INSTRUCTIONS:
Total Marks = 70

Attempt ALL questions.
Marks for parts of each question are shown in brackets next to the question.

The questions are not of equal value.

Do not remove the staple from the examination paper.

Write your answers in the space provided. You may use the back of pages and the two pages at the end of the paper for rough work.

PLEASE NOTE

Examination candidates may only bring authorised materials into the examination room. If a supervisor finds, during the examination, that you have unauthorised material, in whatever form, in the vicinity of your desk or on your person, whether in the examination room or the toilets or en route to/from the toilets, the matter will be reported to the head of school and disciplinary action will normally be taken against you. This action may result in your being deprived of any credit for this examination or even, in some cases, for the whole unit. This will apply regardless of whether the material has been used at the time it is found.

Therefore, any candidate who has brought any unauthorised material whatsoever into the examination room should declare it to the supervisor immediately. Candidates who are uncertain whether any material is authorised should ask the supervisor for clarification.
1. [6 marks] State whether the following limits exist, and, if so, evaluate them. Justify your answer (numerical approximations are not acceptable).

(a) \( \lim_{{x \to 3}} \frac{x^2 - 9}{x - 3} \)

\[ = \lim_{{x \to 3}} \frac{(x+3)(x-3)}{(x-3)} \]
\[ = \lim_{{x \to 3}} (x+3) \]
\[ = 6 \]

(b) \( \lim_{{x \to \infty}} \frac{x^2 + 9}{3x^2 + 7x - 3} \)

\[ = \lim_{{x \to \infty}} \frac{1 + \frac{9}{x^2}}{3 + \frac{7}{x} - \frac{3}{x^2}} \]
\[ = \frac{1}{3} \]
2. [10 marks]

(a) Differentiate the following functions. Do NOT simplify your answers.

(i) \( f(x) = \sqrt{3x^2 + 1} \)  
\[
\frac{d}{dx} \left( \sqrt{3x^2 + 1} \right) = \frac{6x}{2\sqrt{3x^2 + 1}}
\]

(ii) \( f(x) = x^3 e^{2x} \)  
\[
\frac{d}{dx} \left( x^3 e^{2x} \right) = 3x^2 e^{2x} + x^3 e^{2x} \cdot 2
\]
2(a) (Continued)

(iii) \( f(x) = \ln(3 + \sqrt{x}) \) 

\[
\frac{d}{dx} f(x) = \frac{1}{3 + \sqrt{x}} \cdot \frac{1}{2}\sqrt{x}
\]

(b) Let \( f(x) = \sqrt{e^x} \). Show that \( f'(x) = \frac{\sqrt{e^x}}{2} \).

\[
\int e^x = e^{\frac{1}{2}x}
\]

\[
\text{So } f'(x) = \frac{1}{2} e^{\frac{1}{2}x}
\]

\[
\int \frac{e^{\frac{1}{2}x}}{2}
\]

SEE OVER
3. [10 marks] Given that \( f'(0) = 2, g'(0) = 4, f(0) = 3, g(0) = 1, \) find the following:

(a) the derivative of \( f(x) + g(x) \) at \( x = 0. \) [2 marks]

\[
(f + g)'(0) = f'(0) + g'(0) = 2 + 4 = 6
\]

(b) the derivative of \( f(x) \cdot g(x) \) at \( x = 0. \) [4 marks]

\[
(f \cdot g)'(0) = f'(0) \cdot g(0) + f(0) \cdot g'(0) = 2 \cdot 1 + 3 \cdot 4 = 14
\]

(c) the derivative of \( \frac{f(x)}{g(x)} \) at \( x = 0. \) [4 marks]

\[
\left(\frac{f}{g}\right)'(0) = \frac{f'(0) \cdot g(0) - f(0) \cdot g'(0)}{[g(0)]^2} = \frac{2 \cdot 1 - 3 \cdot 4}{1^2} = -10
\]
4. [10 marks] A farmer has $1500 available to build an L-shaped fence along a straight river so as to create two identical rectangular pastures (see figure below). The materials for the side parallel to the river cost $6 per metre and the materials for the three sides perpendicular to the river cost $5 per metre. Denote by $x$ the length of the sides perpendicular to the river, and by $y$ the length of the side parallel to the river.

(a) Show that as a function of $x$ the total area $A(x)$ of the two pastures are

$$A(x) = \frac{15}{6} (100x - x^2)$$

Cost = $1500 = 15x + 6y

so \[ y = \frac{1500 - 15x}{6} \]

Area $A = \pi y$

so \[ A(x) = \pi \cdot \frac{1500 - 15x}{6} \]

\[ = \frac{15}{6} \left( 100x - x^2 \right) \]

QUESTION 4(a) CONTINUES OVER THE PAGE
(b) Find the dimensions of the fence to maximise the area of the pastures. [4 marks]

\[ A'(x) = \frac{15}{6} (100 - 2x) \]

\[ = 0 \]

\[ \Rightarrow x = 50 \]

\[ A''(x) = \frac{15}{6} (-2) < 0 \]

so there is a maximum at \( x = 50 \).

\[ y = \frac{1500 - 750}{6} = 125 \]

\[ x = 50 \text{ m}, \quad y = 125 \text{ m}. \]
4(b) (Continued)

