central elementary abelian subgroup \( N \) such that \( G/N \) is elementary abelian (i.e. \( N \leq \Omega_1(Z(G)) \) and \( \Phi(G) \leq N \)).

Helleloid and Martin (2007) The automorphism group of a finite \( p \)-group is almost always a \( p \)-group.
Constructing \( p \)-groups \( G \) with given action on \( G/\Phi(G) \)

**Notation:**

\( G \) is an \( r \)-generated \( p \)-group,

\( V := G/\Phi(G) = (\mathbb{F}_p)^r \) is an \( r \)-dimensional vector space over \( \mathbb{F}_p \),

\( \alpha: \text{Aut}(G) \to \text{Aut}(G/\Phi(G)) \cong \text{GL}(r, \mathbb{F}_p) \)

\( \Gamma := \alpha(\text{Aut}(G)) \leq \text{GL}(r, \mathbb{F}_p) \) linear gp induced on \( V \) by \( \text{Aut}(G) \).

**Bryant and Kovács (1978)** No restriction on \( \Gamma \leq \text{GL}(r, \mathbb{F}_p) \). (\( \exists \) \( p \)-group \( G \) with \( G/\Phi(G) = (\mathbb{F}_p)^r \) and \( \alpha(\text{Aut}(G)) = \Gamma \).)

[Proof non-constructive. HM: \( \Gamma \leq \text{GL}(r, \mathbb{F}_p) \) usually a unipotent \( p \)-group.]

**Question:** If \( \Gamma \) is a maximal subgroup of \( \text{GL}(r, \mathbb{F}_p) \), or a simple group (e.g. \( J_1, M_{11}, G_2(q) \)), then how do we find a ‘small’ \( p \)-group \( G \) where \( \text{Aut}(G) \) induces \( \Gamma \) on \( G/\Phi(G) \)?

**Idea:** Introduce extra structure.
UCS $p$-groups: definition and properties

**Definition:** A $p$-group $G$ is called a UCS group (for **U**nique **C**haracteristic **S**ubgroup) if it has only one nontrivial proper characteristic subgroup.

**Joint work:** SG, Csaba Schneider, and P$^3$ (= PP Pálfy).

**Convention:** Henceforth $G$ is an $r$-generated UCS $p$-group where $p > 2$.

**Notation:** $V := G/\Phi(G) \cong (\mathbb{F}_p)^r$ an $r$-dimensional vector space over $\mathbb{F}_p$.

1. If $G$ is abelian, then $G \cong (\mathbb{Z}_{p^2})^r$ (Henceforth: $G$ is **nonabelian** UCS);
2. $1 < Z(G) = G' = \Phi(G) < G$ are the only characteristic subgps of $G$;
3. $\exp(G) = p^2$ and $G^p = \Phi(G)$, or $\exp(G) = p$ and $G^p = 1$;
4. if $\exp(G) = p^2$, then $0 \to V \to G \to V \to 0$. Note $(xy)^p = x^py^p[y, x]^{p(p-1)/2} = x^py^p$, so $G/\Phi(G) \to \Phi(G): x\Phi(G) \mapsto x^p$ preserved by $\text{Aut}(G)$, hence $V \to V \cong (\Lambda^2 V)/U$ is a $\Gamma$-module isomorphism.
Exponent-\(p^2\) case

Definition: A \(\Gamma\)-module \(V\) (recall \(\Gamma \leq GL(V)\)) is called an ESQ-module (for Exterior Self-Quotient) if there is a \(\Gamma\)-submodule \(U\) of \(\Lambda^2 V\) such that \(V \cong (\Lambda^2 V)/U\) as \(\Gamma\)-modules. Call \(\Gamma\) an ESQ-group. (\(V\) need not be irred.)

Theorem [SG,P\(^3\),CS]: There exists an \(r\)-generated exponent-\(p^2\) UCS gp \(G\) with \(\alpha(\text{Aut}(G)) = \Gamma\) if and only if \(V = (\mathbb{F}_p)^r\) is an irred ESQ \(\Gamma\)-module.

(1) ESQ-modules closed under field extensions;
(2) ESQ-groups closed under subgroups;
(3) if \(V\) is an ESQ-module, then \(\dim(V) \geq 3\);
(4) an ESQ-group contains no (nontrivial) scalar matrices.
Exponent-$p^2$ case

Ex: $\Gamma = \text{SO}(3, F_p); \ V = F_p^3, \ U = 0 \rightsquigarrow \text{UCS } G$ with $|G| = p^6$.

Ex: $\Gamma = J_1, \ V = (F_{11})^7, \ dim(U) = 14 \rightsquigarrow \text{UCS } G$ with $|G| = 11^{14}$.

Ex: $\Gamma = G_2(q), \ q \text{ odd prime power}, \ V = F_q^7, \ dim(U) = 14$.

Ex: $\Gamma = M_{11}, \ V = (F_3)^5, \ dim(U) = 5 \rightsquigarrow \text{UCS } G$ with $|G| = 3^{10}$.

Question: Can we classify small dimensional ESQ modules?

Theorem [SG,P³,CS]: If $\Gamma \leq \text{GL}(4, F_p)$ is irred and ESQ with $p \neq 2, 5$, then $\Gamma \cong \text{AGL}(2, F_5)$.

Theorem [SG,P³,CS]: If $\Gamma \leq \text{GL}(5, F_p)$ is minimal irred ESQ with $p \neq 2$, then $\Gamma$ cyclic of order 11, or metacyclic of order 55.
Exponent-$p$ case

Recall: $1 = G^p < Z(G) = \Phi(G) = G' < G$ are only characteristic subgroups and $p > 2$.

**Ex:** $G = V \times \Lambda^2 V$ as a set. Define multiplication by

$$(v_1, w_1)(v_2, w_2) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2)$$

Then $G$ is exponent-$p$ UCS and $\Gamma = \text{GL}(V)$. If $g \in \text{GL}(V)$, then $\alpha_g \in \text{Aut}(G)$ where $\alpha_g(u, w) = (ug, w(g \wedge g))$.

**Canonical Ex:** Suppose $V$ an irred $\Gamma$-module where $\Gamma \leq \text{GL}(V)$ and $U$ is a max $\Gamma$-submodule of $\Lambda^2 V$. Define mult’n on the set $G = V \times (\Lambda^2 V)/U$ by

$$(v_1, w_1 + U)(v_2, w_2 + U) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2 + U)$$

Then $G$ is exponent-$p$ UCS and $\Gamma \leq \alpha(\text{Aut}(G))$. Canonical example $\sim$

**Thm [SG,P$^3$,CS]:** $\exists$ bij’n $r$-gen exp-$p$ UCS $p$-gps $G$ with $\Gamma \leq \alpha(\text{Aut}(G))$ $\leftrightarrow$ max $\Gamma$-submodules $U \leq \Lambda^2 V$ where $V = (\mathbb{F}_p)^r$ is an irred $\Gamma$-module.
Theorem [SG,P³,CS]: If $p > 2$ there are precisely 8 non-isomorphic exponent-$p$ UCS groups $G$ with $G/\Phi(G) \cong (\mathbb{F}_p)^4$.

Proof. Geometry of Klein quadric: $Q : \Lambda^2 V \to \Lambda^4 V \cong \mathbb{F}_p$ defined by $Q(w) = w \wedge w$ is a non-degenerate quadratic form. $\text{GL}(V)$ acts on $\Lambda^2 V$. Let $\Gamma = \text{GL}(V)_U$ be the setwise stabilizer of a subspace $U \leq \Lambda^2 V$. Then $G = V \times (\Lambda^2 V/U)$ is a UCS group if and only if $\Gamma$ is irreducible on $V$ and $\Lambda^2 V/U$ if and only if the restriction of $Q$ to $U$ is non-degenerate. There are precisely 8 orbits of such subspaces $U \leq (\Lambda^2 V)/U$ and hence 8 isomorphism classes of exponent-$p^2$ UCS groups of order $p^8$. □
Thank you!