(c) The farmer intends to put a total of fifty cows in the two pastures and animal welfare regulations dictate that there cannot be more than one cow per 100 square metres. Will the farmer be able to satisfy the regulations? [2 marks]

\[
\text{Area required for the cows} = (50)(100) = 5000 \text{ m}^2
\]
\[
\text{Area of pastures} = (50)(125) = 6250 \text{ m}^2
\]

The farmer will be able to satisfy the regulations.
5. [13 marks] A missile is fired up in the air at the edge of a 1000 m high cliff, so it travels up in the air and then falls on the beach below. Let $h(t)$ be the height of the missile above the cliff at time $t$ (in seconds). Then

$$h(t) = 50t - 5t^2, \quad t \geq 0.$$ 

(a) At what velocity is the missile fired? [2 marks]

$$\sqrt{v(t)} = h'(t) = 50 - 10t$$

$$v(0) = 50 \text{ ms}^{-1}$$

(b) What is the maximum height above the cliff that the missile rises to? [4 marks]

$$h'(t) = \sqrt{v(t)} = 50 - 10t = 0$$

$$\Rightarrow t = 5 \text{ s}$$

$$v'(t) = -10 < 0 \Rightarrow \text{maximum}$$

$$h_{\text{max}} = h(5) = 50(5) - 5(5^2)$$

$$= 125 \text{ m}.$$
(c) Find the height of the missile above the beach after 15 s. [2 marks]

\[ h(15) = 50(15) - 5(15)^2 \]
\[ = 750 - 5(225) \]
\[ = 750 - 1125 \]
\[ = -375 \text{ m} \]

So, missile is \(1000 - 375 = 625\) m above the beach.

(d) How long after being fired does the missile land on the beach? [3 marks]

\[ 50t - 5t^2 = -1000 \]
\[ \Rightarrow 5t^2 - 50t - 1000 = 0 \]
\[ \Rightarrow t^2 - 10t - 200 = 0 \]
\[ \Rightarrow (t - 20)(t + 10) = 0 \]
\[ \Rightarrow t = 20 \text{ s} \ (t \geq 0) \]
5(e) (Continued)

(e) With what velocity does the missile land on the beach? Interpret your answer.

\[ v(20) = 50 - 10(20) \]
\[ = -150 \text{ m s}^{-1} \text{ downward.} \]

6. Integrate the following function. Do NOT simplify your answer.

\[ f(x) = e^x + \frac{1}{\sqrt{x}} + \frac{1}{x} + 3 - x^4 \]

\[ \int \left( e^x + \frac{1}{\sqrt{x}} + \frac{1}{x} + 3 - x^4 \right) \, dx \]

\[ = e^x + 2\ln x + \ln x + 3x - \frac{x^5}{5} + c \]
7. Find the area bounded by the graph of \( f(x) = x^2 - 1 \) and the x-axis over the interval \([-1, 2]\). [4 marks]

\[
\text{Area} = - \int_{-1}^{1} (x^2 - 1) \, dx + \int_{1}^{2} (x^2 - 1) \, dx
\]

\[
= - \left[ \frac{3}{3} x^3 - x \right]_{-1}^{1} + \left[ \frac{3}{3} x^3 - x \right]_{1}^{2}
\]

\[
= - \left[ \left( \frac{1}{3} \cdot 1 - (-1) \right) - \left( \frac{1}{3} + 1 \right) \right] + \left( \frac{8}{3} - 2 \right) - \left( \frac{1}{3} \cdot 1 \right)
\]

\[
= \frac{4}{3} + \frac{4}{3}
\]

\[
= \frac{8}{3} \text{ units}^2
\]
8. Consider the functions \( f(x) = x^3 + 1 \) and \( g(x) = 4x + 1 \).

(a) Show that the graphs of \( f \) and \( g \) intersect at \( x = -2, 0, 2 \). [2 marks]

\[
\begin{align*}
\frac{x^3}{3} + 1 &= 4x + 1 \\
\Rightarrow \quad x^3 - 4x &= 0 \\
\Rightarrow \quad x(x^2 - 4) &= 0 \\
\Rightarrow \quad x(x + 2)(x - 2) &= 0 \\
\Rightarrow \quad x &= 0, 2, -2.
\end{align*}
\]

(b) Find the area bounded by the graphs of the functions

\( f(x) = x^3 + 1 \) and \( g(x) = 4x + 1 \)

\[
\begin{align*}
\text{Area} &= \int \frac{x^3}{3} - 4x \, dx \\
&= \left[ \frac{x^4}{4} - 2x^2 \right]_0^2 + \left[ 4x - \frac{x^3}{3} \right]_0^2 \\
&= 0 - (4 - 8) + (8 - 4) = 8 \text{ units}^2
\end{align*}
\]

QUESTION 8(b) CONTINUES OVER THE PAGE
Second Semester Examinations
November 2009

8(b) (Continued)

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

..............................................................

SEE OVER
9. (a) In his lecture, Winthrop Professor Adrian Baddeley discussed several examples of the applications and importance of mathematics in science. State three examples from his lecture. [3 marks]

1. Drug administration
2. Meteorite hurling toward earth
3. Liver function
4. Space shuttle challenger disaster

(b) In half a page, give details of any one of the examples that you have given above. [5 marks]

Drug administered to patients depending on weight of patient, as microgrammes per kilogramme.

QUESTION 9(b) CONTINUES OVER THE PAGE
9(b) (Continued) 

Of patient weight. A common error is to put the decimal place in the wrong place, making a "ten fold error". This can be fatal to the patient, either because of being a lethal dose, or too small a dose to have therapeutic benefit.
Rough Working